Discontinuous Galerkin FE and the incompressible Navier-Stokes equations

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The Navier Stokes Equations

The Navier-Stokes equations *(NSE)* are a system of second order non-linear partial differential equations describing the motion of fluid flow. They are highly interesting from both a mathematical and an engineering perspective. In our study we restrict ourselves to the instationary incompressible NSE at moderate Reynolds numbers and constant viscosity ν:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + f, \quad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad (2)$$

Here ρ denotes the density, $\boldsymbol{u} = (u_1, u_2)^T$ the velocity vector, p the pressure and f a body force per mass unit. The Reynolds number is defined as usual $Re = \frac{u_{max} d}{\nu}$ with maximum inflow velocity u_{max} and cylinder height d in the test setup.

The nonlinearity in the convective term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is resolved using a pressure projection method as proposed in [2][6], the so-called Chorin Scheme.

Chorin Scheme - Continuous Galerkin Weak Formulation

Solve for the intermediate velocity u^*

$$\langle D_t^{n+1}u^*, v\rangle + \nu\langle\nabla u^*, \nabla v\rangle = \langle f^{n+1}, v\rangle - \langle u^n\cdot\nabla u^n, v\rangle, \quad \forall v\in V_h \quad (3)$$

Solve the pressure Poisson equation

$$\langle\nabla p^{n+1}, \nabla q\rangle = -\langle\nabla\cdot u^*, q\rangle/\Delta t, \quad \forall q\in Q_h \quad (4)$$

Solve the corrected velocity,

$$\langle u^{n+1}, v\rangle = \langle u^*, v\rangle - \Delta t\langle\nabla p^{n+1}, v\rangle, \quad \forall v\in V_h \quad (5)$$

where V_h, Q_h are test-/trialspaces, defined below

Initial and boundary conditions

Velocity at the channel inflow and sidewalls is prescribed as Dirichlet conditions. For the outflow, a Neumann condition is used. Furthermore, the Poisson-problem for the pressure is prescribed with Neumann-boundaries. Remark: These are not physically justified.

- $\boldsymbol{u} = \boldsymbol{u}_D$ on $\delta\Omega\setminus\delta\Omega_{out}$
- $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = 0$ on $\delta\Omega_{out}$
- $\frac{\partial p}{\partial \boldsymbol{n}} = 0$ on $\delta\Omega$
- $\boldsymbol{u}(t=0) = \boldsymbol{u_0}, \quad p(t=0) = p_0$

Stability constraints

- To avoid singular systems, one has to make sure that the number of pressure unknowns never exceeds the number of velocity unknowns.
- This is guaranteed if the Brezzi-Babuška conditions [1] are fulfilled.
- A non-sufficient, but generally accepted rule: the approximation order of the pressure has to be at least one lower then for the velocity [5].

FEniCS

For our simulations we use the FEniCS framework [4]. It provides

- automated, efficient solution of differential equations
- problem definitions in terms of variational formulations, easily readable
- implementation of jump and average operators

Test Setup

We implemented a finite elements solver and compared the simulation results of an all-continuous NSE approach and a formulation with discontinuous pressure approximation.

For the velocity we used a ansatz-space V_h of continuous piecewise quadratics (second order Lagrange elements (CG2)). For the pressure Q_h different spaces were considered: continuous piecewise linears (CG1) and discontinuous piecewise linears with interior penalty method (DG1 IP) and with a mixed formulation (DG1 Mix). Additionally we try a lower order method with constant discontinuous pressure (DG0) and linear velocity (CG1).

Test scenarios

As a test scenario we chose the well known Von-Kármán-vortex street behind a square obstacle in two-dimensional channel flow. The obstacle with a blockage ratio of $D/H = 1/8$ is located at 1/4th downstream the channel. Furthermore, we used a Poiseuille channel flow problem with known analytical solution for performance tests.

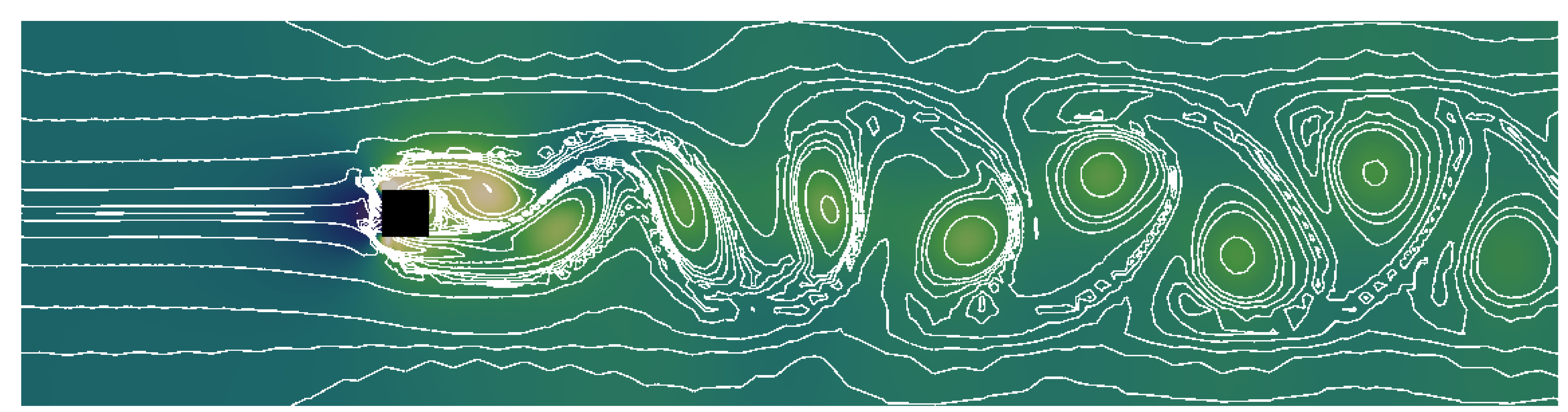

Figure 1: Vorticity isocontours and pressure of the obtained vortex street at $Re = 125$

Results

We successfully tested the used mesh resolution in a grid convergence study. For CG1 and DG1 IP we found good agreement of Stroughal number and drag coefficient with previous results [7], where the continuous pressure approximation had smaller errors (c.f. tab. 1). With the mixed formulation we could not get converged results for the Kármán-street, due to the increased computational cost.

	Re	CG1	DG1 IP
St	125	0.1414	0.1707
St	80	0.1190	0.1663
c_D	125	1.28	1.58
c_D	80	1.31	2.00

Table 1: Stroughal number and averaged drag coefficient c_D on the obstacle in a Von-Kármán street, for different Reynolds numbers

Furthermore we used the Posseuille problem to assess accuracy and computational cost of the various combinations (cf. fig. 2). All methods using CG2 for the velocity result in a similar error. Only the lowest order approximation causes a significantly higher deviation from the analytical solution. Regarding computational cost the lowest runtime was required by the all continuous approach CG1CG2. Especially the mixed DG1CG2 method resulted in high runtime. Note that the plots show functions of number of unknowns, not grid refinement levels.

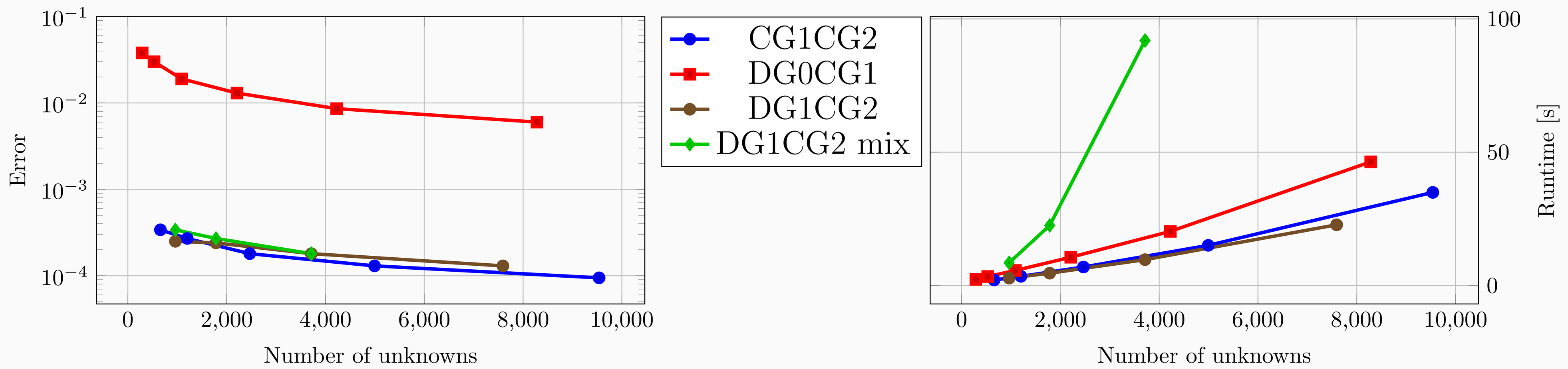

Figure 2: Error and runtime for the different CG/DG versions plotted against the total number of velocity/pressure unknowns

Conclusion

- Successfully implemented a discontinuous Navier-Stokes solver
- For problems with conforming triangulations or no *hp*-adaptivity are CG-methods not as complex
- Even though the mixed formulation is widely promoted in the available literature, we couldn't observe advantages
- A direct comparison between CG and DG is not meaningful, advantages show in different setups

Discontinuous Galerkin

Assume we are looking for an approximation u_h to the solution u of a given problem. Both classical continuous Galerkin (CG) and discontinuous Galerkin (DG) methods enforce the equation on each element of a triangulation using a Galerkin weak formulation. However, in contrast to CG, with DG there are no continuity constraints over the element boundaries, u_h is discontinuous. To link the values of u_h between the elements a suitably defined numerical trace $\widehat{u}_h$ is used.

Advantages and disadvantages [3]

+ Local conservativity
+ Implicit stabilitzation
+ Ability to handle complex geometries and irregular meshes
+ Good parallelizability
+ Ideal for *hp*-adaptive strategies

- More degrees of freedom $\rightarrow$ higher computational cost
- More complex formulation

Discontinuous Pressure Poisson Equation

We consider two different discontinuous approaches for deriving weak formulations of the Poisson equation for the pressure $\Delta p^{n+1} = \frac{\nabla\cdot\boldsymbol{u}^*}{\Delta t}$.

- **Interior penalty approach** [5]: We multiply Δp by a test function v and integrate over the domain.

$$\int_\Omega \Delta p v dx \approx \int_\Omega \nabla p\cdot\nabla v dx - \int_{\Gamma^0}\{\nabla p\}[\![v]\!]dS - \int_{\Gamma^0}\{\nabla v\}[\![p]\!]dS$$
$$-\int_{\Gamma^0}\eta_1[\![p]\!][\![v]\!]dS - \int_{\delta\Omega}\nabla p\cdot n v dS - \int_{\delta\Omega}\nabla v\cdot n p dS - \int_{\delta\Omega}\eta_2 p v dS$$

The first, second and fith term of the right-hand side arise directly from integrating the left hand side by parts. The other terms are added to obtain symmetry and for penalization, thus enforcing stability.

- **Mixed formulation** [3]: rewrite the Poisson problem as a hyperbolic system: $\nabla\cdot\boldsymbol{q} = \frac{\nabla\cdot\boldsymbol{u}^*}{\Delta t} =: f, \quad \boldsymbol{q} = \nabla p$ in Ω. Then multiply both equations by test functions $v\in V_h\subset H^l(\mathcal{T}_h), w\in\Sigma_h\subset(H^l(\mathcal{T}_h))^2$ and integrate over the domain. Combining the equations again, we finally obtain

$$\int_\Omega q\cdot w\, dx - \int_\Omega q\cdot\nabla v dx + \int_\Omega p\nabla\cdot w\, dx - \int_{\Gamma^0}[\![w]\!]\cdot\widehat{p}ds$$
$$+\int_{\Gamma^0}\widehat{q}\cdot[\![v]\!] - \int_{\delta\Omega}\widehat{p}n\cdot w dS + \int_{\delta\Omega}\widehat{q}\cdot n v dS = -\int_\Omega f v dx$$

- In these formulas, $[\![\phi]\!]$ denotes the jump operator and $\{\phi\}$ the average operator. $\widehat{\phi}$ is a general numerical flux term that can be chosen in several ways, usually as a combination of jumps and averages.

References

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