SF2812 Applied linear optimization, final exam
Monday March 9 2020 8.00–13.00

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Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (P) and (D) be defined by

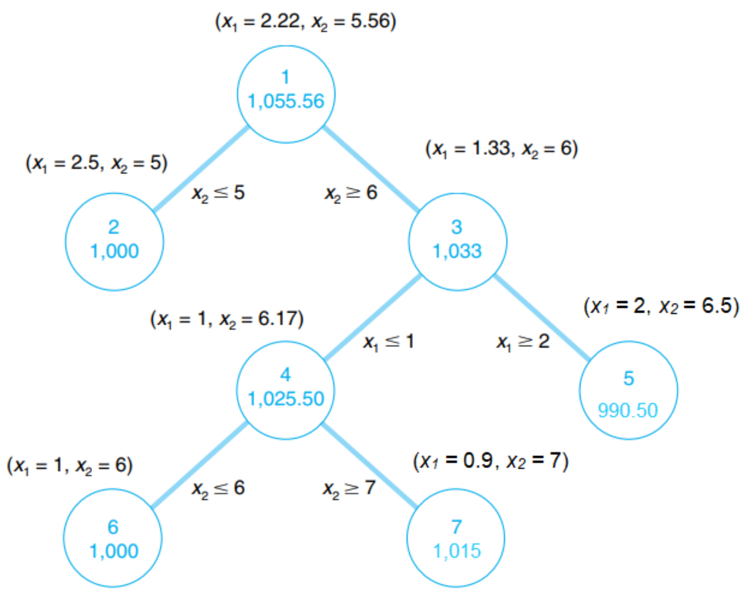
$$(P)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \ge 0. \end{array}$$

Assume that (P) and (D) are both feasible.

(a) Prove that $c^T x \ge b^T y$. .... (5p)

(b) Let x^* be an optimal solution of (P) and let y^* be an optimal solution of (D). Prove that $c^T x^* = b^T y^*$. You may make use of complementary slackness without proof. .... (5p)

2. Observe the following branch-and-bound tree for an integer linear program.



(a) Motivate why the tree corresponds to a maximization problem. ....(3p)

(b) Indicate the current incumbent node. ....(2p)

(c) Indicate the current upper bound (best relaxed solution) node. ....(2p)

(d) Can we conclude if the current incumbent node is an optimal solution? If not, indicate from which node we should continue to iterate and which variable should be use to branch. ....(3p)

3. For $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$, let $\varphi(u) = \min_{x\in I\!R^n}\{f(x) - u^T g(x)\}$ and consider the optimization problem

$$(D) \qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \in I\!R^m. \end{array}$$

(a) Define a subgradient to φ at a point $u \in I\!R^m$. ....(4p)

(b) Assume that u is not optimal to (D). Let s be a subgradient to φ at u and let u^* be optimal to (D). Show that

$$\|u + \theta s - u^*\|_2^2 < \|u - u^*\|_2^2 \qquad \text{for} \quad \theta \in \left(0, \frac{2s^T(u^* - u)}{\|s\|_2^2}\right).$$

....(3p)

(c) Show that $-g(x(u))$ is a subgradient to φ at u, where $x(u)$ is an optimal solution to $\min_{x\in I\!R^n}\{f(x) - u^T g(x)\}$. ....(3p)

4. Consider the linear program (LP) defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 + x_2 = 1, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{array}$$

(a) For a fixed positive barrier parameter μ, formulate the primal-dual system of nonlinear equations corresponding to the problem above. Use the fact that the problem is small to give explicit expressions for the solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the system of nonlinear equations. ....(8p)

Hint: You may for example find an explicit expression for $y(\mu)$ and then express $x(\mu)$ and $s(\mu)$ in terms of this $y(\mu)$.

(b) Calculate optimal solutions to (LP) and the corresponding dual problem by letting $\mu \to 0$ in the expressions given in (4a). Verify optimality. ....(2p)

5. A company is selling kilograms of three different types of cookies: wheat, chocolate and vanilla. To make these products, they have at most 50 man-hours (MH) available a week and the company cannot produce more than 2 kilograms of vanilla cookies, since they do not have enough materials. Let x be the production decision variable vector (x_1 are wheat, x_2 chocolate and x_3 vanilla, in kilograms). Then, the company has been able to formulate the following problem in its standard form:

$$\begin{aligned} \underset{x \in \mathbb{R}^5}{\text{minimize}} \quad & -20x_1 - 15x_2 - 40x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 4x_3 + x_4 = 50 \\ & x_3 + x_5 = 2 \\ & x \geq 0. \end{aligned}$$

The objective is to maximize profits on sales in USD, assuming everything produced is sold. The first restriction refers to the MH capacity and the second is vanilla cookies constraint.

(a) While solving this problem, you finally get the solution: $x = [0, 21, 2, 0, 0]$. Prove this solution is a basic feasible solution and also prove its optimality. .... (2p)

(b) Your supplier is selling you more material for the vanilla cookies, at 10 USD per extra unit. Would you take it and for how many extra units? Justify your answer. .... (2p)

(c) A colleague insists that if wheat cookies improved their profit 50%, the company would start selling them along chocolate and vanilla cookies. What do you think? Justify your answer. .... (2p)

(d) Suppose production has been expanded for N different type of products. Let c_i be the (negative) profits for product i, let d_i be the material supply limitation of product i, let a_i be the amount of MH needed to produce one item i, and let H be the total amount of MH. Then we get the following LP:

$$\begin{aligned} \underset{x \in \mathbb{R}^N}{\text{minimize}} \quad & \sum_{i=1}^{N} c_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{N} a_i x_i \leq H \\ & x_i \leq d_i \qquad\qquad i = 1, \dots, N \\ & x \geq 0. \end{aligned}$$

For large N, the problem size could become an issue. Someone suggested using a decomposition technique: which one would you use and why? .... (2p)

(e) Now suppose x represents boxes of cookies instead of kilograms, so that x has to be integer. For large N this is still an issue, and the technique for continuous problems will not be valid here. What do you suggest instead? .... (2p)

Good luck!