SF2812 Applied linear optimization, final exam
Monday March 9 2020 8.00–13.00
Brief solutions

1. (a) Let x and (y, s) be feasible solutions in (P) respectively (D). Duality gap while taking into account feasibility of (P) and (D) gives

$$c^T x - b^T y = (c - A^T y)^T x = s^T x \geq 0,$$

from which it follows that $c^T x \geq b^T y$.

Alternatively:
Let x be a feasible solution to the primal problem and y be a feasible solution to the dual problem. Let A_j be the j-th column of A and a_i the i-th row of A. Then, we define

$$\begin{aligned} u_i &= y_i(a_i x - b_i), \\ v_j &= (c_j - y' A_j) x_j. \end{aligned}$$

The definition of the dual problem requires the sign of y_i to be the same as the sign of $a_i' x - b_i$, and the sign of $c_j - y' A_j$ to be the same as the sign of x_j. Thus, primal and dual feasibility imply that

$$u_i \geq 0, \forall i,$$

and

$$v_j \geq 0, \forall j.$$

Notice that

$$\sum_i u_i = y' A x - y' b,$$

and

$$\sum_j v_j = c' x - y' A x.$$

We add these two equilities and use nonnegativity of u_i, v_j, to obtain

$$0 \leq \sum_i u_i + \sum_j v_j = c' x - y' b,$$

finishing the proof.

(b) Let x and (y, s) be feasible solutions in (P) respectively (D). Duality gap while taking into account feasibility of (P) and (D) gives

$$c^T x - b^T y = (c - A^T y)^T x = s^T x = 0,$$

where the last equality follows from complementary slackness theorem. Thus $c^T x = b^T y$.
Alternatively:
Let B the optimal basis, then $x_B = B^{-1} b$ be the corresponding vector of basic variables. Then the simplex method terminates, the reduced costs must be nonnegative and we obtain

$$c' - c_B' B^{-1} A \geq 0',$$

where c'_B is the vector of costs of the basic variables and $0'$ the vector of 0's with the corresponding dimension. Let us define a vector y by letting $y' = c'_B B^{-1}$. We then have $y'A \leq c'$, which shows that y is a feasible solution to (D). In addition

$$y'b = c'_B B^{-1} b = c'_B x_B = c'x.$$

it follows that y is an optimal solution to the dual (corollary of the weak duality theorem proven in a), and the optimal dual cost is equal to the optimal primal cost.

2. (a) As we constrain the problem using bounds for the non integer variables, we are making fractional extreme points infeasible, thus we expect the optimal value of the problem to become equal or **worse** (defined whether we are minimizing of maximizing). In the image, as we move deeper into the tree the objective solution diminishes, then we conclude this is a maximization problem.

(b) Node 6, as is the only integer solution node in the tree.

(c) Node 7, being the deepest node with the highest objective value, and higher than the current incumbent.

(d) As the objective value of the current incumbent, i.e. the lower bound, is 1,000 and the upper bound is 1,015, then the optimality gap is higher than 0%. We conclude the problem should continue branching from the upper bound node, i.e. node 7 branching variable x_1. Then, we have not found the optimal solution.

3. Please refer to lesson 12.

4. (a) The primal-dual system obtained is

$$\begin{aligned} x_1 + x_2 &= 1, \\ y_1 + s_1 &= 1, \\ y_1 + s_2 &= 1, \\ x_1 s_1 &= \mu, \\ x_2 s_2 &= \mu. \end{aligned}$$

From the system we conclude that $x_1(\mu) = x_2(\mu) = 1/2, y_1(\mu) = 1-2\mu, s_1(\mu) = s_2(\mu) = 2\mu$.

(b) If $\mu \to 0$ then $\lim_{\mu\to 0} x_1(\mu) = \lim_{\mu\to 0} x_2(\mu) = 1/2, \lim_{\mu\to 0} y_1(\mu) = 1, \lim_{\mu\to 0} s_1(\mu) = \lim_{\mu\to 0} s_2(\mu) = 0$.

5. (a) We can easily check that $x_B \geq 0$ then it is feasible. Since $B = [A_2, A_3]$, then $\bar{c} = c_N - C'_B B^{-1} N = [5/2, 15/2, 10] \geq 0]$ then it is optimal.

(b) Using the optimal basis, let $y = c'_B B^{-1} = [-15/2, -10]$ be the optimal dual solution of our problem. Then, since $y_2 = -10$ we conclude that for an extra unit of supplies for vanilla cookies our objective value decreases in -10 units, but we need to pay 10 units back to the supplier. Thus, as the total gain is

$-10+10=0$, we are economically indifferent in the offer. However, this is true only within the limits set by the feasibility of the current basis. In other words, we need to check that $x_B = B^{-1}\tilde{b} \geq 0$ where $\tilde{b} = b + \epsilon e_2$ for some $\epsilon \in \mathbb{R}$. Then, by solving $B^{-1}\tilde{b} \geq 0$ we obtain the following inequations

$$25 - 4 - 2\epsilon \geq 0, 2 + \epsilon \geq 0.$$

Then, we get $\epsilon \in [-2, 21/2]$.

(c) First, we need to check if the new cost would change the current basis. For that, we check the reduced costs $\bar{c} = \tilde{c}_N - c_B' B^{-1} N \geq 0$ where $\tilde{c}_N = c_N + \epsilon e_1$ for some $\epsilon \in \mathbb{R}$. Therefore, we get the following inequation $\epsilon \geq -5/2$. Since $-20 \times 50\% = -10 < -5/2$, we could conclude the optimal condition would change and x_1 would enter the basis for the next iteration. However, we would have to continue the simplex algorithm to know the final basis and conclude if that production plan would be optimal.

(d) A Dantzig-Wolfe (DW) decomposition would be the best fit, as the problem has the proper structure: one row that binds all variables together and a set of constraints that only relate to a variable for each row. We would need to select a few extreme points, and since we bound every variable to be $0 \leq x_i \leq d_i$ we know that the feasible set is described as a polytope, so there are no extreme rays simplifying the execution of the algorithm further.

(e) In this case, DW would not help as we lose convexity. Therefore, in this case we do have more room to choose. First, the easiest would be to use lagrangian relaxation as it shares the structure we could use. We would need to select which set of constraints to relax: either the complicating constraint, the set of bounds for each variable, or a combination of these. Another technique would be to use the subgradient method, as it shares similarities with the lagrangian relaxation: we would probably choose to relax the bounds, using the first row as a knapsack constraint and then solve the problem of maximizing u by enumerating the possible values of x_i. Both techniques are valid and could potentially solve the problem faster than using branch-and-bound.