KTH ROYAL INSTITUTE OF TECHNOLOGY

Lecture 1

- EQ2810 Estimation theory 6hp (MSc)
- EM3210 Estimation theory 10hp (PhD)

Magnus Jansson

Course homepages

EQ2810: `https://kth.instructure.com/courses/11302`

EM3210:

`https://www.kth.se/social/group/em3210-estimeringste/`

PhD students: Please register today by sending an email to magnus.jansson@ee.kth.se

Course information

- Main course literature: Fundamentals of Statistical Signal Processing: Estimation Theory, Kay, Steven M. ISBN 0133457117.
- Format: Lectures and homework on a weekly basis plus one project. Course contents can be learnt by cooperative discussions, but homework problems should be solved individually and handed in in due time for grading. Please recall the KTH rules for examination.
- For PhD students we will in addition require:
  - peer grading of homework
  - one more project
  - 48 hour take-home examination

Requirements

- Individual solutions to homework problems. (Exam if homework assignments are not properly solved.) Preliminary grading for the masters level course will be: E=60%, D=65% , C=70%, B=80%, A=90% of max score.
- Project assignment
- For PhD students we require at least 80% plus the additional tasks.
- Number of credits:
  Master students: 6 (4.5+1.5) ECTS (graded by A-F)
  PhD students: 10 ECTS (Pass/Fail)

Preliminary Schedule

See course homepages or the KTH central schedule for EQ2810. We have seven slots scheduled, Tuesdays 13.15-15. Rooms will differ. Preliminary we will use six of them for normal lectures.

- **Lec. 1:** Ch. 1-3
- **Lec. 2:** Ch. 4-5
- **Lec. 3:** Ch. 6-7
- **Lec. 4:** Ch. 8-9
- **Lec. 5:** Ch. 10-11
- **Lec. 6:** Ch. 12,14,15
- **Lec. 7:** Reserve, project?

We may need to add some slots later for project presentations or the like.

Course/book Outline

Ch. 1: Introduction
Ch. 2: Minimum Variance Unbiased Estimation
Ch. 3: Cramer Rao Lower Bound
Ch. 4: Linear Models
Ch. 5: General Minimum Variance Unbiased Estimation
Ch. 6: Best Linear Unbiased Estimators
Ch. 7: Maximum Likelihood Estimation
Ch. 8: Least Squares
Ch. 9: Method of Moments
Ch. 10: The Bayesian Philosophy
Ch. 11: General Bayesian Estimators
Ch. 12: Linear Bayesian Estimators
Ch. 14: Summary of Estimators
Ch. 15: Extensions for Complex Data and Parameters

Homework submission

Homework solutions should be submitted in PDF format (scanned handwritten solutions, or typeset). PhD students by email to `magnus.jansson@ee.kth.se` and Master students via Canvas. Make sure that the scanned versions are readable but still of a reasonable file size. It is not necessary to bring paper originals of your solutions to the lectures.

Filenames:
Name your homework solution PDF-file as:

HWX_FN_LN.pdf

where

X is the assignment number,
FN =first name,
LN=last name.

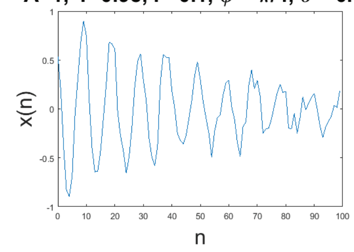
PhD student peer grading

- We will divide PhD students into groups that jointly do the grading of solutions from another group. Further instructions will be sent by email.

Applications

- Radar: Determine range to target, indirectly by estimating time-delays between transmitted pulses and received echoes.
- GPS: Estimate ranges to multiple satellites and use trilateration.
- Pulse meters: Estimate heart-rate from EKG or by using photoplethysmography (led lights)
- Stepcounters: Estimate steps from accelerometer signals
- Dynamic systems: Estimate transfer functions of unknown systems from data.
- Spectral analysis: Estimate power of signals at different frequencies.
- Machine monitoring: Estimate characteristics of signals to monitor faults

Example



A damped sinusoid signal $s(n)$ is observed in noise:

$$x(n) = s(n) + w(n)$$

for $n = 0, 1, \ldots, N-1$. Here

$$s(n) = Ar^n \cos(2\pi fn + \phi)$$

and $w(n)$ is zero mean white Gaussian noise (WGN) with variance σ^2.

Problem: Estimate unknown parameters $\boldsymbol{\theta} = [A \; r \; f \; \phi \; \sigma^2]$, or subsets thereof, from $\mathbf{x} = [x(0) \; x(1) \; \ldots x(N-1)]$.

Example cont'd

- That is, we want to find a function $g(\cdot)$ such that $\hat{\theta} = g(\mathbf{x})$ is a good approximation to the true unknown θ.
- $\hat{\theta} = g(\mathbf{x})$ is generally called an *estimator* when viewed as a function of the random data,
- It is called an *estimate* when it is a function of an observed realization of the data.

Special case: DC-level in noise

$$x(n) = A + w(n); \quad n = 0, 1, \ldots, N-1$$

This means that each data sample is distributed as $x(k) \in N(A, \sigma^2)$, and with probability density function

$$p(x(k); [A \; \sigma^2]) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x(k) - A)^2\right\}$$

Possible estimators of A?

$$\hat{A}_1 = \frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

$$\hat{A}_2 = x(0)$$

Good or bad?

Check mean:

$$E\{\hat{A}_1\} = \frac{1}{N}\sum_{n=0}^{N-1} E\{x(n)\} = A$$

$$E\{\hat{A}_2\} = E\{x(0)\} = A$$

Both have correct mean, but they may still not be good.
Check variance:

$$\text{var}\{\hat{A}_1\} = E\{(\hat{A}_1 - E\{\hat{A}_1\})^2\} = \frac{1}{N^2}\sum_{n=0}^{N-1} \text{var}\{x(n)\} = \frac{\sigma^2}{N}$$

$$\text{var}\{\hat{A}_2\} = \text{var}\{x(0)\} = \sigma^2$$

Unbiased estimator

- Bias is defined as: $b(\theta) = E\{\hat{\theta}\} - \theta = \int g(\mathbf{x})p(\mathbf{x};\theta)d\mathbf{x} - \theta$
- An estimator is unbiased iff $b(\theta) = 0$ for all θ.
- Even if unbiased, an estimator is not necessarily good as we just saw.

Minimum mean square error (MMSE)

A possible optimality criterion is

$$\text{mse}(\hat{\theta}) = E\{(\hat{\theta} - \theta)^2\}$$

A general fact is

$$\begin{aligned}\text{mse}(\hat{\theta}) &= E\{(\hat{\theta} - E\{\hat{\theta}\} + E\{\hat{\theta}\} - \theta)^2\} \\ &= E\{(\hat{\theta} - E\{\hat{\theta}\})^2 + (E\{\hat{\theta}\} - \theta)^2\} \\ &= \text{var}(\hat{\theta}) + b^2(\theta)\end{aligned}$$

To minimize MSE we need to minimize the combined contribution of bias and variance.

Example

For the DC-gain problem, consider the estimator

$$\hat{A}_3 = a\frac{1}{N}\sum_{n=0}^{N-1} x(n)$$

for some constant a. We have $b(A) = (1-a)A$ and $\text{var}(\hat{A}_3) = a^2\sigma^2/N$. Hence,

$$mse(\hat{A}_3) = a^2\sigma^2/N + (a-1)^2A^2$$

The best choice of a in terms of minimizing the MSE is

$$a_{opt} = \frac{A^2}{A^2 + \sigma^2/N} = \frac{1}{1 + \sigma^2/A^2N}$$

However, we cannot implement this estimator since we do not know A^2/σ^2. This is typically true for MMSE estimators, but not always.

Minimum variance unbiased estimators (MVUE)

- Conclusion: Look for unbiased estimators with minimum variance
- Actually we want the estimator to be *uniformly MVU* for all θ.
- Exist? Not always, unfortunately.
- Possible tools to find them: CRLB (Ch. 3), Ch. 5, linear estimators (Ch. 6)

Introd. to Cramér-Rao Lower Bound (CRLB)

- How accurate can we estimate θ?
- Can we say something general?
- In our probabilistic setting, the information is embedded in the pdf $p(x;\theta)$.
- Loosely speaking the more sensitive the pdf is to changes in θ, the more accurate θ can be estimated.

CRLB (scalar parameter)

If the pdf satisfies "regularity conditions" then the variance of any *unbiased* estimator $\hat{\theta}$ satisfies

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-\mathrm{E}\left\{\frac{\partial^2 \ln\left(p(\mathbf{x};\theta)\right)}{\partial\theta^2}\right\}} = \frac{1}{\mathrm{E}\left\{\left(\frac{\partial \ln\left(p(\mathbf{x};\theta)\right)}{\partial\theta}\right)^2\right\}} = \frac{1}{I(\theta)}$$

where the derivatives are evaluated at the true parameter θ and expectation is wrt $p(\mathbf{x};\theta)$. Further, iff

$$\frac{\partial \ln\ p(\mathbf{x};\theta)}{\partial\theta} = I(\theta)(g(\mathbf{x}) - \theta)$$

for some $I(\theta)$ and $g(\mathbf{x})$, the bound can be attained. In fact the estimator is $\hat{\theta} = g(\mathbf{x})$ and its variance is $1/I(\theta)$.
($I(\theta)$ is the Fisher information.)

CRLB comments

- If we find an estimator attaining the CRLB it is also the MVUE. However, an MVUE does not need to attain the CRLB.
- See book for many more comments, examples, and properties of the CRLB. It is an important tool to investigate given estimation problems.
- We will give an alternative proof of the vector parameter case on the board if time permits.