SF2812 Applied linear optimization, final exam
Friday March 8 2019 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

In the discussion below, we let optval$(PLP) = \infty$ if (PLP) is infeasible and analogously optval$(DLP) = -\infty$ if (DLP) is infeasible, where "optval" denotes the optimal value.

Assume that $\widetilde{x}$ is a feasible solution to (PLP).

(a) Give an upper bound on the optimal value of (PLP). .... (2p)

(b) Give an upper bound on the optimal value of (DLP). Is (DLP) necessarily feasible? .... (2p)

(c) Can there exist η and q such that $A^T\eta + q = 0$, $q \geq 0$ and $b^T\eta > 0$? ...... (3p)

(d) Assume that (DLP) has a feasible solution $\widetilde{y}$, $\widetilde{s}$ and in addition assume that $\widetilde{x}^T\widetilde{s} = 1$. Is it possible that $\widetilde{y}$, $\widetilde{s}$ is an optimal solution to (DLP)? .... (3p)

2. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 4 & 5 & -1 & 3 & 1 \\ -1 & 3 & 2 & 0 & 4 & 0 \\ -1 & 2 & 4 & 3 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 3 \\ 1 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 3 & 8 & 4 & 0 & 1 \end{pmatrix}^T.$$

The related barrier transformed problem (P_μ), defined by

$$(P_\mu) \quad \begin{array}{ll} \text{minimize} & c^T x - \mu \sum_{j=1}^{4} \ln x_j \\ \text{subject to} & Ax = b, \\ & (x > 0), \end{array}$$

has an optimal solution $\widetilde{x}$ and Lagrange multiplier vector $\tilde{\lambda}$ for $\mu = 0.01$ which numerically is given by approximately

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xtilde =
    3.0136
    1.9828
    0.0085
    0.0046
    0.0120
    0.9812
lambdatilde =
    0.9898
   -0.9505
    0.9437
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(a) Use the above numbers to give an approximate solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the primal-dual nonlinear equations, associated with a primal-dual interior method for solving (PLP), for $\mu = 0.01$. If there are quantities that you cannot calculate easily without a calculator, explain how you would calculate them. ....(4p)

(b) The above problems (PLP) and (DLP) have optimal solutions which are integer valued. Given this knowledge, use your results from Question 2a to make a qualified guess of optimal solutions to (PLP) and (DLP) respectively. Motivate your guess and verify optimality. ....(4p)

(c) If the simplex method had been used to solve (PLP), would the same optimal solutions to (PLP) and (DLP) that you gave in Question 2b have been obtained? ....(2p)

3. Consider the stochastic program (P) given by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \dots, N, \\ & x \ge 0, \\ & y_i \ge 0, \quad i = 1, \dots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

4. Consider the linear program (LP) given by

$$(LP) \quad \begin{array}{ll} \text{minimize} & -3x_1 - 2x_2 + x_3 + 2x_4 \\ \text{subject to} & 2x_1 + x_2 - 2x_3 - 2x_4 = 2, \\ & -1 \le x_j \le 1, \quad j = 1, \dots, 4. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $2x_1 + x_2 - 2x_3 - 2x_4 = 2$ the complicating constraint. Use the extreme points $v_1 = (1 \;\; 1 \; -1 \; -1)^T$ and $v_2 = (1 \;\; 1 \;\; 1 \;\; 1)^T$ for obtaining an initial basic feasible solution to the master problem.

The subproblem(s) that arise may be solved in any way, that need not be systematic. .... (10p)

5. Consider the integer program (IP) defined by

$$(IP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge b, \\ & Cx \ge d, \\ & x \ge 0, \; x \text{ integer}. \end{array}$$

In the course, Lagrangian relaxation of the constraints $Ax \ge b$ or the constraints $Cx \ge d$ have been considered. The corresponding dual problems (D_1) and (D_2) are given by

$$(D_1) \quad \begin{array}{ll} \text{maximize} & \varphi_1(u) \\ \text{subject to} & u \ge 0, \end{array}$$

where $\varphi_1(u) = \min\{c^T x - u^T(Ax - b) : Cx \ge d, x \ge 0, \; x \text{ integer}\}$, and

$$(D_2) \quad \begin{array}{ll} \text{maximize} & \varphi_2(v) \\ \text{subject to} & v \ge 0, \end{array}$$

where $\varphi_2(v) = \min\{c^T x - v^T(Cx - d) : Ax \ge b, x \ge 0, \; x \text{ integer}\}$. In addition, the optimal value of the Lagrange dual problem has been related to the optimal value of the linear programming relaxation. We will now do some additional work in comparing different Lagrangian relaxations.

For a particular set X of vectors in $I\!R^n$ with integer components, we will be concerned with its *convex hull*, denoted by $\operatorname{conv}(X)$. For a given X, $\operatorname{conv}(X)$ is defined as the smallest convex set that contains X, or equivalently as the set of all convex combinations of points in X. By construction, $X \subseteq \operatorname{conv}(X)$ and any extreme point of $\operatorname{conv}(X)$ belongs to X. Consequently, if minimizing a linear function over X, it is equivalent to minimize the linear function over $\operatorname{conv}(X)$, since there is always a minimizer in $\operatorname{conv}(X)$ which is an extreme point. In addition, for an X formed by vectors in $I\!R^n$ with integer components, $\operatorname{conv}(X)$ is a polytope. These properties may be used without proof in the analysis.

(a) Show that

$$\begin{array}{rll} \operatorname{optval}(D_1) = & \text{minimize} & c^T x \\ & \text{subject to} & Ax \ge b, \\ & & x \in \operatorname{conv}\{x : Cx \ge d, x \ge 0, x \text{ integer}\}, \end{array}$$

and

$$\begin{array}{rll} \operatorname{optval}(D_2) = & \text{minimize} & c^T x \\ & \text{subject to} & Cx \ge d, \\ & & x \in \operatorname{conv}\{x : Ax \ge b, x \ge 0, x \text{ integer}\}. \end{array}$$

....(5p)

Hint: Consider $\operatorname{optval}(D_1)$. Since the set $\operatorname{conv}\{x : Cx \ge d, x \ge 0, x \text{ integer}\}$ is a polytope, we may write

$$\operatorname{conv}\{x : Cx \ge d, x \ge 0, x \text{ integer}\} = \{x : \bar{C}x \ge \bar{d}\},$$

for some matrix $\bar{C}$ and vector $\bar{d}$. In general, $\bar{C}$ and $\bar{d}$ will have an exponential number of rows, so this approach is not practical. It will, however, do for our purposes.

The result may now be shown by motivating the sequence of equalities

$$\begin{aligned} \underset{u \ge 0}{\text{maximize}}\ \varphi_1(u) &= \underset{u \ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^T u + & \text{minimize} & (c - A^T u)^T x \\ & \text{subject to} & \bar{C}x \ge \bar{d} \end{array} \right\} \\ &= \underset{u \ge 0}{\text{maximize}} \left\{ \begin{array}{lll} b^T u + & \text{maximize} & \bar{d}^T \bar{u} \\ & \text{subject to} & \bar{C}^T \bar{u} = c - A^T u,\ \bar{u} \ge 0, \end{array} \right\} \\ &= \begin{array}{ll} \text{maximize} & b^T u + \bar{d}^T \bar{u} \\ \text{subject to} & A^T u + \bar{C}^T \bar{u} = c,\ u \ge 0,\ \bar{u} \ge 0, \end{array} \\ &= \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \ge b, \\ & \bar{C}x \ge \bar{d}. \end{array} \end{aligned}$$

If these equalities are used, each step must be motivated.

(b) An alternative to forming Lagrangian duals (D_1) or (D_2) would be to use so-called *variable splitting* or *Lagrangian decomposition* and rewrite (IP) as the equivalent problem

$$(IP')\qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cy \geq d, \\ & x = y, \\ & x \geq 0,\ x \text{ integer}, \\ & y \geq 0,\ y \text{ integer}, \end{array}$$

where additional variables y have been introduced. We may now do Lagrangian relaxation of the constraint $x - y = 0$ in (IP'), which gives the Lagrangian dual problem

$$(D_3) \qquad \underset{w}{\text{maximize}} \quad \varphi_3(w)$$

where

$$\begin{array}{rlll} \varphi_3(w) = & \text{minimize} & c^T x - w^T(x - y) & \\ & \text{subject to} & Ax \geq b, & \\ & & Cy \geq d, & \\ & & x \geq 0,\ x \text{ integer}, & \\ & & y \geq 0,\ y \text{ integer}, & \\ = & \text{minimize} & (c - w)^T x & + \quad \text{minimize} \quad w^T y \\ & \text{subject to} & Ax \geq b, & \qquad\ \ \text{subject to} \quad Cy \geq d, \\ & & x \geq 0, x \text{ integer} & \qquad\qquad\qquad\qquad y \geq 0, y \text{ integer}. \end{array}$$

Show that

$$\begin{array}{rll} \text{optval}(D_3) = & \text{minimize} & c^T x \\ & \text{subject to} & x \in \text{conv}\{x : Ax \geq b, x \geq 0, x \text{ integer}\}, \\ & & x \in \text{conv}\{x : Cx \geq d, x \geq 0, x \text{ integer}\}. \end{array}$$

In addition, motivate $\text{optval}(D_3) \geq \text{optval}(D_1)$ and $\text{optval}(D_3) \geq \text{optval}(D_2)$.
....(5p)

Hint: You may use technique analogous to what was suggested for Question 5a.

Comment 1: The implication is that (D_3) always gives at least as tight lower bound on the optimal value of (IP) as the best of (D_1) and (D_2). This may appear strange at first sight, but note that evaluation of the objective function in (D_3) means solving both a problem in x analogous to evaluation of the objective function in (D_2), and a problem in y analogous to evaluation of the objective function in (D_1).

Comment 2: Note that (IP) may be written as

$$(IP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \text{conv}\{x : Ax \geq b, Cx \geq d, x \geq 0, x \text{ integer}\}. \end{array}$$

Good luck!