SF2822 Applied nonlinear optimization, final exam
Friday May 31 2019 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear program

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, 2, \\ & x \in I\!R^3, \end{array}$$

where $f : I\!R^3 \to I\!R$ and $g_i : I\!R^3 \to I\!R$, $i = 1, 2$, are twice-continuously differentiable. For $x^* = (1\ 1\ 1)^T$, it is known that

$$f(x^*) = 0, \quad \nabla f(x^*) = \begin{pmatrix} 1 & -2 & 0 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_1(x) = a^T x - b, \text{ with } a = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T \text{ and } b = 1,$$

$$g_2(x^*) = 0, \quad \nabla g_2(x^*) = \begin{pmatrix} -1 & -1 & 0 \end{pmatrix}^T,$$

(a) Does x^* satisfy the first-order necessary optimality conditions for (NLP)? (3p)

(b) Are there conditions on $\nabla^2 g_2(x^*)$, which guarantee that x^* is a local minimizer to (NLP)? If so, which conditions? .... (4p)

(c) Are there conditions on $\nabla^2 g_2(x)$, $x \in I\!R^3$, which guarantee that x^* is a global minimizer to (NLP)? If so, which conditions? These conditions should not include any properties related to f, a or b. .... (3p)

2. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

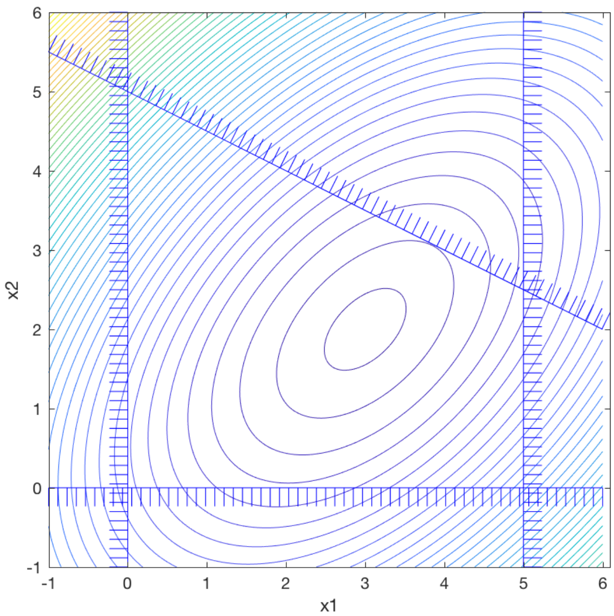
3. Consider the quadratic program (QP) defined by

$$(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -4 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ -2 & -4 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -5 \\ -20 \end{pmatrix}.$$

The problem is illustrated geometrically in the figure below.



(a) Solve (QP) by an active-set method. Start at the point $x = (5 \quad 0)^T$ with exactly one constraint in the working set, namely $-x_1 \geq -5$. You need not compute any numerical values, but you may utilize the fact that the problem is two-dimensional and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Question 3a, which is appended at the end of the exam. Motivate each step carefully. . . . . (5p)

(b) Solve (QP) with the same method as in Question 3a and with the same starting point, $x = (5 \quad 0)^T$, but with $x_2 \geq 0$ as the only constraint in the working set instead. Illustrate your iterations in the figure corresponding to Question 3b, which is appended at the end of the exam. Motivate each step carefully. . (5p)

4. Consider the nonlinear program

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \ i = 1, 2, 3, \\ & x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $g_i : I\!R^2 \to I\!R$, $i = 1, 2, 3$, are twice-continuously differentiable. Assume specifically that $x^{(0)} = (0\ 0)^T$, at which it holds that

$$f(x^{(0)}) = 0, \qquad \nabla f(x^{(0)}) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \qquad \nabla^2 f(x^{(0)}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$g_1(x^{(0)}) = 2, \qquad \nabla g_1(x^{(0)}) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_1(x^{(0)}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

$$g_2(x^{(0)}) = -1, \qquad \nabla g_2(x^{(0)}) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T, \qquad \nabla^2 g_2(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$g_3(x^{(0)}) = -1, \qquad \nabla g_3(x^{(0)}) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \qquad \nabla^2 g_3(x^{(0)}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We want to solve (NLP) by sequential quadratic programming. In addition to $x^{(0)}$ given above, let $\lambda^{(0)} = (2\ 0\ 0)^T$ and perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. .... (10p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

5. Consider the optimization problem

$$(NLP) \qquad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & \frac{1}{2} \sum_{i=1}^{m} (p_i^T x - u_i)_+^2 \\ \text{subject to} & x \geq 0, \end{array}$$

where the subscript $''+''$ denotes the positive part, i.e., $x_+ = \max(x, 0)$. The constants u_i, $i = 1, \ldots, m$, are known as well as the constant vectors p_i, $i = 1, \ldots, m$. This means that we pay a quadratic penalty cost for violating upper bounds u_i, $i = 1, \ldots, m$, respectively.

The formulation (NLP) is straightforward, but a drawback is that the objective function is not twice-continuously differentiable. Your task is to show that we may obtain a smooth problem by introducing additional variables and constraints.

(a) Show that the objective function of (NLP) has continuous gradient but discontinuities in the Hessian. .... (2p)

(b) Show that (NLP) is equivalent to the quadratic programming problem

$$(QP) \qquad \begin{array}{ll} \underset{x \in I\!R^n, y \in I\!R^m}{\text{minimize}} & \frac{1}{2} \sum_{i=1}^{m} y_i^2 \\ \text{subject to} & y_i \geq p_i^T x - u_i, \ i = 1, \ldots, m, \\ & x \geq 0. \end{array}$$

Do so by showing that minimization over y in (QP) for a given x gives $y_i = (p_i^T x - u_i)_+$, $i = 1, \ldots, m$. .... (4p)

(c) For a given positive barrier parameter μ, formulate the perturbed first-order optimality conditions that are to be solved approximately if a primal-dual interior method is applied to (QP). .... (4p)

Note: The motivation for considering this reformulation is that we obtain a smooth problem. The increased dimensionality introduced by the y variables can be eliminated in the linear equations that are solved in a primal-dual interior method.

Good luck!

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Figure for Question 3a:

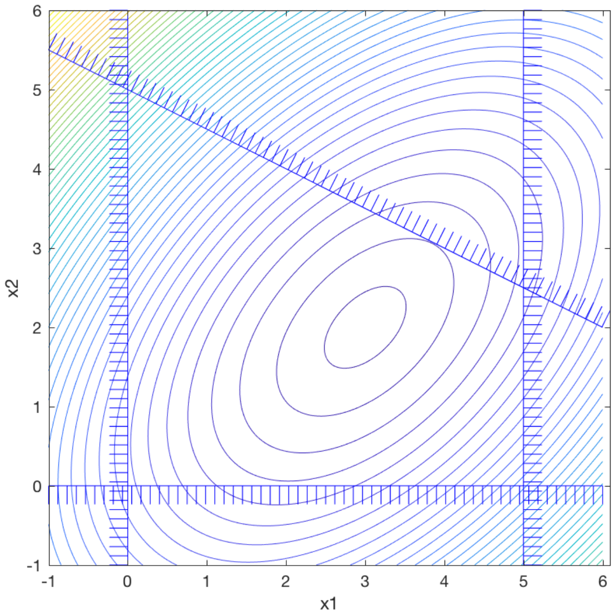

Figure for Question 3b: