Lecture 2 (adapted from K. Jöns, Optical and Quantum Optical Properties of Site-controlled and Strain-tuned Quantum Dots. Verlag Dr. Hut; 2013 and from S. M. Ulrich, Non-Classical and Stimulated Photon Emission Processes from Self-Assembled Semiconductor Quantum Dots. Verlag Dr. Hut; 2007)

Second-order Intensity Correlation Function

As we learned in lecture 2 different light states are defined by their underlying photon statistics, photon probability distribution function and in the fluctuations of the photon numbers. With the help of the second-order correlation function $g^{(2)}(\tau)$ introduced by Glauber in 1963 [Glauber1963], we can classify the different light states. First, we briefly introduce the classical second-order intensity correlation function:

$$g^{(2)}_{class.}(\tau) = \frac{\langle I(t)\, I(t+\tau)\rangle}{\langle I(t)\rangle^2} = \frac{\langle E^*(t)\, E^*(t+\tau)\, E(t+\tau)\, E(t)\rangle}{\langle E^*(t)\, E(t)\rangle^2}$$

with $I \propto |E(t)|^2$ and $I(t+\tau)$ being the averaged intensities of the mode at a given time. Based on this definition the $g^{(2)}(\tau)$ function describes the correlation between two temporally separated intensity signals with time difference $\tau = t_2 - t_1$ from one light source. Using the transformation formalism from classical field quantities into equivalent quantum mechanical operators using the second quantization, we can rewrite the electric field *E*(t) of a mode *k* with the help of annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators [Loudon1983]:

$$\hat{E}_k(t) = \hat{E}_k^{(+)}(t) + \hat{E}_k^{(-)}(t)$$

with

$$\begin{aligned} \hat{E}_k^{(+)}(t) &\propto \hat{a}_k \cdot \exp\left(-i\left(\omega_k t - \vec{k}\cdot\vec{r}\right)\right) \\ \hat{E}_k^{(-)}(t) &\propto (\hat{a}_k)^\dagger \cdot \exp\left(+i\left(\omega_k t - \vec{k}\cdot\vec{r}\right)\right) \end{aligned}$$

representing the 'positive' and 'negative' ω_k frequency parts of the mode. For a single mode we can rewrite the $g^{(2)}(\tau)$ function using the commutator relation.

$$\begin{aligned} g^{(2)}_{QM}(\tau) &= \frac{\left\langle \hat{E}_k^{(-)}(t)\, \hat{E}_k^{(-)}(t+\tau)\, \hat{E}_k^{(+)}(t+\tau)\, \hat{E}_k^{(+)}(t)\right\rangle}{\left\langle \hat{E}_k^{(-)}(t)\, \hat{E}_k^{(+)}(t)\right\rangle^2} \\ &\overset{(\tau\to 0)}{=} \frac{\left\langle (\hat{a}_k)^\dagger (\hat{a}_k)^\dagger \hat{a}_k \hat{a}_k\right\rangle}{\left\langle (\hat{a}_k)^\dagger \hat{a}_k\right\rangle^2} = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2} \end{aligned}$$

In important difference between the classical and quantum mechanical description is that in the latter case the detection of a photon at *t* reduces the amount of photons a *t + τ*, since the photons are the smallest quanta of the radiation field and cannot be divided. In the classical field theory this effect is not taken into account, a reason why non-classical light states (as defined in lecture 2) cannot be correctly described.

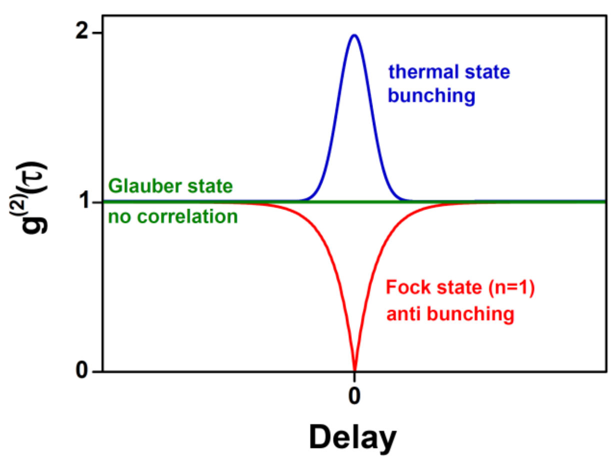
Of particular interest is $g^{(2)}(0)$, since it represents the conditional probability how likely is it to detect a second photon at the same time one photon was already detected. Thus it is a measure of the temporal photon coincidences, required to distinguish between different light states. Using the 2nd factorial moment (see appendix) and the variance we can simplify the equation:

$$\begin{aligned} g^{(2)}_{QM}(0) &= \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} \\ &= \frac{(\Delta n)^2 + \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} \\ &= 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2} \end{aligned}$$

With the given variance of the different light states (see lecture 2), we can now calculate the $g^{(2)}(0)$ value for the three different light states categorized in lecture 2:

$$\begin{aligned} &(\Delta n)^2_{thermal} = \langle n^2 \rangle + \langle n \rangle && \Rightarrow \quad g^{(2)}_{QM}(0) = 2 \\ &(\Delta n)^2_{coherent} = \langle n \rangle && \Rightarrow \quad g^{(2)}_{QM}(0) = 1 \\ &(\Delta n)^2_{Fock} = 0 && \Rightarrow \quad g^{(2)}_{QM}(0) = 1 - \frac{1}{n} \qquad (n \geq 1) \\ & && \Rightarrow \quad g^{(2)}_{QM}(0) = 0 \qquad\quad\; (n = 0) \end{aligned}$$

Since in the Glauber state photon emission is completely uncorrelated, the $g^{(2)}(\tau)$ function is unity for all delay times τ ,given an infinite coherence time of the state. As we can see from equation above, the thermal state has a higher probability to emit more than one photon at the same time. However, this happens only in time periods shorter than the coherence time, which is typically very short for thermal/chaotic light. This effect is called photon bunching. In contrast, Fock states give $g^{(2)}_{Fock}(0) < 1$, leading to a reduced probability to emit two photons at the same time. This effect is called photon antibunching.
The graph below depicts the $g^{(2)}(\tau)$ function for three light states: thermal, coherent, and Fock state with a photon number of n = 1. If in this state a single photon is annihilated (e.g. detected), there is no photon left and no second photon can be detected. Fock states are therefore called non-classical light states and the first demonstration of photon antibunching [Kimble1977] was proof of the non-classical nature of light. Photon sources with n = 1 are single photon sources and important for the realization of different applications in photonic quantum technologies.



Appendix

In quantum statistics we can characterize the fluctuations in the photon number *n* of a single mode by the variance $(\Delta n)^2$ defined as the weighted sum of the squared deviations of the photon number *n* with respect to the mean occupation $\langle n \rangle$:

$$\begin{aligned}(\Delta n)^2 &:= \sum_{n=0}^{\infty} (n - \langle n \rangle)^2 \cdot P(n) \\ &= \langle n^2 \rangle - \langle n \rangle^2\end{aligned}$$

The above expression can be further derived by using the general definition of the k-th order factorial moment:

$$\langle n(n-1)(n-2) \cdot \ldots \cdot (n-k+1) \rangle := \sum_{n=0}^{\infty} n(n-1)(n-2) \cdot \ldots (n-k+1) \cdot P(n)$$

In particular, for the 2nd factorial we get:

$$\begin{aligned}\langle n(n-1) \rangle &= \sum_{n=0}^{\infty} n(n-1) \cdot P(n) \\ &= \sum_{n=0}^{\infty} n^2 \cdot P(n) - \sum_{n=0}^{\infty} n \cdot P(n) \\ &= \langle n^2 \rangle - \langle n \rangle\end{aligned}$$

References

[Glauber1963] R. J. Glauber, Quantum theory of optical coherence, Phys. Rev. 130, 2529 (1963)

[Loudon1983] R. Loudon, The quantum theory of light, Oxford University Press (1983)

[Kimble1977] H. J. Kimble, M. Dagenais, L. Mandel, Photon Antibunching in Resonance Fluorescence, Phys. Rev. Lett. 39, 691 (1977)