Lecture 2 (adapted from K. Jöns, Optical and Quantum Optical Properties of Site-controlled and Strain-tuned Quantum Dots. Verlag Dr. Hut; 2013)

Photon Statistics

Introduction

This lecture discusses what is light and how can we can categorize different light sources. Interestingly, when Maxwell published his famous four equations in the years 1861 and 1862, it seemed that light was finally fully understood. However, the wave-particle duality of light, already discussed in the 17th century by Huygens and Newton, was still unsolved. Planck's formula for black-body radiation in 1901 [Planck1900, Planck1901] and Einstein's interpretation of it as a quantized electromagnetic radiation [Einstein1905] made a new quantum mechanical description of light necessary. Dirac gave the first quantum mechanical solution for the interaction between atoms and the light field [Dirac1927] and shortly after, Fermi [Fermi1932] gave a complete review of quantum electrodynamics. In quantum electrodynamics the electromagnetic field is quantized and the photon is the smallest quantum of light. A complete derivation of the quantization of the light field can be found in several textbooks [Loudon1983, Fox2006]. In this lecture, the focus lies on photon statistics where the quantum description of light is required to fully classify different states of light. Therefore, we give a brief introduction to the quantization of the electromagnetic field. In simplified terms we can replace the field amplitudes in the classical electrodynamics description with bosonic creation $\hat{a}^\dagger$ and annihilation operators $\hat{a}$. These operators follow the bosonic commutator relation. Each mode of the light field (k, λ) can be independently described by a harmonic oscillator. The consecutive operation of first the annihilation and then the creation operator is equal to an operation of a new quantum mechanical operator $\hat{n}$, the so-called photon number operator which gives the number of photons in one mode: $\hat{n} = \hat{a}^\dagger\hat{a}$. The quantum mechanical approach allows to sum up all modes as independent quantum mechanical harmonic oscillators. The Hamiltonian of the electromagnetic field $\hat{H}_{em}$ for an arbitrary number of field modes is given by:

$$\hat{H}_{\mathrm{em}} = \sum_k \sum_\lambda \frac{1}{2}\hbar\omega_k \left(\hat{a}_{k\lambda}\hat{a}^\dagger_{k\lambda} + \hat{a}^\dagger_{k\lambda}\hat{a}_{k\lambda}\right)$$

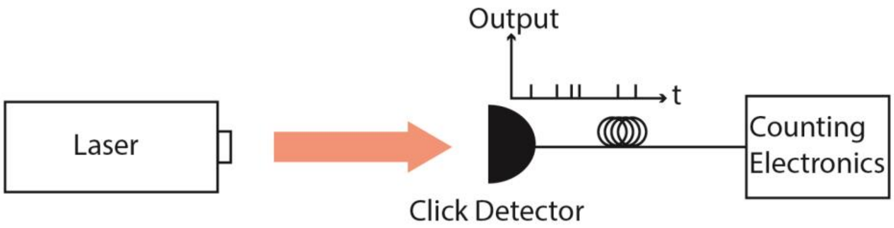
If we consider only the fundamental mode, we can rewrite the Hamiltonian with the help of the photon number operator:

$$\hat{H}_{\mathrm{em}} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

Similar to the quantum mechanical harmonic oscillator, there is a ground state with finite energy, called vacuum state. The photon number operator $\hat{n}$ states how many photons with an energy $\hbar\omega$ are in the fundamental mode. The mean photon number of a mode $\langle n \rangle$ is an important figure of merit for the characterization of the light states. Together with the photon number variance $(\Delta n)^2$, we will later identify different light states and use it to categorize light sources.

Number of Photons in a given Volume

To give an intuitive introduction to the topic of photon statistics we will take a look at a simple experiment adapted from Ref. [Fox2006]. A light source emits photons which are detected on an ideal click detector. Such a detector is sensitive down to the single photon level and produces a short electrical pulse in response to a single photon absorbed on the detector. The counting electronics registers the number of electrical pulses within a defined time interval *T*. The detection scheme is similar to the principles of a Geiger counter.



For the example we start with a perfectly monochromatic beam of angular frequency ω and constant intensity *I* and calculate the mean number of photons <n(*T*)> passing through a cross section *A* of the beam in a given time interval *T*, set by the counting electronics.

$$< n(T) > = \frac{IAT}{\hbar\omega} \equiv \frac{PT}{\hbar\omega} \ ,$$

where *P* is the power of the light beam. Remember that this equation represents the average properties of the beam and any given light source will have fluctuations at short time scales. These fluctuations or differences in statistics of the photon numbers are used to characterize different light sources. Coming back to our example we can now calculate the constant photon flux Φ, i.e. the mean number of photons in a given cross-section of a beam of light of photon energy 1.0eV with an average power of 1nW:

$$\Phi = \frac{< n(T) >}{T} = \frac{P}{\hbar\omega} = \frac{10^{-9}}{1 \times \ (1.6 \times 10^{-19})} = 6.2 \ \times \ 10^{9} \ \frac{photons}{s}$$

Now consider that light travels with approximately $3 \times 10^{8}\ m/s$ you will realize that the photons are quite spread in space during this one second. (A beam segment with a length of $3 \times 10^{8}\ m$ contains $6.2 \ \times \ 10^{9}$ photons). If we now reduce the volume we are interested, let us say down to one meter instead of $3 \times 10^{8}\ m$ we have on average 20.67 photons in volume. Since photons are the smallest quantum of light and are discrete 20.67 mean photons does not make sense physically. Instead there will be fluctuations in the number of photons and these fluctuations in mean photon numbers will increase the smaller we make the volume we are looking at. Similarly speaking one can also look at shorter and shorter time windows. Let us look at a segment of 1.5m which on average contains 31 photons. We will divide the 1.5m in 31 sub-segments. There are different options how the photons will be distributed within these sub-segments:

0	0	2	1	0	4	0	3	1	0	1	0	2	0	2	0	1	0	0	2	1	0	3	0	0	5	0	1	1	0	1
0	1	0	0	2	1	0	2	1	4	0	2	1	0	1	0	0	1	1	0	1	2	1	1	0	2	1	3	0	2	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Even though in all 3 cases the mean photon number <n> for each sub-segment is 1, the distribution function for each case is fundamentally different. In the first row it is super-Poissonian, second row Poissonian, and in the third row sub-Poissonian. We will now define these distributions.

Different light states

Glauber state

The Glauber state, or coherent state $|\hat{\alpha}_i\rangle$, describes the electromagnetic wave of a laser mode *i*. In 1963, Glauber provided a complete quantum mechanical description of these light states [180]. The Glauber state is an eigenstate of the annihilation operator â:

$$\hat{a}_i |\alpha_i\rangle = \alpha_i |\alpha_i\rangle$$

Being an eigenstate of â, a coherent state remains unchanged by the annihilation of a photon. Additionally, since the vacuum state can be written as an eigenstate of the annihilation operator with α = 0, all coherent states have the same minimal uncertainty as the vacuum state. Linear superposition of these states allows for an expression of the Glauber state in the basis of the photon number operator $\hat{n}$ [Loudon1983]:

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^2}{\sqrt{n!}} |n_i\rangle$$

We can now calculate the probability distribution *P* for *n* photons in a given mode *i*.

$$\begin{aligned} P_{coherent}(n) &= |\langle n \mid \alpha_i \rangle|^2 \\ &= \left| \exp\left(-\tfrac{1}{2}|\alpha_i|^2\right) \cdot \sum_m \frac{\alpha_i^m}{\sqrt{m!}} \langle n \mid m \rangle \right|^2 \\ &= \exp\left(-|\alpha_i|^2\right) \cdot \frac{|\alpha_i|^{2n}}{n!} \end{aligned}$$

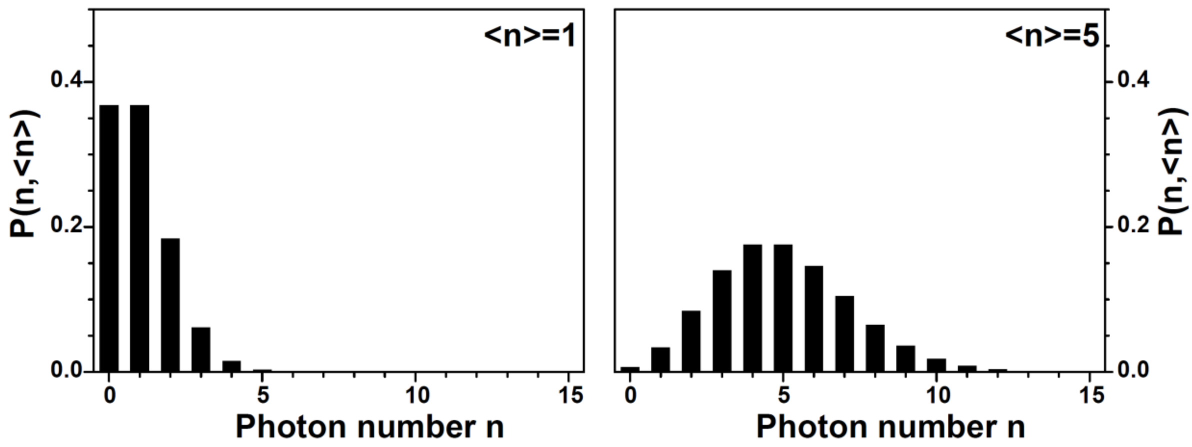
This is the characteristic Poisson statistics used to describe coherent light states. The expected value of the photon number in one mode of a coherent state is therefore:

$$\langle n \rangle = \langle \alpha | \hat{n} | \alpha \rangle = |\alpha|^2$$

The variance of a Glauber (coherent) state is given as:

$$(\Delta n)^2_{\text{Glauber}} = \left\langle n^2 \right\rangle - \langle n \rangle^2 = |\alpha|^2 = \langle n \rangle$$

For the coherent state the maximum probability to find *n* photons in a mode is at the expected value $\langle n \rangle$. The probability distribution of the photon number obeys a Poisson distribution. Any light state with larger (smaller) variance is called super (sub)- Poissonian light. The photon number distribution for a Glauber state is plotted below for 2 different mean photon numbers.



Fock state

The Fock or photon number state $|\hat{n}_i\rangle$ results directly from the quantization of the electromagnetic field, since the Fock state is the eigenstate of the photon number operator $\hat{n}_i$:

$$\hat{n}_i \left| n_i \right\rangle = n_i \left| n_i \right\rangle$$

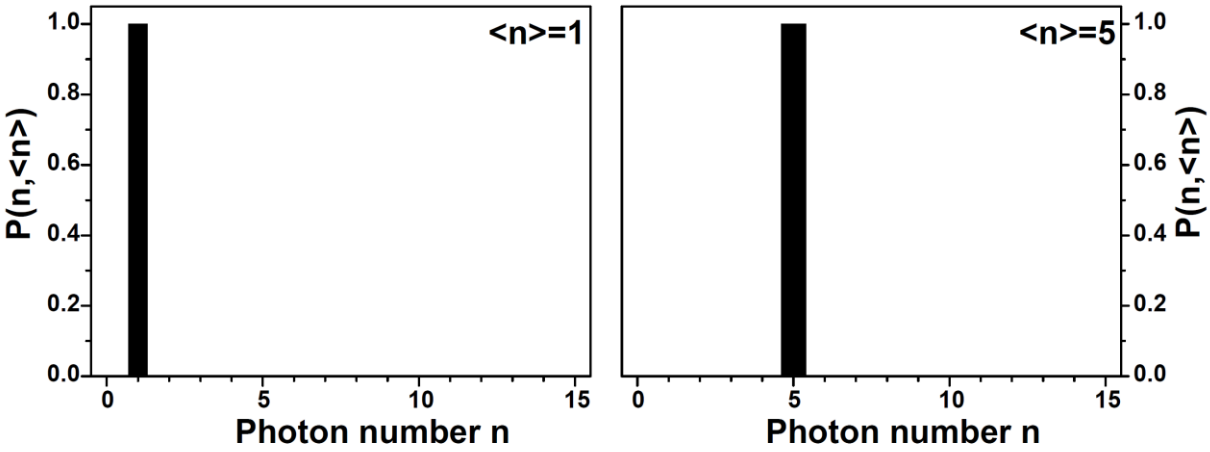
The eigenvalue n_i of the photon number operator describes the number of photons in a specific mode *i*. The probability $P_{Fock}(n)$ to find n_i photons in one mode is either 1 for $n = n_i$ or 0 for $n \neq n_i$. This is a special characteristic of the Fock state: The photon number is fully determined. Thus. the probability distribution of the photon number follows a δ-distribution. The expected value of the photon number in a Fock state is equal to the number of photons in the state:

$$\langle n \rangle = \langle n_i | \, \hat{n} \, | n_i \rangle = n_i$$

For the Fock state the variance is therefore:

$$(\Delta n)^2_{\mathrm{Fock}} = \left\langle n^2 \right\rangle - \langle n \rangle^2 = 0$$

The Fock state fulfills the inequality $\Delta n < \sqrt{n}$, showing a variance smaller than the Glauber state. Such sub-Poisson statistics cannot be described by classical electromagnetic theory; thus such light is classified as non-classical light. The figure below shows the photon number distribution for two Fock states. Light emitters with a Fock state n = 1 are called single photon sources, since they can only emit one single photon at a time.



Thermal state

The thermal state which is well described by the black-body radiation [Planck1900,Planck1901], is an incoherent mixture of different photon number states $|\hat{n}_i\rangle$. A quantum mechanical description of the thermal state takes advantage of the density matrix notation. The thermal state, being a quantum mechanical mixed state, can be written as the sum over all possible photon number states $|\hat{n}_i\rangle$ weighted with their occurrence probability:

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n(n) \left|n\right\rangle \left\langle n\right|$$

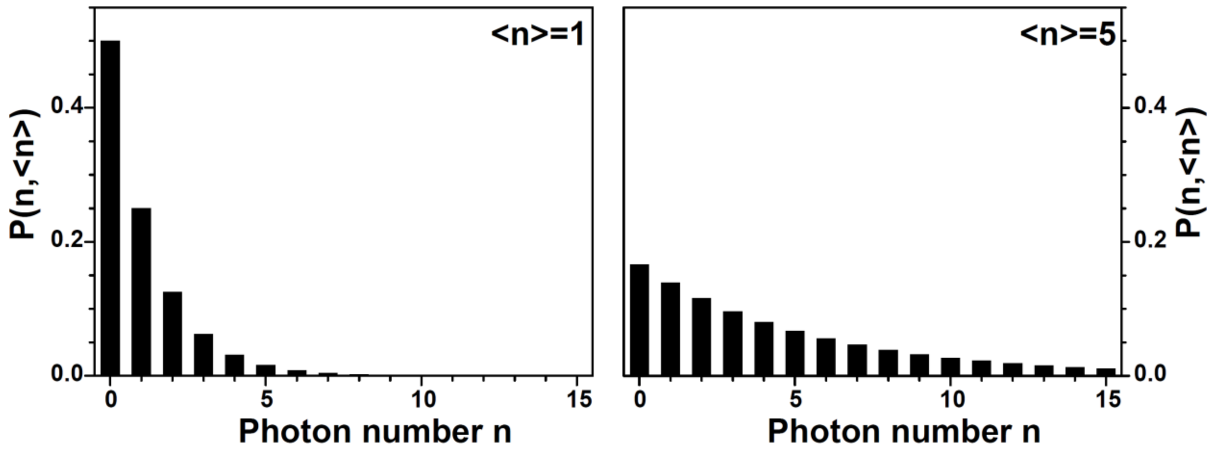
where $\hat{\rho}$ is the density matrix operator and P(n) gives the probability of finding n photons in a certain mode *i* of the thermal state. This is identical to the probability of having a certain photon number state occupied. P(n) can be expressed as a function of n and $\langle n\rangle$ for a single mode:

$$P(n, \langle n\rangle) = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}$$

P(n, $\langle n\rangle$) has the form of a Bose-Einstein distribution; the state with the maximum probability is always the vacuum state with n = 0. The variance for a thermal state is given by:

$$(\Delta n)^2_{\text{Thermal}} = \langle n\rangle^2 + \langle n\rangle$$

It follows that the fluctuation in the photon number is typically larger than the mean photon number. Therefore, thermal light states are also called chaotic light. Since $\Delta n > \sqrt{n}$ one often describes thermal state statistics as super-Poissonian statistics. The thermal state distribution for 2 different mean photon numbers is plotted below. Finding zero photons in the mode always has the highest probability.



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