KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday January 10 2013 8.00–13.00
Brief solutions

---

1. (a) There is at least one optimal solution, which is integer valued. However, if the optimal solution is nonunique, there will also be noninteger optimal solutions.

(b) Since $\widehat{X}$ is nonnegative, summation of rows and columns of $\widehat{X}$ shows that $\widehat{X}$ is feasible. If we let the matrix S denote the dual slacks, i.e., $s_{ij} = c_{ij} - \widehat{u}_i - \widehat{v}_j$, then

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

as stated in the hint. Consequently, S has nonnegative components. In addition, complementarity holds, since $\widehat{x}_{ij} s_{ij} = 0$, $i = 1, \ldots, 3$, $j = 1, \ldots, 4$. This means that we have optimal solutions to the two problems.

(c) The nonzero components of the given W correspond to strictly positive components of $\widehat{X}$. By the properties of W, it follows that $\widehat{X} + \alpha W$ is optimal as long as $\widehat{X} + \alpha W$ is nonnegative. The most limiting positive and negative values of α are 0.5 and -1.5 respectively. These values correspond to two integer valued optimal solutions:

$$\widehat{X} - 1.5W = \begin{pmatrix} 6 & 0 & 0 & 2 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 3 & 7 \end{pmatrix} \quad \text{and} \quad \widehat{X} + 0.5W = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{pmatrix}.$$

(d) Since $\widehat{X}$ is not an extreme point, it is not provided as a solution by the simplex method.

2. (a) We may rewrite the linear program as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & x_i k + l + z \geq y_i, \quad i = 1, \ldots, m, \\ & -x_i k - l + z \geq -y_i, \quad i = 1, \ldots, m. \end{array}$$

The dual may for example be derived via Lagrangian relaxation. For nonnegative Lagrange multipliers $u \in I\!R^m$ and $v \in I\!R^m$ we obtain

$$\text{minimize} \quad z - \sum_{i=1}^m u_i(x_i k + l + z - y_i) - \sum_{i=1}^m v_i(-x_i k - l + z + y_i),$$

which may be rewritten as

$$\sum_{i=1}^m y_i u_i - \sum_{i=1}^m y_i v_i + \quad \text{minimize} \quad \begin{array}{l}\{(-\sum_{i=1}^m x_i u_i + \sum_{i=1}^m x_i v_i)k \\ +(-\sum_{i=1}^m u_i + \sum_{i=1}^m v_i)l \\ +(1 - \sum_{i=1}^m u_i - \sum_{i=1}^m v_i)z\}\,.\end{array}$$

The dual (DLP) then becomes

$$(DLP) \qquad \begin{array}{ll} \text{maximize} & \sum_{i=1}^m y_i(u_i - v_i) \\ \text{subject to} & \sum_{i=1}^m x_i(u_i - v_i) = 0, \\ & \sum_{i=1}^m (u_i - v_i) = 0, \\ & \sum_{i=1}^m (u_i + v_i) = 1, \\ & u_i \ge 0, \quad i = 1, \dots, m, \\ & v_i \ge 0, \quad i = 1, \dots, m. \end{array}$$

(b) We need to show that (LP) has an optimal solution with at least three active constraints, corresponding to at least three different points. Basically, (LP) is a three-dimensional problem and hence an extreme point has at least three active constraints. Note that in (LP), $-z \le kx_i + l - y_i \le z$, $i = 1, \dots, m$. Hence, an active constraint corresponds to $|kx_i + l - y_i| = z$. If $z = 0$, all constraints in (LP) are active. If $z > 0$, for each i, at most one of the constraints $-z \le kx_i + l - y_i$ and $kx_i + l - y_i \le z$ an be active. Hence, an optimal extreme point has at least three active constraints corresponding to three different indices, which means at least three different indices for which $|kx_i + l - y_i| = z$, i.e., at least three points at which $|kx_i + l - y_i| = z$.

In the above, we have implicitly assumed that (LP) is three-dimensional, which corresponds to the constraint matrix in (DLP) having full column rank. To be precise, we should also show that this is the case, so that the standard analysis applies. This is more of a technicality. To see that the constraint matrix of (DLP) has full row rank, assume that there is a linear combination of the rows of the constraint matrix which gives the zero vector, i.e., there are α, β and γ such that

$$\begin{aligned} x_i\alpha + \beta + \gamma &= 0, \quad i = 1, \dots, n, \\ -x_i\alpha - \beta + \gamma &= 0, \quad i = 1, \dots, n. \end{aligned}$$

We now need to show that $\alpha = \beta = \gamma = 0$. Adding the two equations for a given i gives $\gamma = 0$. Taking two different indices i and j gives $(x_i - x_j)\alpha = 0$. Consequently, $\alpha = 0$, since $x_i \ne x_j$ by the statement. Thus, $\beta = 0$, and we conclude that the constraint matrix has full row rank.

Since (LP) is feasible with bounded optimal value, it follows by strong duality that (DLP) is feasible with the same optimal value. Hence, if we solve (DLP) by the simplex method, we obtain a final basic feasible solution with a basis matrix of dimension 3×3. Corresponding to this matrix, there are three constraints in the primal that are satisfied with equality. The above argument thus applies.

3. (See the course material.)

4. (a) For $u = 1$, the resulting Lagrangian relaxed problem becomes

$$
(IP_1) \quad \begin{array}{ll} \text{minimize} & -2x_1 - 1x_2 - 3x_3 \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & x_j \in \{0,1\}, \quad j = 1,\dots,n. \end{array}
$$

By enumeration, we find two optimal solutions, $x(1) = (1\ 1\ 0)^T$ and $x(1) = (0\ 0\ 1)^T$.

(b) If $x(1)$ is an optimal solution to the Lagrangian relaxed problem for $u = 1$, a subgradient is given by $3x_1(1) + 6x_2(1) + 7x_3(1) - 8$. Hence, $x(1) = (1\ 1\ 0)^T$ gives a subgradient $s_1 = 1$ and $x(1) = (0\ 0\ 1)^T$ gives a subgradient $s_2 = -1$.

(c) Since $0 = 1/2s_1 + 1/2s_2$, the zero vector is a subgradient to $\varphi(u)$ at $u = 1$. Hence, $u = 1$ is an optimal solution to the dual problem.

5. (a) For the given cut patterns, we obtain

$$
B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_B = B^{-1}b = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix}, \quad y = B^{-T}e = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{pmatrix},
$$

with $e = (1\ 1\ 1)^T$. As $y \geq 0$ no slack variables enters the basis.
The subproblem is given by

$$
\begin{array}{ll} 1 - \frac{1}{6}\text{maximize} & 2\alpha_1 + 3\alpha_2 + 6\alpha_3 \\ \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 11, \\ & \alpha_i \geq 0, \text{ integer}, \quad i = 1,2,3. \end{array}
$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value of the subproblem is $\alpha^* = (2\ 1\ 0)^T$ with optimal value $-1/6$. Hence, $a_4 = (2\ 1\ 0)^T$ and the maximum step is given by

$$
0 \leq x = B^{-1}b - \eta B^{-1}a_4 = \begin{pmatrix} 20 \\ 25 \\ 40 \end{pmatrix} - \eta \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}.
$$

Hence, $\eta_{\max} = 30$ and x_1 leaves the basis, so that the basic variables are given by $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$. The reduced costs are given by

$$
y = B^{-T}e = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
$$

which gives $y_1 = 1/4$, $y_2 = 1/2$ and $y_3 = 1$.
The subproblem is given by

$$
\begin{array}{ll} 1 - \frac{1}{4}\text{maximize} & \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ \text{subject to} & 3\alpha_1 + 5\alpha_2 + 9\alpha_3 \leq 11, \\ & \alpha_i \geq 0, \text{ integer}, \quad i = 1,2,3. \end{array}
$$

We may enumerate the feasible solutions for this small problem to conclude that the optimal value is zero, so that the linear program has been solved. The optimal solution is $x_2 = 10$, $x_3 = 40$ and $x_4 = 30$, with $a_2 = (0\ 2\ 0)^T$, $a_3 = (0\ 0\ 1)^T$ and $a_4 = (2\ 1\ 0)^T$.

(b) The solution given by the linear programming relaxation happens to be integer valued. This means that we have solved the original problem as well. The optimal solution is to use 80 W-rolls, with 10 rolls cut according to pattern $(0\ 2\ 0)^T$, 40 rolls cut according to pattern $(0\ 0\ 1)^T$ and 30 rolls cut according to pattern $(2\ 1\ 0)^T$.

(Note that this is very special. In general one can not expect to obtain an optimal integer solution in this way.)