SF2822 Applied nonlinear optimization, final exam
Wednesday August 14 2019 08.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the inequality-constrained quadratic program (IQP) defined by

$$(IQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

with

$$H = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ -3 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \end{pmatrix}.$$

In this question, you may base your arguments on the fact that the problem has only one constraint. The linear systems of equations that may arise need not be solved in a systematic way.

(a) For the given H and c, consider the unconstrained quadratic program

$$(QP) \qquad \text{minimize} \quad \frac{1}{2}x^T Hx + c^T x.$$

Is there a point that satisfies the second-order necessary optimality conditions for (QP)? .... (3p)

(b) For the given H, c, A and b, consider the equality-constrained quadratic program

$$(EQP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Hx + c^T x \\ \text{subject to} & Ax = b. \end{array}$$

Is there a point that satisfies the second-order necessary optimality conditions for (EQP)? .... (3p)

(c) Does (IQP) have a local minimizer? .... (2p)

(d) Does (IQP) have a global minimizer? .... (2p)

2. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}(x_2 + 2)^2 \\ \text{subject to} & -3(x_1 + x_2 - 2)^2 - (x_1 - x_2)^2 + 6 = 0. \end{array}$$

You have obtained a printout from an SQP solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Six iterations, without linesearch, have been performed. The printout reads:

It	x_1	x_2	λ	$\|\nabla f(x) - \nabla g(x)\lambda\|$	$\|g(x)\|$
0	0	0	0	2.2361	6
1	0.75	-0.25	0.14583	0.74361	1.75
2	0.5285	0.050045	0.20644	0.098113	0.29052
3	0.57728	0.041731	0.21804	0.0044016	0.0081734
4	0.57666	0.043089	0.21854	$4.1731 \cdot 10^{-6}$	$5.5421 \cdot 10^{-6}$
5	0.57666	0.043089	0.21854	$3.9569 \cdot 10^{-12}$	$4.8512 \cdot 10^{-12}$
6	0.57666	0.043089	0.21854	$1.1102 \cdot 10^{-15}$	$1.7764 \cdot 10^{-15}$

(a) Formulate the first QP problem. Verify that the solution to this QP problem is given by the printout above. .... (6p)

Hint: The exact value of λ after the first iteration is 7/48, which is approximately 0.14583.

(b) How would the iterates change if the constraint in (NLP) would be changed to $-3(x_1 + x_2 - 2)^2 - (x_1 - x_2)^2 + 6 \geq 0$? .... (2p)

(c) For the original problem (NLP), show that in this case the iterates converge to a global minimizer. (You need not verify the numerical values.) ...... (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function.

3. Consider the QP-problem (QP) defined as

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 2. \end{array}$$

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytical expression for this small problem. .... (5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in Question 3a converge to the optimal solution and Lagrange multiplier respectively of (QP). .... (3p)

(c) For μ small and positive, use your results of Question 3b to give an estimate of $x(\mu) - x^*$ in terms of μ, where x^* denotes the optimal solution to (QP). Is this as expected? .... (2p)

4. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

5. Consider the optimization problem (P) defined by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & x_j \in \{0,1\}, \quad j = 1,\dots,n, \end{array}$$

where H is an indefinite symmetric matrix. Problems of this type arise within combinatorial optimization, and the interest is to find a global minimizer.

One may compute lower bounds on the optimal value of (P) by considering relaxed problems.

(a) One way to relax (P) is to replace the constraints $x_j \in \{0,1\}$, $j = 1,\dots,n$, with $0 \le x_j \le 1$, $j = 1,\dots,n$. This gives a relaxed problem without discrete variables, according to

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T H x \\ \text{subject to} & 0 \le x_j \le 1, \quad j = 1,\dots,n, \end{array}$$

Explain way this relaxed problem is not very interesting in practise. .... (3p)

(b) An alternative way to create a relaxation to (P) is to introduce a symmetric matrix Y and formulate the semidefinite programming problem

$$(SDP) \qquad \begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2}\,\text{trace}(HY) \\ \text{subject to} & \begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & Y = Y^T, \\ & y_{jj} = x_j, \quad j = 1,\dots,n. \end{array}$$

Show that if the constraint $Y = xx^T$ is added to (SDP), one obtains a problem which is equivalent to (P). ....(7p)

Hint: The following two results, which may be used without proof, might be useful:

(i) If H is an $n \times n$-matrix and x is an n-vector, then $\text{trace}(Hxx^T) = x^T H x$.

(ii) If Y is a symmetric $n \times n$-matrix and x is an n-vector, then

$$\begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if and only if} \quad Y - xx^T \succeq 0.$$

Good luck!