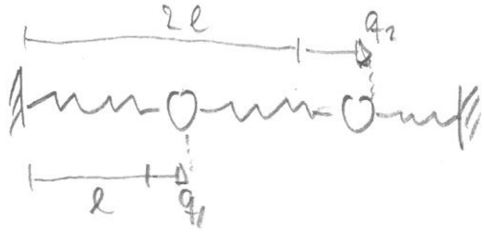
SG2150

2018-12-20

Lösningsförslag

1]



$$T = \frac{m\dot{q}_1^2}{2} + \frac{m\dot{q}_2^2}{2}$$

$$V = \frac{k}{2} q_1^2 + \frac{ak}{2}(q_2 - q_1)^2 + \frac{k}{2}(-q_2)^2$$

Eq point: $\frac{\partial V}{\partial q_1} = k\left[(1+a)q_1^* - a q_2^*\right] = 0 \quad \Rightarrow \quad q_1^* = q_2^* = 0$

$\frac{\partial V}{\partial q_2} = k\left[-a q_1^* + (1+a) q_2^*\right] = 0$

Mass matrix $M = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Stiffness matrix $K = k\begin{bmatrix} 1+a & -a \\ -a & 1+a \end{bmatrix}$

Eigenvalue problem $(K - \lambda M)\varphi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

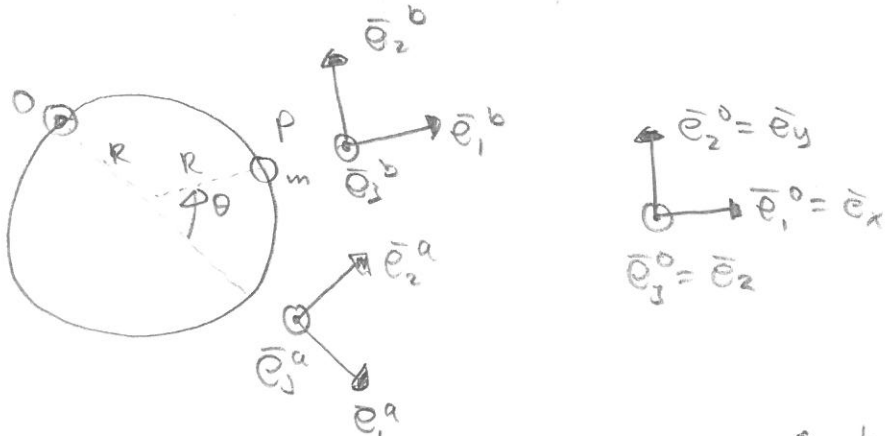
Symmetry $q_1 \to -q_2$, $q_2 \to -q_1$ gives modes are symmetric: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or anti-symmetric: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\varphi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \lambda_1 = \omega_1^2 = \frac{k}{m}$

$\varphi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \lambda_2 = \omega_2^2 = (1+2a)\frac{k}{m}$

If $\omega_2 = 2\omega_1$ then $1 + 2a = 4 \Rightarrow a = \frac{3}{2}$

2]



${}^{a}\bar{\omega}^{b} = \dot{\theta}\,\bar{e}_z \qquad {}^{0}\bar{\omega}^{b} = {}^{0}\bar{\omega}^{a} + {}^{a}\bar{\omega}^{b}$

${}^{0}\bar{\omega}^{a} = \omega\,\bar{e}_z$

$$\bar{r}_{OP} = R(\bar{e}_1^a + \bar{e}_1^b) \qquad \bar{V}_P = \frac{{}^{0}d}{dt}\bar{r}_{OP} = R\left[\frac{{}^{0}d\bar{e}_1^a}{dt} + \frac{{}^{0}d\bar{e}_1^b}{dt}\right] =$$

$$= R\left[\frac{{}^{a}d\bar{e}_1^a}{dt} + {}^{0}\bar{\omega}^{a}\times\bar{e}_1^a + \frac{{}^{b}d\bar{e}_1^b}{dt} + {}^{0}\bar{\omega}^{b}\times\bar{e}_1^b\right] =$$

$$= R\left[\bar{0} \times \omega\bar{e}_3^a\times\bar{e}_1^a + \bar{0} + (\omega+\dot{\theta})\bar{e}_3^b\times\bar{e}_1^b\right] = R\left[\omega\bar{e}_2^a + (\omega+\dot{\theta})\bar{e}_2^b\right]$$

$$T = \frac{m}{2}\bar{V}_P\cdot\bar{V}_P = \frac{mR^2}{2}\left[\omega^2 + (\omega+\dot{\theta})^2 + 2\omega(\omega+\dot{\theta})\,\bar{e}_2^a\cdot\bar{e}_2^b\right] =$$

$$= \frac{mR^2}{2}\left[\omega^2 + (\omega+\dot{\theta})^2 + 2\omega(\omega+\dot{\theta})c_\theta\right]$$

$V = 0 \quad L = T - V = T$ No non-conservative forcos.

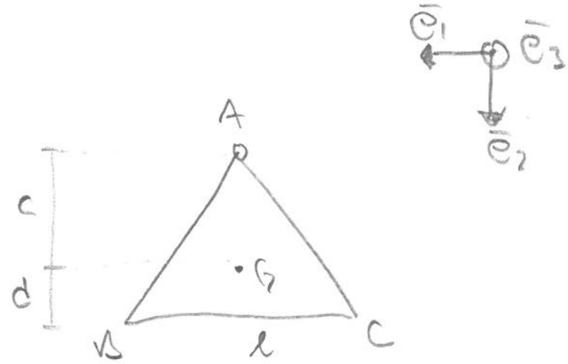
$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mR^2\left[(\omega+\dot{\theta}) + \omega c_\theta\right] \qquad \dot{p}_\theta = mR^2\left[\ddot{\theta} - \omega s_\theta\dot{\theta}\right]$$

$$\frac{\partial L}{\partial\theta} = -mR^2\left[\omega(\omega+\dot{\theta})s_\theta\right] \qquad \dot{p}_\theta - \frac{\partial L}{\partial\theta} = mR^2\left[\ddot{\theta} + \omega^2 s_\theta\right] = 0$$

Differential equation is $\ddot{\theta} + \omega^2 s_\theta = 0$. Same D.E. as for planar pendulum with $l = \frac{g}{\omega^2}$.

Constant $\theta(t) = \theta_0$ gives $0 + \omega^2 s_{\theta_0} = 0 \Rightarrow \theta_0 = 0$ ($\theta_0 = \pi$ collides with axis)

3



$d = \frac{1}{3}\,\frac{\sqrt{3}}{2}\,\ell = \frac{1}{2\sqrt{3}}\,\ell$

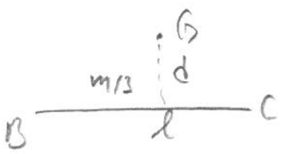
$c = 2d = \frac{1}{\sqrt{3}}\,\ell$

Planes through the point A with normals $\bar{e}_1$ or $\bar{e}_3$ are planes of mirror symmetry. Thus 1, 2, 3 are principal axes for the point A.

First use the point G. Three-fold rotation symmetry about the $\bar{e}_3$ axis and planar mass distribution gives

$$J_{G_1} = J_{G_2} = \frac{1}{2} J_{G_3}$$

Consider the rod BC.



$$J_{G_3}^{(BC)} = \frac{m}{3}\frac{\ell^2}{12} + \frac{m}{3} d^2 = \frac{m}{3}\left(\frac{\ell^2}{12} + \frac{\ell^2}{12}\right) = \frac{m}{3}\frac{\ell^2}{6}$$

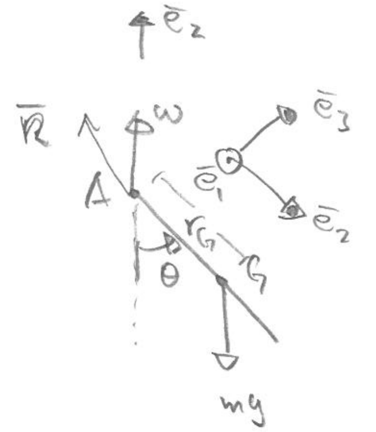
$$J_{G_3} = 3 \cdot J_{G_3}^{(BC)} = \frac{1}{6} m\ell^2 \Rightarrow J_{G_1} = J_{G_2} = \frac{1}{12} m\ell^2$$

$$J_{A_1} = J_{G_1} + mc^2 = \frac{1}{12} m\ell^2 + \frac{1}{3} m\ell^2 = \frac{5}{12} m\ell^2$$

$$J_{A_2} = J_{G_2} + m\,0^2 = \frac{1}{12} m\ell^2$$

$$J_{A_3} = J_{G_3} + mc^2 = \frac{1}{6} m\ell^2 + \frac{1}{3} m\ell^2 = \frac{1}{2} m\ell^2$$

4]



For the given motion, $\bar{e}_1$ stays horizontal and θ is constant.

1,2,3 are principal axes.

$$\bar{\omega} = \omega \bar{e}_z = \omega(-c_\theta \bar{e}_2 + s_\theta \bar{e}_3)$$

$$\bar{L}_A = -J_2 \omega c_\theta \bar{e}_2 + J_3 \omega s_\theta \bar{e}_3$$

$$\dot{\bar{L}}_A = \frac{d^{(1,2,3)} \bar{L}_A}{dt} + \bar{\omega} \times \bar{L}_A = \bar{0} + \omega(-c_\theta e_2 + s_\theta \bar{e}_3) \times \omega(-J_2 c_\theta \bar{e}_2 + J_3 s_\theta \bar{e}_3) =$$

$$= -(J_3 - J_2)\omega^2 s_\theta c_\theta \bar{e}_1 \,.$$ Planar mass $\Rightarrow J_3 - J_2 = J_1$

$$\bar{M}_A = \bar{0} \times \bar{R} + r_G e_2 \times (-mg\bar{e}_z) = -mg r_G s_\theta \bar{e}_1$$

A is fixed point: $\dot{\bar{L}}_A = \bar{M}_A \Rightarrow -J_1 \omega^2 s_\theta c_\theta \bar{e}_1 = -mg r_G s_\theta \bar{e}_1$

This is fulfilled if $J_1 \omega^2 s_\theta c_\theta = mg r_G s_\theta$. (1)

For the triangular frame: $J_1 = \frac{5}{12} m l^2 \quad r_G = \frac{1}{\sqrt{3}} l$

For $\theta = \pi/6$: $s_\theta \neq 0 \quad c_\theta = \frac{\sqrt{3}}{2}$

(1) $\Rightarrow \quad \frac{5}{12} m l^2 \omega^2 s_\theta \frac{\sqrt{3}}{2} = mg \frac{1}{\sqrt{3}} l s_\theta \Rightarrow$

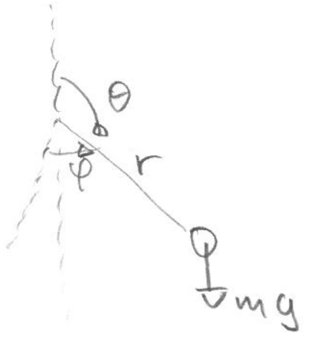
$$\omega^2 = \frac{8}{5} \frac{g}{l}$$

5] $\frac{\partial L}{\partial \varphi} = 0$, φ is cyclic $\Rightarrow$ $\dot{p}_\varphi = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 s_\theta^2 \dot{\varphi}$ is constant.

$\frac{\partial L}{\partial t} = 0$; L is time independent $\Rightarrow$ $p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - L =$

$= \frac{mr^2}{2}\left[\dot{\theta}^2 + s_\theta^2 \dot{\varphi}^2\right] + mgrc_\theta$ is constant.

The Lagrange function is that of a spherical pendulum.



6

Let $S' = \int_{t_0}^{t_1} \left[\left(\sum_a p_a \dot{q}_a \right) - H(q, p, t) \right] dt$

First order variation:

$$\delta S' = \sum_a \int_{t_0}^{t_1} \left[\left(\dot{q}_a - \frac{\partial H}{\partial p_a} \right) \delta p_a + p_a \delta \dot{q}_a - \frac{\partial H}{\partial q_a} \delta q_a \right] dt$$

Partial integration $\Rightarrow$ $\int_{t_0}^{t_1} p_a \delta \dot{q}_a \, dt = \left[p_a \delta q_a \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}_a \delta q_a \, dt$

Fixed endpoints $\Rightarrow$ $\delta q_a(t_0) = \delta q_a(t_1) = 0$

$$\delta S' = \sum \int_{t_0}^{t_1} \left[\left(\dot{q}_a - \frac{\partial H}{\partial p_a} \right) \delta p_a + \left(-\dot{p}_a - \frac{\partial H}{\partial q_a} \right) \delta q_a \right] dt$$

$\delta S' = 0$ for arbitrary $\delta q_a, \delta p_a$ $\Rightarrow$

$\dot{q}_a - \frac{\partial H}{\partial p_a} = 0 \quad -\dot{p}_a - \frac{\partial H}{\partial q_a} = 0$ for all $t_0 \leq t \leq t_1$