1. We have

\[ f(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2, \quad g(x) = x_1 + x_2 + x_3^2 + 2, \]

\[ \nabla f(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \nabla g(x) = \begin{pmatrix} 1 \\ 1 + 2x_2 \end{pmatrix}, \]

\[ \nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \nabla^2 g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \]

(a) Insertion of numerical values in the expressions above gives the first QP-problem according to

\[
\begin{aligned}
& \min \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 \\
\text{subject to} & \quad p_1 + p_2 = -2.
\end{aligned}
\]

This is a convex QP-problem with a globally optimal solution given by

\[ p_1 - \lambda = 0, \]

\[ p_2 - \lambda = 0, \]

\[ p_1 + p_2 = -2. \]

The solution is given by \( p_1 = -1, p_2 = -1 \) and \( \lambda = -1 \), which agrees with the printout from the SQP-solver.

(b) We can see that \( \nabla^2 f(x) \) is positive definite and \( \nabla^2 g(x) \) is positive semidefinite, independently of \( x \). Moreover \( \lambda \) is non-positive in all iterations. This implies that the solution to each QP subproblem is optimal also for the case when the equality constraint is changed to a less than or equal constraint. Hence, the iterates would not change at all if the constraint was changed as suggested.

(c) The inequality-constrained problem is a convex problem, and in addition a relaxation of the original problem. Hence we get convergence towards a global minimizer of this problem, which is also a global minimizer of \((NLP)\).

2. (a) The problem \((QP)\) is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

\[
(P_\mu) \quad \min \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - \mu \ln(x_1 + x_2 - 2)
\]

under the implicit condition that \( x_1 + x_2 - 2 > 0 \). The first-order optimality conditions of \((P_\mu)\) gives

\[
\frac{x_1(\mu)}{x_1(\mu) + x_2(\mu) - 2} - \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} = 0,
\]

\[
x_2(\mu) - \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} = 0.
\]

These equations are symmetric in \( x_1(\mu) \) and \( x_2(\mu) \). Hence, \( x_1(\mu) = x_2(\mu) \). This means that \( 2x_1(\mu)^2 - 2x_1(\mu) - \mu = 0 \), from which it follows that

\[
x_1(\mu) = x_2(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu}{2}} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\mu}.
\]
where the plus sign has been chosen for the square root to enforce \( x_1(\mu) + x_2(\mu) - 2 > 0 \). Since \((P)\) is a convex problem, this is a global minimizer.

The dual part of the trajectory, i.e. \( \lambda(\mu) \), is normally given by \( \lambda_i(\mu) = \mu / g_i(x(\mu)), \)
\( i = 1, \ldots, m \). Here we only have one constraint, so
\[
\lambda(\mu) = \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} = \frac{\mu}{\sqrt{1 + 2\mu} - 1} = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 2\mu}.
\]

(b) As \( \mu \to 0 \) it follows that \( x(\mu) \to (1 1)^T \) and \( \lambda(\mu) \to 1 \). Let \( x^* = (1 1)^T \) and \( \lambda^* = 1 \). Then \( x^* \) and \( \lambda^* \) satisfy the first-order optimality conditions of \((QP)\).

Since \((QP)\) is a convex problem, this is sufficient for global optimality of \((QP)\).

(c) We have
\[
x_1(\mu) - x^*_1 = x_2(\mu) - x^*_2 = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 2\mu} = \frac{1}{2} \mu + o(\mu).
\]

This is as expected. We would expect \( \|x(\mu) - x^*\|_2 \) to be of the order \( \mu \) near an optimal solution where regularity holds.

3. (See the course material.)

4. (a) The objective function is \( f(x) = e^{x_1} + x_1 x_2 + x_2^2 - 2x_2 x_3 + x_3^2 - 2x_1 - x_2 - x_3 \).

Differentiation gives
\[
\nabla f(x) = \begin{pmatrix} e^{x_1} + x_2 - 2 \\ x_1 + 2x_2 - 2x_3 - 1 \\ -2x_2 + 2x_3 - 1 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} e^{x_1} & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.
\]

In particular, \( \nabla f(\bar{x}) = (0 -1 -1)^T \). With \( g_1(x) = -x_1^2 - x_2^2 - x_3^2 + 5 \) we get \( g_1(\bar{x}) = 3 \), which means that constraint 1 is not active at \( \bar{x} \). Since \( \nabla f(\bar{x}) \neq 0 \), constraint 2 must be active for \( \bar{x} \) to possibly satisfy the first-order necessary optimality conditions. These conditions require the existence of a \( \lambda_2 \) such that
\[
\nabla f(\bar{x}) = a_2 \lambda_2
\]
and it cannot be fulfilled with \( \lambda_2 = 0 \). Hence, \( \lambda_2 > 0 \), and we obtain \( a_1 = 0, a_2 = a_3 = -1/\lambda_2 \). The condition \( -2/\lambda_2 + 2 = 0 \) so that \( \lambda_2 = 1 \). Hence, \( a = (0 -1 -1)^T \).

If \( a = (0 -1 -1)^T \), then \( \bar{x} \) fulfills the first-order necessary optimality conditions together with \( \lambda = (0 1)^T \).

(b) As we only have one active linear constraint at \( \bar{x} \) we obtain
\[
\nabla^2_{2,2} \mathcal{L}(\bar{x}, \lambda) = \nabla^2 f(\bar{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.
\]
Since $\tilde{\lambda}_2 > 0$, we also have that $A_+ (\tilde{x}) = a^T$, where we can let $a^T = (N \ B)$ for $B = -1$ and $N = (0 \ -1)$. We then obtain a matrix whose columns form a basis for the null space of $A_+ (\tilde{x})$ as

$$Z_+ (\tilde{x}) = \begin{pmatrix} I \\ -B^{-1}N \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix},$$

which gives

$$Z_+ (\tilde{x})^T \nabla^2 f (\tilde{x}) Z_+ (\tilde{x}) = \begin{pmatrix} 1 & 1 \\ 1 & 8 \end{pmatrix},$$

which is a positive definite matrix. Hence, $\tilde{x}$ fulfils the second-order sufficient optimality conditions and is therefore a local minimizer.

5. (a) The function $f(y) = y^2$ has derivative $f'(y) = 0$ for $y < 0$ and $f'(y) = 2y$ for $y > 0$. Hence, $f'(y)$ is continuous with $f'(0) = 0$. The second derivative is given by $f''(y) = 0$ for $y < 0$ and $f''(y) = 2$ for $y > 0$. Hence, $f''$ is discontinuous at $y = 0$. As a consequence, the objective function has discontinuous Hessian at points where $p_i^T x = u_i$ for some $i \in U$ or $p_i^T x = l_i$ for some $i \in L$.

(b) Consider a fixed $x$ and minimize over $y$ in $(QP)$. We want to show that $y_i = (p_i^T x - u_i)_+$, $i \in U$, and $y_i = (l_i - p_i^T x)_+$, $i \in L$. Assume that $p_i^T x - u_i < 0$ for some $i \in U$. Then, $y_i = 0$, since $y_i = 0$ is the the minimizer of $y_i^2$. Similarly, if $p_i^T x - u_i \geq 0$, the optimal choice of $y_i$ is $y_i = p_i^T x - u_i$, as $y_i^2$ is a strictly increasing function for $y_i > 0$. Hence, $y_i = (p_i^T x - u_i)_+$, $i \in U$, as required. The argument for $i \in L$ is analogous.

(c) We may write the Lagrangian function as

$$l(x, y, \lambda, \eta) = \frac{1}{2} \sum_{i \in U} y_i^2 + \frac{1}{2} \sum_{i \in L} y_i^2 - \sum_{i \in U} \lambda_i (y_i - p_i^T x + u_i) - \sum_{i \in L} \lambda_i (y_i + p_i^T x - l_i) - x^T \eta,$$

for Lagrange multipliers $\lambda_i \geq 0$, $i \in U \cup L$, and $\eta \geq 0$. Let $P_U$ be the matrix whose rows comprise $p_i^T$, $i \in U$, and analogously for $P_L$. Let subscripts "U" and "L" respectively denote the vectors with components in the two sets. Also, let $A_U = \text{diag}(\lambda_U)$, $Y_U = \text{diag}(y_U)$, $A_L = \text{diag}(\lambda_L)$, $Y_L = \text{diag}(y_L)$, $X = \text{diag}(x)$ and $N = \text{diag}(\eta)$. For a positive barrier parameter $\mu$, the perturbed first-order optimality conditions may be written

$$P_U^T A_U - P_L^T A_L - \eta = 0,$$

$$y_U - \lambda_U = 0,$$

$$y_L - \lambda_L = 0,$$

$$A_U (y_U - P_U (x + u_U)) = \mu e,$$

$$A_L (y_L + P_L (x - l_L)) = \mu e,$$

$$N x = \mu e.$$