Rigid Body Dynamics (SG2150)
Solution to Exam, 2019-10-21, 14.00-18.00

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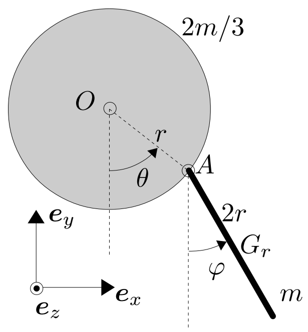
Problem 1.



Use θ and φ as generalised coordinates.

To find equilibrium points, we study the potential energy. Only the gravity force on the rod contributes. Denoting the centre of mass of the rod as G_r, we can write $\boldsymbol{r}_{OG_r} = \boldsymbol{r}_{OA} + \boldsymbol{r}_{AG_r}$. From the vertical component of this vector, we get

$$V = -mgr\left[\cos(\theta) + \cos(\varphi)\right].$$

Equilibrium points are found from

$$\begin{aligned}\frac{\partial V}{\partial \theta} &= mgr\sin(\theta) = 0\\ \frac{\partial V}{\partial \varphi} &= mgr\sin(\varphi) = 0.\end{aligned}$$

The first equation gives $\theta = 0$ or $\theta = \pi$. The second equation gives $\varphi = 0$ or $\varphi = \pi$. Of the four combinations only

$$\theta^* = 0,\ \varphi^* = 0$$

gives a positive definite stiffness matrix, which is

$$\boldsymbol{K} = mgr\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

To compute the kinetic energy we use the instantaneous centre of rotation O for the disc, and the "two parts" formula for the rod. Thus we need the velocity of the centre of mass G_r and the angular velocities $\boldsymbol{\omega}_d$ and $\boldsymbol{\omega}_r$ for the disc and rod, respectively. For the angular velocities, we find $\boldsymbol{\omega}_d = \dot{\theta}\boldsymbol{e}_z$, $\boldsymbol{\omega}_r = \dot{\varphi}\boldsymbol{e}_z$. As we are considering small oscillations about the equilibrium point, it is enough to compute the velocities in the equilibrium point configuration where O, A, and G_r lie below each other. The velocity connection formula then gives

$$\boldsymbol{v}_A = \boldsymbol{v}_O + \boldsymbol{\omega}_d \times \boldsymbol{r}_{OA} = r\dot{\theta}\boldsymbol{e}_x.$$

and

$$\boldsymbol{v}_{G_r} = \boldsymbol{v}_A + \boldsymbol{\omega}_r \times \boldsymbol{r}_{AG_r} = r\left(\dot{\theta} + \dot{\varphi}\right)\boldsymbol{e}_x.$$

Together with the moments of inertia of a disc $J_d = (2m/3)r^2/2$ and rod $J_r = m(2r)^2/12$ about their respective centres of mass, we get a total kinetic energy of

$$T^* = \frac{1}{2}J_d\dot{\theta}^2 + \frac{1}{2}m|\boldsymbol{v}_{G_r}|^2 + \frac{1}{2}J_r\dot{\varphi}^2 = mr^2\left[\frac{2}{3}\dot{\theta}^2 + \dot{\theta}\dot{\varphi} + \frac{2}{3}\dot{\varphi}^2\right].$$

(Had we computed T for an arbitrary configuration, we would have found

$$T = mr^2\left[\frac{2}{3}\dot{\theta}^2 + \dot{\theta}\dot{\varphi}\cos(\theta - \varphi) + \frac{2}{3}\dot{\varphi}^2\right]$$

instead). This gives the mass matrix

$$\boldsymbol{M}_1 = mr^2\begin{bmatrix}\frac{4}{3} & 1\\ 1 & \frac{4}{3}\end{bmatrix}.$$

The eigenvalue problem now becomes

$$(\boldsymbol{K} - \lambda\boldsymbol{M})\,\boldsymbol{a} = \boldsymbol{0}.$$

With $\lambda = g\beta/r$ we get

$$mgr\begin{bmatrix}1 - \frac{4}{3}\beta & -\beta\\ -\beta & 1 - \frac{4}{3}\beta\end{bmatrix}\boldsymbol{a} = \boldsymbol{0}.$$

The matrix is singular when the determinant is zero, which gives

$$(1 - \frac{4}{3}\beta)^2 - \beta^2 = 0 \Rightarrow \beta = \frac{12 \pm 9}{7}.$$

As expected, both β values are positive, confirming that the equilibrium point is stable. We get the answer

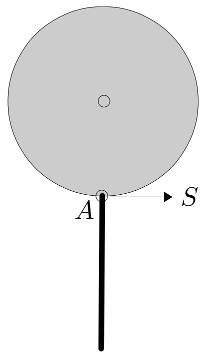
$$\omega_1^2 = \lambda_1 = \frac{3}{7}\frac{g}{r}, \quad \omega_2^2 = 3\frac{g}{r}.$$

Note that this system has a curious symmetry where θ and φ can be exchanged. This means that synchronised motion $\theta(t) = \varphi(t)$ or $\theta(t) = -\varphi(t)$ is possible. For the eigenvalue eigenvalue problem the symmetry gives that the mode shapes must be

$$\boldsymbol{a}_1 = \begin{bmatrix}1\\ -1\end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$$

from which the eigenvalues can be directly read off whithout solving a second order equation. Both of these modes extend to finite amplitudes.

Problem 2.



We note that the system is in the equilibrium configuration from Problem 1. Thus we can use the same generalised coordinates, and reuse the kinetic energy. Since the system is at rest, $\dot{\theta}_i = \dot{\varphi}_i = 0$. Lagrange's equations for impact thus becomes

$$mr^2 \begin{bmatrix} \frac{4}{3} & 1 \\ 1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \dot{\theta}_f - 0 \\ \dot{\varphi}_f - 0 \end{bmatrix} = \begin{bmatrix} I_\theta \\ I_\varphi \end{bmatrix}.$$

It remains to compute the generalised impulses. The physical impulse is $\boldsymbol{S} = S\boldsymbol{e}_x$ acting on the point A. Again from Problem 1 we know that

$$\boldsymbol{v}_A = r\dot{\theta}\boldsymbol{e}_x$$

so we can read off the tangent vectors for $\boldsymbol{v}_A$ as

$$\boldsymbol{\tau}_\theta^{\boldsymbol{v}_A} = r\boldsymbol{e}_x, \quad \boldsymbol{\tau}_\varphi^{\boldsymbol{v}_A} = \boldsymbol{0}.$$

Thus

$$I_\theta = \boldsymbol{\tau}_\theta^{\boldsymbol{v}_A} \bullet \boldsymbol{S} = rS, \quad I_\varphi = \boldsymbol{\tau}_\varphi^{\boldsymbol{v}_A} \bullet \boldsymbol{S} = 0.$$

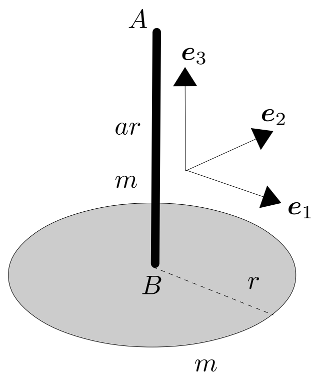
Now we solve

$$mr^2 \begin{bmatrix} \frac{4}{3} & 1 \\ 1 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} \dot{\theta}_f \\ \dot{\varphi}_f \end{bmatrix} = rS \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to get

$$\dot{\theta}_f = \frac{12}{7}\frac{S}{mr}, \quad \dot{\varphi}_f = -\frac{9}{7}\frac{S}{mr}.$$

Problem 3.



By continuous rotation symmetry, the symmetry axis is a principal axis and so is any perpendicular axis. Take the 3-axis to be along the symmetry axis.

$$J_{A33} = J_{A33,\text{disc}} + J_{A33,\text{rod}} = \frac{mr^2}{2} + 0 = \frac{mr^2}{2}.$$

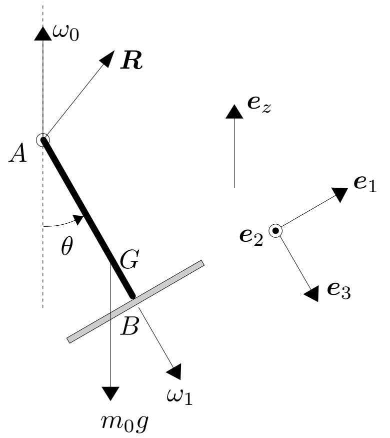
$$J_{A11} = J_{A11,\text{disc}} + J_{A11,\text{rod}} = J_{B11,\text{disc}} + m|\boldsymbol{r}_{BA}|^2 + \frac{m(ar)^2}{3} = \frac{mr^2}{4} + ma^2r^2 + \frac{ma^2r^2}{3} = \left(\frac{1}{4} + \frac{4}{3}a^2\right) mr^2.$$

By symmetry $J_{A22} = J_{A11}$.

To get all principal moments of inertia to be equal, we need

$$\frac{1}{4} + \frac{4}{3}a^2 = \frac{1}{2} \Rightarrow a^2 = \frac{3}{16}.$$

Problem 4.



Use a basis triad with $\boldsymbol{e}_3$ in the $\boldsymbol{r}_{AB}$ direction and $\boldsymbol{e}_2$ horizontal. For the considered motion the angular velocity of the triad is

$$\boldsymbol{\omega}^b = \omega_0 \boldsymbol{e}_z, \quad \boldsymbol{e}_z = -c_\theta \boldsymbol{e}_3 + s_\theta \boldsymbol{e}_1,$$

and the angular velocity of the body is

$$\boldsymbol{\omega} = \boldsymbol{\omega}^b + \omega_1 \boldsymbol{e}_3 = s_\theta \omega_0 \boldsymbol{e}_1 + (\omega_1 - c_\theta \omega_0)\, \boldsymbol{e}_3.$$

Thus the angular momentum of the body about the point A is

$$\boldsymbol{L}_A = J_1 s_\theta \omega_0 \boldsymbol{e}_1 + J_3 (\omega_1 - c_\theta \omega_0)\, \boldsymbol{e}_3$$

where the moments of inertia J_1 and J_3 are computed in Problem 3. The time derivative is

$$\dot{\boldsymbol{L}}_A = \frac{{}^b d\boldsymbol{L}_A}{dt} + \boldsymbol{\omega}^b \times \boldsymbol{L}_A = \omega_0 (-c_\theta \boldsymbol{e}_3 + s_\theta \boldsymbol{e}_1) \times (J_1 s_\theta \omega_0 \boldsymbol{e}_1 + J_3 (\omega_1 - c_\theta \omega_0)\, \boldsymbol{e}_3) = -s_\theta \omega_0 \left[J_3 \omega_1 + (J_1 - J_3)\, c_\theta \omega_0\right] \boldsymbol{e}_2$$

Considering forces

$$\boldsymbol{M}_A = \boldsymbol{r}_{AG} \times m_0 g\, (-\boldsymbol{e}_z)\,, \ \boldsymbol{r}_{AG} = l\boldsymbol{e}_3 \Rightarrow \boldsymbol{M}_A = -m_0 g l s_\theta \boldsymbol{e}_2,$$

where $m_0 = 2m$ is the body mass and l gives the position of the centre of mass for the composite body:

$$l = \frac{1}{2}\left(ar + \frac{1}{2}ar\right) = \frac{3}{4}ar.$$

Since A is a fixed point, angular momentum balance gives

$$\dot{\boldsymbol{L}}_A = \boldsymbol{M}_A \Rightarrow -s_\theta\omega_0\left[J_3\omega_1 + (J_1 - J_3)\, c_\theta\omega_0\right]\boldsymbol{e}_2 = -m_0 g l s_\theta \boldsymbol{e}_2.$$

We see this is fulfilled if

$$s_\theta\omega_0\left[J_3\omega_1 + (J_1 - J_3)\, c_\theta\omega_0\right] = m_0 g l s_\theta,$$

or with values from Problem 3 inserted

$$s_\theta\omega_0\left[\frac{1}{2}\omega_1 + \left(\frac{4}{3}a^2 - \frac{1}{4}\right)c_\theta\omega_0\right] = \frac{3}{2}s_\theta a\frac{g}{r}.$$

If $a^2 = 3/16$ then for any angle $0 < \theta < \pi$ this simplifies to

$$\omega_0\omega_1 = \frac{3\sqrt{3}}{4}\frac{g}{r}.$$

Note that, apart from the particular values of J_1, J_3, m_0, and l, this is precisely the precession of a heavy symmetry top, as treated in the text book.

Problem 5. Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = 0.$$

Thus p is constant (q is a cyclic coordinate) and by the initial conditions $p(t) = p_0$. Inserting this into the other equation, and using the initial conditions gives $q(t) = q_0 + \frac{p_0}{m}t$, so q has a linear drift.

A concrete example is a particle of mass m on a smooth track with all applied forces perpendicular to the track, if the q coordinate measures length along the track.

Problem 6. Let $L = T - V$. First we note that θ is a cyclic coordinate, so

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant}.$$

Secondly, since L is time independent and of type $L_2 + L_0$, we have that

$$E = L_2 - L_0 = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{mGM}{r} = \text{constant}.$$

Thus we have the required two expressions.

To find the values of p_θ and E, we eliminate $\dot{\theta}$ to get

$$E = \frac{1}{2}m\dot{r}^2 + \frac{p_\theta^2}{2mr^2} - \frac{mGM}{r}.$$

At time point t_0 of minimal value of $r(t)$ we must have $\dot{r}(t_0) = 0$, $r(t_0) = r_0$. Inserting gives

$$E = \frac{p_\theta^2}{2mr_0^2} - \frac{mGM}{r_0}.$$

Similarly we get at the time point of maximal $r(t)$

$$E = \frac{p_\theta^2}{8mr_0^2} - \frac{mGM}{2r_0}.$$

Solving for E and p_θ^2, we find

$$E = -\frac{1}{3}\frac{mGM}{r_0}, \quad p_\theta^2 = \frac{4}{3}m^2 GMr_0.$$

Note that E and p_θ have the interpretation of mechanical energy and angular momentum, respectively. Thus the negative value of E was expected for this bounded orbit.