**SF2812 Applied linear optimization, final exam**
**Wednesday June 5 2019 14.00–19.00**

*Examiner:* Anders Forsgren, tel. 08-790 71 27.

*Allowed tools:* Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

*Solution methods:* Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

*Note!* Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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**1.** Consider a transportation problem $(TP)$ defined as

$$(TP)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1,2,3, \\ & \displaystyle\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3,4, \\ & x_{ij} \ge 0, \quad i = 1,2,3, \ j = 1,2,3,4, \end{array}$$

where

$$C = \begin{pmatrix} 4 & 2 & 5 & 1 \\ 7 & 4 & 7 & 5 \\ 7 & 5 & 6 & 2 \end{pmatrix}, \quad a = \begin{pmatrix} 10 \\ 12 \\ 10 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 8 \\ 7 \\ 9 \end{pmatrix}.$$

The dual problem associated with $(TP)$ may be written as

$$(DTP)\quad \begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{3} a_i u_i + \sum_{j=1}^{4} b_j v_j \\ \text{subject to} & u_i + v_j \le c_{ij}, \quad i = 1,2,3, \quad j = 1,2,3,4. \end{array}$$

You have been given $\widehat{X}$, $\widehat{u}$ and $\widehat{v}$ as

$$\widehat{X} = \begin{pmatrix} 8 & 1.5 & 0 & 0.5 \\ 0 & 6.5 & 5.5 & 0 \\ 0 & 0 & 1.5 & 8.5 \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 0 \end{pmatrix}.$$

(a) A friend of yours, who has not taken this course, claims that $\widehat{X}$ cannot be optimal to $(TP)$, since the transportation problem should have integer valued optimal solutions when $a$ and $b$ are integers. Comment on your friend's claim. ............ (2p)

(b) Verify that $\widehat{X}$ is optimal to $(TP)$ and that $\widehat{u}$, $\widehat{v}$ is optimal to $(DTP)$. ... (3p)

(c) Find, using $\widehat{X}$, two integer valued optimal solutions to $(TP)$. ........... (3p)
*Hint:* It holds that $\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}u_{ij} = 0$, $\sum_{j=1}^{4} u_{ij} = 0$, $i = 1,2,3$, and $\sum_{i=1}^{3} u_{ij} = 0$, $j = 1,2,3,4$, for

$$U = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(d) Explain why you would not obtain $\widehat{X}$ as an answer if you used the simplex method to solve $(TP)$. ................................................ (2p)

**2.** Consider a mixed-integer linear programming problem with one integer variable (but a large number of continuous variables). Assume that this problem is solved by branch-and-bound with linear programming relaxation at the nodes. Show that the branch-and-bound tree will have at most three nodes. You may assume that the linear programs that arise have unique optimal solutions. ................... (10p)

**3.** Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 3 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} -3 & 3 & 2 & 0 \end{pmatrix}^T.$$

An optimal basic feasible solution has been computed for $b_2 = -1$. This solution is $\widetilde{x} = (1\ 0\ 0\ 1)^T$. The corresponding dual optimal solution is $\widetilde{y} = (-1\ 1)^T$ and $\widetilde{s} = (0\ 1\ 1\ 0)^T$.

Unfortunately, the value of $b_2$ was not correct. The correct value is $b_2 = 3$. Now, $\widetilde{x}$ is not feasible to the correct primal problem, whereas $\widetilde{y}$ and $\widetilde{s}$ are feasible to the correct dual problem. Solve the correct problem by the dual simplex method, starting from $\widetilde{y}$ and $\widetilde{s}$. ............................................................ (10p)

**4.** Let $(P)$ and $(D)$ be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

For a fixed positive barrier parameter $\mu$, consider the primal-dual nonlinear equations

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

where we in addition require $x > 0$ and $s > 0$. Here, $X = \mathrm{diag}(x)$, $S = \mathrm{diag}(s)$ and $e$ is an $n$-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive $\mu$, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to $(P)$ and $y(\mu), s(\mu)$ are feasible to $(D)$ with duality gap $n\mu$. ..............(3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton's method. .......................... (5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of $\mu$? ......................(2p)

**5.** Consider the integer linear programming problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} f_j z_j \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\dots,n, \\ & \sum_{i=1}^{n} a_i x_{ij} \le b_j z_j, \quad j = 1,\dots,n, \\ & x_{ij} \in \{0,1\}, \quad i = 1,\dots,n,\ j = 1,\dots,n, \\ & z_j \in \{0,1\}, \quad j = 1,\dots,n, \end{array}$$

where $a_i$, $i = 1,\dots,n$, $b_j$, $j = 1,\dots,n$, $f_j$, $j = 1,\dots,n$, and $c_{ij}$, $i = 1,\dots,n$, $j = 1,\dots,n$, are nonnegative integer constants.

(a) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\dots,n,$$

are relaxed by Lagrangian relaxation. ...................................(2p)

(b) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{i=1}^{n} a_i x_{ij} \le b_j z_j, \quad j = 1,\dots,n,$$

are relaxed by Lagrangian relaxation. ..................................(2p)

(c) Describe how the Lagrangian relaxed problems can be solved in the two cases. ....................................................................(4p)

(d) Discuss which of the two Lagrangian relaxations that would give the best underestimate of the optimal value of the original problem when the corresponding dual problem is solved. ................................................ (2p)

*Good luck!*