1. (a) This claim is true.
   
   The output of “active=find(g<sqrt(eps))” shows that constraints 1 and 3 are active at \( x^* \).
   
   In addition, the command “\texttt{rank(A(active,:))}” gives 2, showing that the constraint gradients of the active constraints are linearly independents, i.e., \( x^* \) is a regular point.
   
   (b) This claim is true.
   
   The output of “\texttt{norm(gradf-A'*lambdastar)}” is of the order of machine precision, i.e., \( \nabla f(x^*) - A(x^*)^T\lambda^* \) is numerically zero.
   
   In addition, the output of “[g lambdastar]” shows that \( g(x^*) \geq 0, \lambda^* \geq 0 \) and \( g_i(x^*)\lambda_i^* = 0, i = 1, \ldots, 24 \). Consequently, \( x^* \) together with \( \lambda^* \) satisfy the first-order necessary optimality conditions.
   
   (c) This claim is true.
   
   The additional requirement to first-order necessary optimality conditions is that the reduced Hessian \( Z^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z \) is positive semidefinite. The output of “\texttt{eig(Z'*HessL*Z)}” shows that \( Z^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z \) has all eigenvalues nonnegative, hence being positive semidefinite, where \( Z \) is a matrix whose columns form a basis for the nullspace of the Jacobian of the active constraints.
   
   (d) This claim is false.
   
   We have strict complementarity. Hence, in addition to existence of Lagrange multipliers, the second-order sufficient optimality conditions require the reduced Hessian \( Z^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z \) positive definite. This is not true, since one eigenvalue of \( Z^T\nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)Z \) is zero.
   
   (e) This claim is true.
   
   Since constraints 6, 7, \ldots, 24 are inactive at \( x^* \), they may be omitted from the problem without affecting the local optimality conditions. The resulting problem is then convex, since \( f \) and \(-g_i, i = 1, \ldots, 5\), are convex on \( \mathbb{R}^9 \).

   Therefore, first-order necessary optimality conditions are sufficient to ensure global minimality.

2. If the problem is put on the form

   \[
   \begin{align*}
   \text{minimize} & \quad f(x) \\
   \text{subject to} & \quad g(x) \geq 0, \; x \in \mathbb{R}^2,
   \end{align*}
   \]

   we obtain

   \[
   \begin{align*}
   \nabla f(x)^T &= \begin{pmatrix} x_1 + x_2 + \frac{3}{2} & x_1 + x_2 - \frac{3}{2} \end{pmatrix}, \\
   \nabla g(x)^T &= \begin{pmatrix} x_2 & x_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
   \nabla^2_{xx}\mathcal{L}(x,\lambda) &= \begin{pmatrix} 1 & 1 - \lambda_1 \\ 1 - \lambda_1 & 1 \end{pmatrix}.
   \end{align*}
   \]
With $x^{(0)} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}^T$ and $\lambda^{(0)} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, the first QP-problem becomes

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 4 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
\text{subject to} & \quad \begin{pmatrix} \frac{1}{2} & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ -2 \\ -\frac{1}{2} \end{pmatrix}.
\end{align*}
\]

The optimal solution of the QP-problem is given by the feasible point which is closest, in 2-norm, to $(-4, 2)^T$. This may for example be solved graphically:

The solution is $p^{(0)} = (-2, 2)^T$ with constraint 2 active. The Lagrange multiplier $\lambda^{(1)}_2$ of the active constraint is given by

\[
\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^{(1)}_2,
\]

i.e., $\lambda^{(1)}_2 = 2$. Thus, we have $\lambda^{(1)} = (0, 2, 0)^T$, and $x^{(1)}$ is given by $x^{(1)} = x^{(0)} + p^{(0)} = (0, 5/2)^T$.

3. (a) The problem $(QP)$ is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

\[
(P_{\mu}) \quad \min \quad \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 - \mu \ln(x_1 - 1)
\]
under the implicit condition that $x_1 + 1 > 0$. The first-order optimality conditions of $(P_{\mu})$ gives

\[
x_1(\mu) - \frac{\mu}{x_1(\mu) - 1} = 0, \quad x_2(\mu) = 0,
\]

Since $(QP)$ is a convex problem, $(P_{\mu})$ is an unconstrained convex problem, taking into account the implicit constraint $x_1 - 1 > 0$. Therefore, the first-order necessary optimality conditions are sufficient for global optimality. The first-order optimality conditions give $x_2(\mu) = 0$, and $x_1(\mu)$ is given by

\[
x_1^2(\mu) - x_1(\mu) - \mu = 0.
\]
i.e.,
\[ x_1(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}, \]
where the plus sign has been chosen for the square root to enforce \( x_1(\mu) - 1 > 0 \).
The dual part of the trajectory, i.e. \( \lambda(\mu) \), is normally given by \( \lambda_i(\mu) = \mu / g_i(x(\mu)) \), \( i = 1, \ldots, m \). Here we only have one constraint, so
\[ \lambda(\mu) = \frac{\mu}{x_1(\mu) - 1} = \frac{\mu}{-\frac{1}{2} + \sqrt{\frac{1}{4} + \mu}} = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}. \]

(b) As \( \mu \to 0 \) it follows that \( x(\mu) \to (1 \ 0)^T \) and \( \lambda(\mu) \to 1 \). Let \( x^* = (1 \ 0)^T \) and \( \lambda^* = 1 \). Then \( x^* \) and \( \lambda^* \) satisfy the first-order optimality conditions of \((QP)\). Since \((QP)\) is a convex problem, this is sufficient for global optimality of \((QP)\).

(c) We have
\[ x_1(\mu) = \lambda(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \mu}, \quad x_1^* = \lambda^* \quad \text{and} \quad x_2(\mu) = x_2^* = 0. \]
Therefore, \( \|x(\mu) - x^*\|_2 = \|\lambda(\mu) - \lambda^*\| \), and it suffices to consider \( \|x(\mu) - x^*\|_2 \).
The expression from above gives
\[ \|x(\mu) - x^*\|_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu} = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\mu} = \mu + o(\mu). \]
This is as expected. We would expect \( \|x(\mu) - x^*\|_2 \) and \( \|\lambda(\mu) - \lambda^*\|_2 \) to be of the order \( \mu \) near an optimal solution where regularity holds.

4. (a) We may write \( A = (I - e) \), with \( e = (1 \ 1 \ 1 \ 1)^T \). Then, a matrix whose columns form a basis for the nullspace of \( A \) is given by \( Z = (-e^T)1^T = (1 \ 1 \ 1 \ 1 \ 1)^T \).
(b) The step to the minimizer of the new problem can be written as \( p = Zp_Z \), where
\[ Z^T H Z p_Z = -Z^T (Hx^* + c + 20e_1). \]
As \( x^* \) is optimal to the original problem we have \( Z^T (Hx^* + c) = 0 \), so that \( Z^T H Z p_Z = -20Z^T e_1 \). Insertion of numerical values gives \( 10p_z = -20 \), i.e., \( p_Z = -2 \). Hence, if the optimal solution to the new problem is denoted by \( \bar{x} \), we obtain
\[ \bar{x} = x^* + Zp_z = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \]
(c) As \( x_5 < 0 \), \( \bar{x} \) is not feasible to the third problem. When finding \( \bar{x} \), we computed \( p \) as the first step in an active-set method for solving the third problem. The maximum steplength is given by the maximum \( \alpha \) such that \( x^* + \alpha p \geq 0 \). We obtain \( \alpha = 1/2 \). The new point, \( \hat{x} \), becomes \( \hat{x} = x^* + 1/2p = (4 \ 3 \ 2 \ 1 \ 0)^T \). This point is in fact optimal, as the Lagrange multiplier of an added constraint will
become positive. If the constraint \( x_5 \geq 0 \) is added as a fifth constraint, this can be verified algebraically by solving

\[
H\hat{x} + c = \begin{pmatrix} 20 \\ 1 \\ 2 \\ -5 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \\ \hat{\lambda}_4 \\ \hat{\lambda}_5 \end{pmatrix},
\]

to obtain the Lagrange multipliers. We obtain \( \hat{\lambda}_1 = 20, \hat{\lambda}_2 = 1, \hat{\lambda}_3 = 2, \hat{\lambda}_4 = -5, \hat{\lambda}_5 = 10. \) As \( \hat{\lambda}_5 \geq 0 \), the solution is optimal.

5. (See the course material.)