SF2822 Applied nonlinear optimization, final exam
Wednesday August 14 2019 08.00–13.00
Brief solutions

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1. (a) The first-order necessary optimality conditions for (EQP) are given by $Hx+c=0$. As H is nonsingular, there is a unique solution given by $x^1 = (1\ 1\ 1)^T$.

The matrix H is not positive semidefinite, since the leading two-by-two principal submatrix is indefinite. With $d = (1\ \ -1\ 0)^T$, we obtain $d^T H d = -2$. Consequently, x^1 does not satisfy the second-order necessary optimality conditions to (EQP).

Consequently, there is no point that satisfies the second-order necessary optimality conditions for (EQP).

(b) The first-order necessary optimality conditions for (EQP) are given by

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -\lambda \end{pmatrix} = \begin{pmatrix} -c \\ b \end{pmatrix}$$

which has unique solution $x^2 = (0\ 3\ 1)^T$, $\lambda^2 = 3$. We may for example form a matrix Z whose columns form a basis for $\text{null}(A)$ as

$$Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for which $Z^T H Z = I$. Hence, x^2 satisfies the second-order necessary optimality conditions.

(c) Since A has only one row, a local minimizer to (IQP) has to be a local minimizer to (QP) or a local minimizer to (EQP). Since x^1 does not satisfy the second-order necessary optimality conditions to (QP), it is not a local mininimizer to (QP). Hence, it is not a local minimizer to (IQP). Since x^2 satisfies the second-order sufficient optimality conditions to (EQP), it is a local minimizer to (EQP). In addition, since $\lambda^2 > 0$, it is also a local minimizer to (IQP).

(d) Let $q(x) = \frac{1}{2}x^T H x + c^T x$. With d given as in (1a), it follows that $q(x^1 + \alpha d)$ and $q(x^1 - \alpha d)$ tend to minus infinity as $\alpha \to \infty$. Since we have only one constraint, at least one of $x^1 + \alpha d$ and $x^1 - \alpha d$ must remain feasible in (IQP) as $\alpha \to \infty$. We conclude that no global minimizer can exist.

2. We have

$$f(x) = \frac{1}{2}(x_1+1)^2 + \frac{1}{2}(x_2+2)^2, \qquad g(x) = -3(x_1+x_2-2)^2 - (x_1-x_2)^2 + 6,$$

$$\nabla f(x) = \begin{pmatrix} x_1+1 \\ x_2+2 \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -8x_1 - 4x_2 + 12 \\ -4x_1 - 8x_2 + 12 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -8 & -4 \\ -4 & -8 \end{pmatrix}.$$

(a) Insertion of numerical values in the expressions above gives the first QP-problem according to

$$\begin{array}{ll} \min & \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + p_1 + 2p_2 \\ \text{subject to} & 12p_1 + 12p_2 = 6. \end{array}$$

This is a convex QP-problem with a globally optimal solution given by

$$\begin{aligned} p_1 - 12\lambda &= -1, \\ p_2 - 12\lambda &= -2, \\ 12p_1 + 12p_2 &= 6. \end{aligned}$$

The prinout from the SQP-solver suggests $p_1 = 3/4$, $p_2 = -1/4$ and $\lambda = 7/48$. Insertion of these values show that they satisfy the optimality conditions.

(b) We can see that $\nabla^2 f(x)$ is positive definite and $\nabla^2 g(x)$ is negative definite, independently of x. Moreover λ is nonnegative in all iterations. This implies that the solution to each QP subproblem is locally optimal also for the case when the equality constraint is changed to a greater than or equal constraint. Hence, the iterates would not change at all if the constraint was changed as suggested.

(c) The inequality-constrained problem is a convex problem, and in addition a relaxation of the original problem. Hence we get convergence towards a global minimizer of this problem, which is also a global minimizer of (NLP).

3. (a) The problem (QP) is a convex quadratic program. The primal part of the trajectory is obtained as minimizer to the barrier-transformed problem

$$(P_\mu) \qquad \min \ \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 - \mu \ln(x_1 + x_2 - 2)$$

under the implicit condition that $x_1 + x_2 - 2 > 0$. The first-order optimality conditions of (P_μ) gives

$$\begin{aligned} x_1(\mu) - \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} &= 0, \\ x_2(\mu) - \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} &= 0. \end{aligned}$$

These equations are symmetric in $x_1(\mu)$ and $x_2(\mu)$. Hence, $x_1(\mu) = x_2(\mu)$. This means that $2x_1(\mu)^2 - 2x_1(\mu) - \mu = 0$, from which it follows that

$$x_1(\mu) = x_2(\mu) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu}{2}} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\mu}.$$

where the plus sign has been chosen for the square root to enforce $x_1(\mu) + x_2(\mu) - 2 > 0$. Since (P_μ) is a convex problem, this is a global minimizer.

The dual part of the trajectory, i.e. $\lambda(\mu)$, is normally given by $\lambda_i(\mu) = \mu/g_i(x(\mu))$, $i = 1, \dots, m$. Here we only have one constraint, so

$$\lambda(\mu) = \frac{\mu}{x_1(\mu) + x_2(\mu) - 2} = \frac{\mu}{\sqrt{1 + 2\mu} - 1} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\mu}.$$

(b) As $\mu \to 0$ it follows that $x(\mu) \to (1\ 1)^T$ and $\lambda(\mu) \to 1$. Let $x^* = (1\ 1)^T$ and $\lambda^* = 1$. Then x^* and λ^* satisfy the first-order optimality conditions of (QP). Since (QP) is a convex problem, this is sufficient for global optimality of (QP).

(c) We have

$$x_1(\mu) - x_1^* = x_2(\mu) - x_2^* = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 2\mu} = \frac{1}{2}\mu + o(\mu).$$

This is as expected. We would expect $\|x(\mu) - x^*\|_2$ to be of the order μ near an optimal solution where regularity holds.

4. (See the course material.)

5. (a) The relaxed problem is a non-convex quadratic programming problem. To obtain a lower bound of the original problem we do need to calculate a global minimizer of this non-convex relaxed problem, which in general is not computationally tractable.

(b) If we let (SDP') be the problem arising as the constraint $Y = xx^T$ is added to (SDP) we can replace Y with xx^T, which by (i) gives

$$(SDP') \qquad \begin{array}{ll} \min & c^Tx + \frac{1}{2}x^THx \\ \text{subject to} & \begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ & x_j^2 = x_j, \quad j = 1, \ldots, n. \end{array}$$

By hint (ii) we can see that the constraint

$$\begin{pmatrix} xx^T & x \\ x^T & 1 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is always fulfilled, hence (SDP') may be written as

$$(SDP') \qquad \begin{array}{ll} \min & c^Tx + \frac{1}{2}x^THx \\ & x_j^2 = x_j, \quad j = 1, \ldots, n. \end{array}$$

But $x_j^2 = x_j$ if and only if $x_j \in \{0, 1\}$. Hence, (SDP') and (P) are equivalent.