SF2812 Applied linear optimization, final exam
Thursday March 17 2016 8.00–13.00
Brief solutions

1. As $\widehat{x}_j > 0$, $j = 1, 2, 4$, the active constraints at $\widehat{x}$ are given by

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \\ \widehat{x}_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 16 \\ 0 \end{pmatrix}.$$

These constraints remain active for $\widehat{x} + \alpha p$, where p satisfies

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the given hint we obtain $p = (-1 \;\; 3 \;\; 0 \;\; 4)^T$. The additional requirement $\widehat{x} + \alpha p \geq 0$ gives

$$\begin{pmatrix} 1 \\ 9 \\ 0 \\ 4 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It follows that $\widehat{x} + \alpha p \geq 0$ for $-1 \leq \alpha \leq 1$. In addition, it holds that $c^T p = 0$, so that $\widehat{x} + \alpha p$ has the same objective function value as $\widehat{x}$ for all α. By taking the limiting values of α, we get two new points at which four constraints are active, namely

$$x^{(1)} = \widehat{x} - p = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \quad x^{(2)} = \widehat{x} + p = \begin{pmatrix} 0 \\ 12 \\ 0 \\ 8 \end{pmatrix}.$$

It follows that $\widehat{x} = 1/2 x^{(1)} + 1/2 x^{(2)}$. As there are four active constraints at these points, we expect them to be basic feasible solutions. By assuming that x_1 and x_2 are basic variables, we may compute y and s from $B^T y = c_B$, $s = c - A^T y$, i.e.,

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

with solution $y = (-1 \;\; 1)^T$, so that $s = c - A^T y = (0 \;\; 0 \;\; 1 \;\; 0)^T$. As $s \geq 0$, we have verified optimality of $x^{(1)}$, and $c^T p = 0$ implies that $\widehat{x}$ and $x^{(2)}$ are optimal as well. It is straightforward to verify that $x^{(2)}$ is also a basic feasible solution at which x_2 and x_4 are basic variables.

2. (a) With $X = \operatorname{diag}(x)$ and $S = \operatorname{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.9 \\ -0.9 \\ -0.9 \\ -0.9 \end{pmatrix}.$$

(b) If we compute $\alpha_{\max}$ as the largest step α for which $x+\alpha\Delta x \geq 0$ and $s+\alpha\Delta s \geq 0$ we see that the limiting step is given by $x_4 + \alpha_{\max}(\Delta x)_4 = 0$, i.e.,

$$\alpha_{\max} \approx \frac{1}{1.0756} \approx 0.9297.$$

As $\alpha_{\max} < 1$ we cannot accept the unit step. If we let $\alpha = 0.99\alpha_{\max}$ the new iterate becomes $x+\alpha\Delta x \approx (0.6161\ \ 0.7980\ \ 0.0302\ \ 0.0100)^T$, $y+\alpha\Delta y \approx (1.1414\ \ -0.8384)^T$, and $s + \alpha\Delta s \approx (0.5555\ \ 0.3737\ \ 1.1414\ \ 1.1616)^T$.

(The numeric value of $\alpha_{\max}$ is not required. It suffices to note how $\alpha_{\max}$ is determined and that it is less than one. Analogously, the numeric values of the new iterates are not required, it suffices to state how they are computed.)

3. (See the course material.)

4. The suggested initial extreme points $v_1 = (-1\ \ -1\ \ -1\ \ -1)^T$ and $v_2 = (1\ \ -1\ \ -1\ \ 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (1\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (-5\ \ -1)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -3 \end{pmatrix}.$$

By forming $c^T - y_1 A = (2/3 \;\; 4/3 \;\; 5/3 \;\; {-2/3})$ we obtain the subproblem

$$3+ \quad \begin{array}{ll} \text{minimize} & \frac{2}{3}x_1 + \frac{4}{3}x_2 + \frac{5}{3}x_3 - \frac{2}{3}x_4 \\ \text{subject to} & -1 \le x_j \le 1, \quad j = 1,2,3. \end{array}$$

An optimal extreme point to the subproblem is given by $v_3 = (-1 \;\; -1 \;\; -1 \;\; 1)^T$ with optimal value -4/3. Hence, α_3 should enter the basis. The corresponding column in the master problem is given by

$$\begin{pmatrix} Av_3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The change to the basic variables is given by

$$\begin{pmatrix} -3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Finding the maximum step η for which $\alpha + \eta p \ge 0$ gives

$$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} + \eta \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $\eta = 1/2$ so that α_1 leaves the basis.

Hence, the new basis corresponds to v_2 and v_3 so that

$$B = \begin{pmatrix} Av_3 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (1 \; 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_3 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_3 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_3 \;\; c^T v_2) = (-5 \;\; -1)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

By forming $c^T - y_1 A = (0 \;\; 1 \;\; 2 \;\; -1)$ we obtain the subproblem

$$4+ \quad \begin{array}{ll} \text{minimize} & x_2 + 2x_3 - x_4 \\ \text{subject to} & -1 \le x_j \le 1, \quad j = 1,2,3. \end{array}$$

Both v_2 and v_3 are optimal extreme points to the subproblem, so the optimal value of the subproblem is 0. Hence, the master problem has been solved. The solution to the original problem is given by

$$v_3\alpha_3 + v_2\alpha_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

The optimal value is -3.

5. *Remark:* This example comes from
A. Fredriksson, *A characterization of robust radiation therapy treatment planning methods—from expected value to worst case optimization*, Medical Physics 39(8): 5169–5181, 2012.

(a) For $\alpha = 1$, the last constraint reads $0 \le \pi_s \le p_s$. If $\pi_{s'} < p_{s'}$ for some scenario s', then $\sum_{s=1}^S \pi_s < \sum_{s=1}^S p_s$. But since p_s, $s = 1, \dots, S$, is a given probability distribution, we have $p_s \ge 0$, $s = 1, \dots, S$, and $\sum_{s=1}^S p_s = 1$. Hence, if $\pi_{s'} < p_{s'}$ for some s', we conclude that $\sum_{s=1}^S \pi_s < 1$ so that π is infeasible. It follows that $\pi_s = p_s$, $s = 1, \dots, S$, is the only feasible solution so that $P_{expected}$ and (P_α) are equivalent.

(b) For $\alpha \le \min_{s=1,\dots,S} p_s$, the constraints $\pi_s \le \frac{1}{\alpha} p_s$, $s = 1, \dots, S$, become redundant in (P_α), as they are dominated by $\pi_s \le 1$. Hence, in the inner maximization problem, it is optimal to let $\pi_{s'} = 1$ for one scenario s' such that $f(x, s') = \max_{s=1,\dots,S} f(x, s)$, and $\pi_s = 0$, $s \ne s'$. Therefore, (P_α) is equivalent to (P_{robust}).

(c) For a given x, the inner maximization problem is a linear program in the form

$$(PLP_\alpha) \quad \begin{array}{ll} \underset{\pi \in I\!R^S}{\text{maximize}} & \sum_{s=1}^S \pi_s f(x, s) \\ \text{subject to} & \sum_{s=1}^S \pi_s = 1, \\ & \pi_s \le \frac{1}{\alpha} p_s, \quad s = 1, \dots, S, \\ & \pi \ge 0. \end{array}$$

We may state the dual linear program (DLP_α) directly from known results. For completeness, we derive it by Lagrangian relaxation. The first step is to form the Lagrangian relaxation problem

$$\underset{\pi \ge 0}{\text{maximize}} \quad \sum_{s=1}^S \pi_s f(x, s) + \lambda\Big(1 - \sum_{s=1}^S \pi_s\Big) + \sum_{s=1}^S \mu_s\Big(\frac{1}{\alpha} p_s - \pi_s\Big)$$

where λ and μ_s, $s = 1, \dots, S$, are Lagrange multipliers. We must require $\mu_s \ge 0$, $s = 1, \dots, S$, to obtain a relaxation. This problem may be rewritten as

$$\begin{aligned} &\lambda + \frac{1}{\alpha} \sum_{s=1}^S \mu_s p_s + \sum_{s=1}^S \underset{\pi_s \ge 0}{\text{maximize}} \, (f(x, s) - \lambda - \mu_s)\pi_s \\ &= \begin{cases} \lambda + \frac{1}{\alpha} \sum_{s=1}^S p_s \mu_s & \text{if } f(x, s) - \lambda - \mu_s \le 0, \quad s = 1, \dots, S, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The dual problem may now be written as

$$(DLP_\alpha) \quad \begin{array}{ll} \underset{\lambda \in I\!R, \mu \in I\!R^S}{\text{minimize}} & \lambda + \frac{1}{\alpha} \sum_{s=1}^S p_s \mu_s \\ \text{subject to} & \lambda + \mu_s \ge f(x, s), \quad s = 1, \dots, S, \\ & \mu_s \ge 0, \quad s = 1, \dots, S. \end{array}$$

By strong duality for linear programming, the optimal values of (PLP_α) and (DLP_α) are equal. Hence, the original problem may be written as

$$\begin{array}{lll} \underset{x \in \mathcal{X}}{\text{minimize}} & \underset{\lambda \in I\!R, \mu \in I\!R^S}{\text{minimize}} & \lambda + \frac{1}{\alpha} \sum_{s=1}^S p_s \mu_s \\ & \text{subject to} & \lambda + \mu_s \ge f(x, s), \quad s = 1, \dots, S, \\ & & \mu_s \ge 0, \quad s = 1, \dots, S. \end{array}$$

This may now be written as one minimization problem on the form

$$
\begin{array}{ll}
\text{minimize} & \lambda + \frac{1}{\alpha} \sum_{s=1}^{S} p_s \mu_s \\
\text{subject to} & \lambda + \mu_s - f(x, s) \geq 0, \quad s = 1, \dots, S, \\
& \mu_s \geq 0, \quad s = 1, \dots, S, \\
& x \in \mathcal{X}.
\end{array}
$$