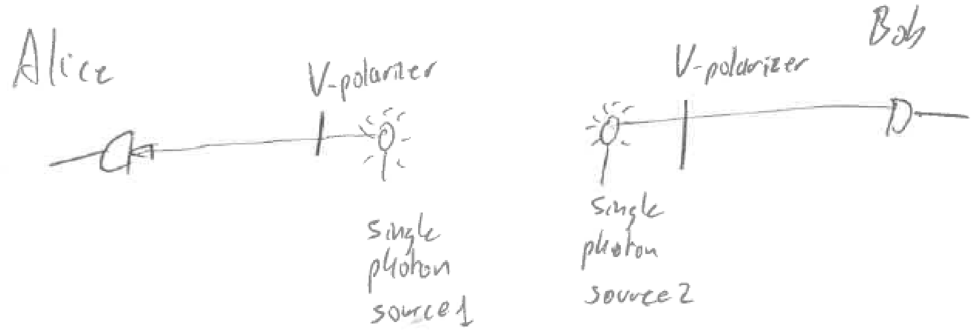
Lecture 8

Bell's theorem

We need an experimental test to identify quantum entanglement.

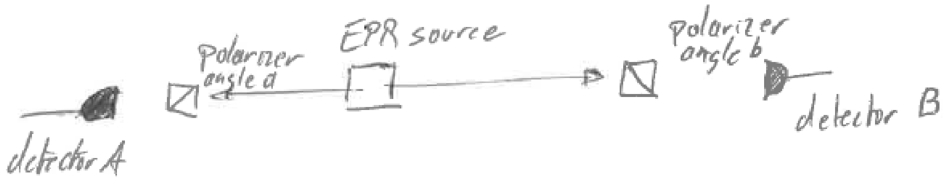
Correlations in one basis are not enough to identify quantum entanglement: imagine we have two independent single photon sources, each with a polarizer, the photons emitted by these two independent sources would not be entangled, yet the polarizers could be set so that photons measured by Alice and Bob have the same polarization.



The (desperate) attempt shown above to fake entanglement would be easy to identify: in this case Alice always gets a V polarized photon and so does Bob. But when measuring in the AD basis, their measurements would be random, the correlations would be gone. For an entangled state, the correlations would also be there in any basis.

To distinguish quantum entanglement, we need to measure in different bases: as we have seen, an entangled state in the HV basis is also entangled in the AD basis. Another crucial question is whether there are hidden variables.

John Bell came up with an answer: the Bell inequality that can test experimentally the question whether hidden variables set the outcome of a measurement.



We assume that the measurement is a function of the photon polarization angle and the hidden variable λ.

Result of the measurement on the photon in direction A (a,λ)= +- 1

Result of the measurement on the photon in direction B (b,λ)= +- 1

Principle of locality

The result of a measurement on photon A does not depend on the setting b, provided the distance is large enough. And symmetrically, B (b,λ) does not depend on a.

A (a,b,λ) = A (a,λ)

B (a,b,λ) = B (b,λ)

Note that we make no assumption on λ! It could beat special relativity and propagate arbitrarily fast.

In the lab, we can measure correlations:

$$C(a,b) = \int A(a,\lambda)\, B(b,\lambda)\, \rho(\lambda)\, d\lambda$$

probability density of hidden variables, normalized:

$$\int \rho(\lambda)\, d\lambda = 1$$

Possible outcomes:

A(a) = +1 for a vertically polarized photon detection.

A(a) = -1 for a horizontally polarized photon detection.

B(b) = +1 for a vertically polarized photon detection.

B(b) = -1 for a horizontally polarized photon detection.

The probability of getting a +1 outcome is equal to getting a -1 outcome.

What correlations can we expect among different polarizations?

$$\begin{aligned} |C(a,b) - C(a,b')| &\leq \int \left| A(a,\lambda) B(b,\lambda) - A(a,\lambda) B(b',\lambda) \right| \rho(\lambda)\, d\lambda \\ &\leq \int \left| A(a,\lambda) \left[B(b,\lambda) - B(b',\lambda) \right] \right| \rho(\lambda)\, d\lambda \\ &\leq \int \left| B(b,\lambda) - B(b',\lambda) \right| \rho(\lambda)\, d\lambda \qquad (1) \end{aligned}$$

$$|c(a',b)+c(a',b')| \leqslant \int |A(a',\lambda)[B(b,\lambda)+B(b',\lambda)]|\, \rho(\lambda)\, d\lambda$$

$$\leqslant \int |B(b,\lambda)+B(b',\lambda)|\, \rho(\lambda)\, d\lambda \qquad (2)$$

We add (1) and (2)

$$|c(a,b)-c(a,b')|+|c(a',b)+c(a',b')| \leqslant \int \left[|B(b,\lambda)-B(b',\lambda)| + |B(b,\lambda)+B(b',\lambda)| \right] \rho(\lambda)\, d\lambda \qquad (3)$$

B only has values ± 1

$$|B(b,\lambda)-B(b',\lambda)|+|B(b,\lambda)+B(b',\lambda)| = 2 \qquad (4)$$

We plug (4) into (3) and remember $\int \rho(\lambda)\, d\lambda = 1$

$$\boxed{|c(a,b)-c(a,b')|+|c(a',b)+c(a',b')| \leqslant 2}$$

This is Bell's inequality, denoted S.

We have followed the CHSH (Clauser, Horne, Shimony and Holt) derivation introduced in 1969 where measurements only have two possible outcomes: +1 and -1. It can be generalized to the case of additional possible outcomes. For instance, in the case where a photon is not detected (in the unlikely case where hidden variables would set photon detection). This opens the way to loopholes which are alternative explanations.

For example, in the detection loophole. We assume fair sampling: the detected photons give the same results than the undetected photons would give. But it could be that only photons that look 'entangled' are detected and that measuring all photons would not violate Bell's inequality anymore, this is the detection loophole, where one needs to measure nearly all the photons to close the loophole.

Communication loophole: the setting on B might influence the measurement outcome on A, even for large distances. This was closed by Aspect et al in 1981 using fast changing random polarizers that changed faster than light takes to travel from A to B. But one can still question how random the settings can be.

What angles should one choose for a, b, a' and b'?

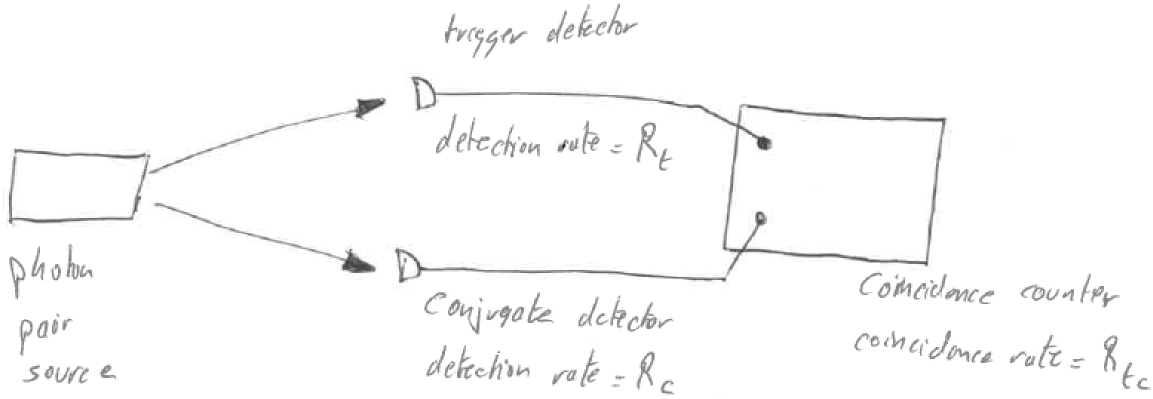
Taking $\theta_1 = 0$, $\theta_2 = \frac{3\pi}{8}$, $\theta_1' = -\frac{\pi}{4}$ and $\theta_2' = \frac{\pi}{8}$ gives the highest possible value.

$$|c(\theta_1,\theta_2) - c(\theta_1',\theta_2')| + |c(\theta_1',\theta_2) + c(\theta_1',\theta_2')| = 2\sqrt{2}$$

For values $2 < S < 2\sqrt{2}$ we violate Bell's inequality and therefore demonstrate that no local theory can explain the correlations.

Another use for two photon states: absolute quantum efficiency measurements

The idea was first suggested by Klyshko in 1980 and was implemented in 1993 by Kwiat at al. (Phys. Rev. A 48, R867). It is based on a source of pairs of photons:



We generate R photon pairs per second, η is the quantum efficiency of the detector: the probability for the detector to detect a photon. η_t is the probability for the trigger detector to detect a photon, η_c is the probability for the conjugate detector to detect a photon. These quantum efficiency measurements are hard to obtain with good statistics because light sources tend to fluctuate in intensity over time and attenuation also introduce uncertainties.

We have:

$R_t = R\eta_t$

$R_c = R\eta_c$

$R_{tc} = R\eta_t\eta_c$

Very simple math then gives us:

$$\eta_t = R_{tc} / R\eta_c = R_{tc} / R_c$$

The quantum efficiency is given by the ratio between the coincidence rate and the detection rate of the other detector! We do not need a low fluctuation light source to obtain a precise measurement of the quantum efficiency, only a source of pairs of photons (that do not need to be entangled, just pairs) and we don't even need to know the precise detection efficiency of a reference detector.

Note that we do not take into account dark counts (detection events by the detector in the absence of an incoming photons) in this calculation, but that can also be done. The dark count rate for the two detectors are A and B, we then get the expression:

$$\eta_c = (R_{tc} - A) / (R_t - B)$$

One remaining issue with this approach to determine the quantum efficiency is that that the value we get encompasses all losses on the way: not just the inefficiency of the detector but also losses in transmission between the light source and the detector (lenses, filters, optical fibers..).