SF2822 Applied nonlinear optimization, final exam
Friday May 31 2019 8.00–13.00
Brief solutions

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1. (a) Both constraints are active at x^*. The first-order necessary optimality conditions then require the existence of nonnegative λ_1^* and λ_2^* such that

$$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \lambda_1^* + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \lambda_2^*.$$

There is a unique solution with $\lambda_1^* = 3$ and $\lambda_2^* = 2$, so that x^* satisfies the first-order necessary optimality conditions together with λ^*.

(b) Both Lagrange multipliers are strictly positive, so that strict complementarity holds. A matrix $Z_+(x^*)$ whose columns form a basis for the nullspace of the matrix formed of the constraint gradients of the constraints with positive Lagrange multipliers, evaluated at x^*, is given by $Z_+(x^*) = (0\ 0\ 1)^T$. In addition to the first-order necessary optimality conditions, the second-order sufficient optimality conditions require

$$Z_+(x^*)^T \left(\nabla^2 f(x^*) - \lambda_2^* \nabla^2 g_2(x^*)\right) Z_+(x^*) \succ 0,$$

which gives

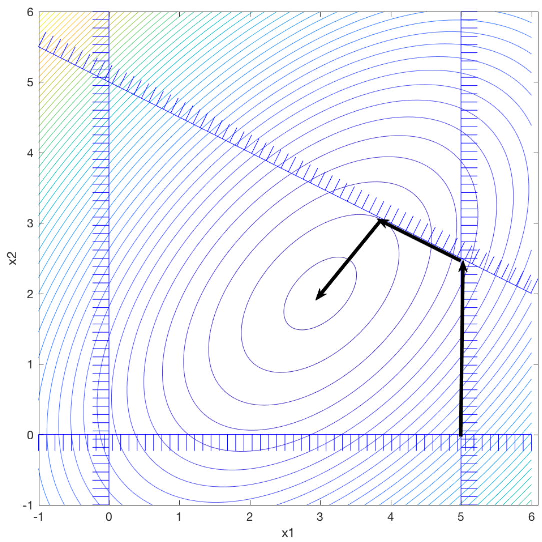
$$-1 - 2\nabla^2 g_2(x^*)_{33} > 0.$$

Hence, x^* is a local minimizer if $\nabla^2 g_2(x^*)_{33} < -1/2$.

(c) Since conditions on f are only known at x^*, it is not sufficient to put any conditions on $\nabla^2 g_2(x)$ to ensure global minimality.

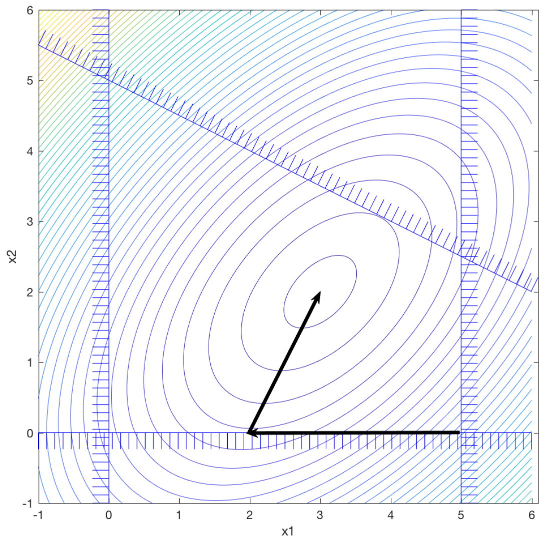
2. (See the course material.)

3. (a) The iterates are illustrated in the figure below:



At the first iteration constraint 3 is in the working set. The direction points at $(5\ 3)^T$, which is infeasible. The maximum step gives the new point $(5\ \frac{5}{2})^T$. Constraint 4 is added, which gives a vertex and hence a zero step. Constraint 3 has a negative multiplier, –4, and it is hence deleted. The direction points at $(\frac{27}{7}\ \frac{43}{14})^T$, which is feasible. Constraint 4 has a negative multiplier, $-\frac{9}{28}$, and it is hence deleted. The direction points at (3 2) which is feasible. No constraints are active, and we have found the optimal solution.

(b) The iterates are illustrated in the figure below:



At the first iteration constraint 2 is in the working set. The direction points at $(2\ 0)^T$, which is feasible. Constraint 2 has negative multiplier, –3, and it is hence deleted. The direction points at (3 2) which is feasible. No constraints are active, and we have found the optimal solution.

4. The QP subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^T p \\ \text{subject to} & \nabla g_i(x^{(0)})^T p \geq -g_i(x^{(0)}), \quad i = 1,2,3. \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \min & p_1^2 + p_2^2 \\ \text{subject to} & p_1 + p_2 \geq -2, \\ & p_1 \geq 1, \\ & p_2 \geq 1. \end{array}$$

If we let $p^{(0)}$ denote the optimal solution of the QP subproblem, we obtain $x^{(1)} = x^{(0)} + p^{(0)}$. We obtain $\lambda^{(1)}$ as the Lagrange multipliers of the QP subproblem.

The quadratic program is convex, and the optimal solution is given by $p^{(0)} = (1\ 1)^T$, so that $x^{(2)} = x^{(0)} + p^{(0)} = (1\ 1)^T$. The Lagrange multiplier of the quadratic program is given by$\lambda^{(1)} = (0\ 2\ 2)^T$.

5. (a) The function $f(y) = y_+^2$ has derivative $f'(y) = 0$ for $y < 0$ and $f'(y) = 2y$ for $y > 0$. Hence, $f'(y)$ is continuous with $f'(0) = 0$. The second derivative is given by $f''(y) = 0$ for $y < 0$ and $f''(y) = 1$ for $y > 0$. Hence, f'' is discontinuous at $y = 0$. As a consequence, the objective function has discontinuous Hessian at points where $p_i^T x = u_i$ for some i.

(b) Consider a fixed x and minimize over y in (QP). We want to show that $y_i = (p_i^T x - u_i)_+$, $i = 1, \dots, m$. Assume that $p_i^T x - u_i < 0$ for some i. Then, $y_i = 0$, since $y_i = 0$ is the unconstrained minimizer of y_i^2. Similarly, if $p_i^T x - u_i \geq 0$, the optimal choice of y_i is $y_i = p_i^T x - u_i$, as y_i^2 is a strictly increasing function for $y_i > 0$. Hence, $y_i = (p_i^T x - u_i)_+$, $i = 1, \dots, m$, as required.

(c) We may write the Lagrangian function as

$$l(x, y, \lambda, \eta) = \tfrac{1}{2} \sum_{i=1}^{m} y_i^2 - \sum_{i=1}^{m} \lambda_i (y_i - p_i^T x + u_i) - \eta^T x,$$

for Lagrange multiplier vectors $\lambda \geq 0$ and $\eta \geq 0$. Let P be the matrix whose rows comprise p_i^T, $i = 1, \dots, m$. Also, let $\Lambda = \operatorname{diag}(\lambda)$, $X = \operatorname{diag}(x)$ and $N = \operatorname{diag}(\eta)$. Finally, let e denote the vector of ones. For a positive barrier parameter μ, the perturbed first-order optimality conditions may be written

$$\begin{aligned} P^T \lambda - \eta &= 0, \\ y - \lambda &= 0, \\ \Lambda(y - Px + u) &= \mu e, \\ Nx &= \mu e. \end{aligned}$$