1. Consider the inequality-constrained quadratic program \((IQP)\) defined by

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^T H x + c^T x \\
\text{subject to} & \quad Ax \geq b,
\end{align*}
\]

with

\[
H = \begin{pmatrix}
1 & 2 & 0 \\
2 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad c = \begin{pmatrix}
-3 \\
-3 \\
-1
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
0
\end{pmatrix}.
\]

In this question, you may base your arguments on the fact that the problem has only one constraint. The linear systems of equations that may arise need not be solved in a systematic way.

(a) For the given \(H\) and \(c\), consider the unconstrained quadratic program \((QP)\)

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^T H x + c^T x.
\end{align*}
\]

Is there a point that satisfies the second-order necessary optimality conditions for \((QP)\)? ................................................................. (3p)

(b) For the given \(H\), \(c\), \(A\) and \(b\), consider the equality-constrained quadratic program

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^T H x + c^T x \\
\text{subject to} & \quad Ax = b.
\end{align*}
\]

Is there a point that satisfies the second-order necessary optimality conditions for \((EQP)\)? ................................................................. (3p)

(c) Does \((IQP)\) have a local minimizer? ........................................ (2p)

(d) Does \((IQP)\) have a global minimizer? ....................................... (2p)

2. Consider the nonlinear optimization problem \((NLP)\) defined as

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} (x_1 + 1)^2 + \frac{1}{2} (x_2 + 2)^2 \\
\text{subject to} & \quad -3(x_1 + x_2 - 2)^2 - (x_1 - x_2)^2 + 6 = 0.
\end{align*}
\]
You have obtained a printout from an SQP solver for this problem. The initial point is \( x = (0 \ 0)^T \) and \( \lambda = 0 \). Six iterations, without linesearch, have been performed. The printout reads:

<table>
<thead>
<tr>
<th>It</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( \lambda )</th>
<th>( |\nabla f(x) - \nabla g(x)\lambda| )</th>
<th>( |g(x)| )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2.2361</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>0.75</td>
<td>-0.25</td>
<td>0.14583</td>
<td>0.74361</td>
<td>1.75</td>
</tr>
<tr>
<td>2</td>
<td>0.5285</td>
<td>0.050045</td>
<td>0.20644</td>
<td>0.098113</td>
<td>0.29052</td>
</tr>
<tr>
<td>3</td>
<td>0.57728</td>
<td>0.041731</td>
<td>0.21804</td>
<td>0.0044016</td>
<td>0.0081734</td>
</tr>
<tr>
<td>4</td>
<td>0.57666</td>
<td>0.043089</td>
<td>0.21854</td>
<td>4.1731 \cdot 10^{-6}</td>
<td>5.5421 \cdot 10^{-6}</td>
</tr>
<tr>
<td>5</td>
<td>0.57666</td>
<td>0.043089</td>
<td>0.21854</td>
<td>3.9569 \cdot 10^{-12}</td>
<td>4.8512 \cdot 10^{-12}</td>
</tr>
<tr>
<td>6</td>
<td>0.57666</td>
<td>0.043089</td>
<td>0.21854</td>
<td>1.1102 \cdot 10^{-15}</td>
<td>1.7764 \cdot 10^{-15}</td>
</tr>
</tbody>
</table>

(a) Formulate the first QP problem. Verify that the solution to this QP problem is given by the printout above. \( \text{(6p)} \)

\text{Hint:} The exact value of \( \lambda \) after the first iteration is \( \frac{7}{48} \), which is approximately 0.14583.

(b) How would the iterates change if the constraint in (NLP) would be changed to \(-3(x_1 + x_2 - 2)^2 - (x_1 - x_2)^2 + 6 \geq 0\)? \( \text{.................(2p)} \)

(c) For the original problem (NLP), show that in this case the iterates converge to a global minimizer. (You need not verify the numerical values.) \( \text{.................(2p)} \)

\text{Note:} According to the convention of the book we define the Lagrangian \( \mathcal{L}(x, \lambda) \) as \( \mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x) \), where \( f(x) \) the objective function and \( g(x) \) is the constraint function.

3. Consider the QP-problem \((QP)\) defined as

\[
(QP) \quad \text{minimize} \quad \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \\
\text{subject to} \quad x_1 + x_2 \geq 2.
\]

(a) For a given positive barrier parameter \( \mu \), find the corresponding optimal solution \( x(\mu) \) and the corresponding multiplier estimate \( \lambda(\mu) \) to the barrier-transformed problem. It is possible to obtain an analytical expression for this small problem. \( \text{..........................(5p)} \)

(b) Show that \( x(\mu) \) and \( \lambda(\mu) \) which you obtained in Question 3a converge to the optimal solution and Lagrange multiplier respectively of \((QP)\). \( \text{.................(3p)} \)

(c) For \( \mu \) small and positive, use your results of Question 3b to give an estimate of \( x(\mu) - x^* \) in terms of \( \mu \), where \( x^* \) denotes the optimal solution to \((QP)\). Is this as expected? \( \text{..........................(2p)} \)

4. Derive the expression for the symmetric rank-1 update, \( C_k \), in a quasi-Newton update \( B_{k+1} = B_k + C_k \). \( \text{..........................(10p)} \)
5. Consider the optimization problem \((P)\) defined by

\[
\begin{align*}
\text{(P)} & \quad \text{minimize} \quad c^T x + \frac{1}{2} x^T H x \\
& \quad \text{subject to} \quad x_j \in \{0, 1\}, \quad j = 1, \ldots, n,
\end{align*}
\]

where \(H\) is an indefinite symmetric matrix. Problems of this type arise within combinatorial optimization, and the interest is to find a global minimizer.

One may compute lower bounds on the optimal value of \((P)\) by considering relaxed problems.

(a) One way to relax \((P)\) is to replace the constraints \(x_j \in \{0, 1\}, \quad j = 1, \ldots, n\), with \(0 \leq x_j \leq 1, \quad j = 1, \ldots, n\). This gives a relaxed problem without discrete variables, according to

\[
\begin{align*}
\text{minimize} & \quad c^T x + \frac{1}{2} x^T H x \\
\text{subject to} & \quad 0 \leq x_j \leq 1, \quad j = 1, \ldots, n,
\end{align*}
\]

Explain way this relaxed problem is not very interesting in practise. . . . . \(3\text{p}\)

(b) An alternative way to create a relaxation to \((P)\) is to introduce a symmetric matrix \(Y\) and formulate the semidefinite programming problem

\[
\begin{align*}
\text{(SDP)} & \quad \text{minimize} \quad c^T x + \frac{1}{2} \text{trace}(HY) \\
& \quad \text{subject to} \quad \begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
& \quad Y = Y^T, \\
& \quad y_{jj} = x_j, \quad j = 1, \ldots, n.
\end{align*}
\]

Show that if the constraint \(Y = xx^T\) is added to \((SDP)\), one obtains a problem which is equivalent to \((P)\). \(7\text{p}\)

\textbf{Hint:} The following two results, which may be used without proof, might be useful:

(i) If \(H\) is an \(n \times n\)-matrix and \(x\) is an \(n\)-vector, then \(\text{trace}(Hxx^T) = x^T H x\).

(ii) If \(Y\) is a symmetric \(n \times n\)-matrix and \(x\) is an \(n\)-vector, then

\[
\begin{pmatrix} Y & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

if and only if \(Y - xx^T \succeq 0\).

\textit{Good luck!}