SF2812 Applied linear optimization, final exam
Wednesday June 5 2019 14.00–19.00
Brief solutions

1. (a) There is at least one optimal solution, which is integer valued. However, if the optimal solution is nonunique, there will also be noninteger optimal solutions.

(b) Since $\widehat{X}$ is nonnegative, summation of rows and columns of $\widehat{X}$ shows that $\widehat{X}$ is feasible. If we let the matrix S denote the dual slacks, i.e., $s_{ij} = c_{ij} - \widehat{u}_i - \widehat{v}_j$, then

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 0 \end{pmatrix}.$$

Consequently, S has nonnegative components. In addition, complementarity holds, since $\widehat{x}_{ij}s_{ij} = 0$, $i = 1, \ldots, 3$, $j = 1, \ldots, 4$. This means that we have optimal solutions to the two problems.

(c) The nonzero components of the given U correspond to strictly positive components of $\widehat{X}$. By the properties of U, it follows that $\widehat{X} + \alpha U$ is optimal as long as $\widehat{X} + \alpha U$ is nonnegative. The most limiting positive and negative values of α are 0.5 and -1.5 respectively. These values correspond to two integer valued optimal solutions:

$$\widehat{X} - 1.5U = \begin{pmatrix} 8 & 0 & 0 & 2 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 3 & 7 \end{pmatrix} \quad \text{and} \quad \widehat{X} + 0.5U = \begin{pmatrix} 8 & 2 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{pmatrix}.$$

(In this case, $\widehat{X} - 0.5U$ is also an integer valued optimal solution.)

(d) Since $\widehat{X}$ is not an extreme point, it is not provided as a solution by the simplex method.

2. Let z denote the integer variable and let x denote the continuous variables.

At node 0, the LP relaxation of the original problem is solved. Let x_0, z_0 denote an optimal solution to this linear program. If z_0 is integer, we have solved the original problem at node 0, i.e., the original problem.

If z_0 is noninteger, there will be two new nodes in the search tree, one node (node 1) with the additional constraint $z \geq \text{ceil}(z_0)$, and one node (node 2) with the constraint $z \leq \text{floor}(z_0)$, where floor means rounding down to the nearest integer and ceil means rounding up to the nearest integer.

At node 1, the LP relaxation is solved. If this LP is infeasible, the problem at node 1 is infeasible and the node is fathomed. Otherwise, let x_1, z_1 denote an optimal solution. If $z_1 = \text{ceil}(z_0)$, then the problem at node 1 has been solved, and the node is fathomed. If $z_1 > \text{ceil}(z_0)$, then the constraint $z \leq \text{ceil}(z_0)$ is inactive at x_1, z_1. Hence, x_1, z_1 is optimal to the LP relaxed problem of node 0 as well. By assumption that this LP had a unique optimal solution, this cannot happen.

The same argument can be applied to node 2, with the constraint $z \geq \text{ceil}(z_0)$ replaced by $z \leq \text{floor}(z_0)$.

Hence, it takes at most three nodes.

(The assumption about unique LP solutions is merely to avoid some technicalities. Note that if $z_1 > \text{ceil}(z_0)$, then

$$\begin{pmatrix} x_0 \\ z_0 \end{pmatrix} + \frac{\text{ceil}(z_0) - z_0}{z_1 - z_0} \begin{pmatrix} x_1 - x_0 \\ z_1 - z_0 \end{pmatrix}$$

is an optimal solution to the LP relaxation at node 0 where the z component is integer, $\text{ceil}(z_0)$. Hence, the integer program has been solved. The result for node 2 is analogous.)

3. The basis corresponding to $\widetilde{y}$ and $\widetilde{s}$ is $\mathcal{B} = \{1, 4\}$. Let $y = \widetilde{y}$ and $s = \widetilde{s}$. It is straightforward to verify that $B^T y = c_B$ and $s = c - A^T y \geq 0$. Hence, y and s are dual feasible. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

which gives $x_1 = -1/3$, $x_4 = 7/3$. As $x_1 < 0$, the dual solution is not optimal. Consequently, since $x_1 < 0$, x_1 becomes nonbasic, and as x_1 is the first basic variable, the step in the y-direction is given by

$$\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

which gives $q_1 = -1/3$, $q_2 = 1/3$. With $y \leftarrow y + \alpha q$, dual feasibility requires $s \leftarrow s + \alpha\eta$, with $A^T q + \eta = 0$ and $s + \alpha\eta \geq 0$. Consequently, the nonnegativity of s requires $s - \alpha A^T q \geq 0$, i.e.,

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The maximum value of α is given by $\alpha_{\max} = 3/2$ making component 2 of $s - \alpha A^T q$ zero, so that the new basis becomes $\mathcal{B} = \{2, 4\}$. The basic variables are given by

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

which gives $x_2 = 1/2$, $x_4 = 3/2$. As $x \geq 0$, an optimal solution has been obtained. Together with $y + \alpha_{\max} q$ and $s - \alpha_{\max} A^T q$ the primal and dual optimal solutions are given by

$$x = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}, \quad y = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

4. (See the course material.)

5. (a) For a fixed vector $u \in I\!R^n$, Lagrangian relaxation of the first set of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\left(-u_i + \sum_{j=1}^{n}(u_i - c_{ij})x_{ij}\right) + \sum_{j=1}^{n} f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} a_i x_{ij} \le b_j z_j, \quad j = 1,\dots,n, \\ & x_{ij} \in \{0,1\}, \quad i = 1,\dots,n,\ j = 1,\dots,n, \\ & z_j \in \{0,1\}, \quad j = 1,\dots,n, \end{array}$$

where a_i, $i = 1,\dots,n$, b_j, $j = 1,\dots,n$, f_j, $j = 1,\dots,n$, and c_{ij}, $i = 1,\dots,n$, $j = 1,\dots,n$, are nonnegative integer constants.

(b) For a fixed nonnegative vector $v \in I\!R^n$, Lagrangian relaxation of the second group of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i v_j - c_{ij})x_{ij} + \sum_{j=1}^{n}(f_j - b_j v_j) z_j \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\dots,n, \\ & x_{ij} \in \{0,1\}, \quad i = 1,\dots,n,\ j = 1,\dots,n, \\ & z_j \in \{0,1\}, \quad j = 1,\dots,n, \end{array}$$

where a_i, $i = 1,\dots,n$, b_j, $j = 1,\dots,n$, f_j, $j = 1,\dots,n$, and c_{ij}, $i = 1,\dots,n$, $j = 1,\dots,n$, are nonnegative integer constants.

(c) The first relaxation decomposes into one separate problem for each j according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n}(u_i - c_{ij})x_{ij} + f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} a_i x_{ij} \le b_j z_j, \\ & x_{ij} \in \{0,1\}, \quad i = 1,\dots,n, \\ & z_j \in \{0,1\}, \end{array}$$

for $j = 1,\dots,n$. We can here solve two problems, for $z_j = 0$ and $z_j = 1$, and then take the minimum. For $z_j = 0$, the solution is given by $x_{ij} = 0$, $j = 1,\dots,n$. For $z_j = 1$, we obtain a binary knapsack problem, which may for example be solved using dynamical programming.

The second relaxation decomposes into trivial problems. For the z-variables we obtain for each i according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{n}(f_j - b_j v_j) z_j \\ \text{subject to} & z_j \in \{0,1\}, \quad j = 1,\dots,n, \end{array}$$

which can be solved directly with $z_j = 1$ if $f_j - b_j v_j < 0$ and $z_j = 0$ if $f_j - b_j v_j \ge 0$

for $j = 1, \dots, n$. For the x-variables we obtain

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^{n} (a_i v_j - c_{ij}) x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1, \\ & x_{ij} \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

for $i = 1, \dots, n$. These can be solved directly by noting which x_{ij}-variable having the smallest coefficient in the objective function.

(d) The second relaxation gives a relaxed problem which gives integer optimal solutions even if one relaxes the integer constraint. Hence, the corresponding dual underestimation becomes identical with the one obtained if performing an LP-relaxation.

The first relaxation gives a more complicated relaxed problem, and here one can expect the underestimation to be better than one would obtain with an LP-relaxation.