Chapter 3

Photon Statistics

We discus the nature of light, how to measure its statistics and show that light sources can be categorized along their photon emission statistics in three states: coherent, Fock and thermal states. We then consider the case of single photon sources and the measurement that can reveal the single photon nature of a light source.

When Maxwell published his four famous equations it seemed that the nature of light was finally fully understood: it was made of rather simple electromagnetic waves. However, the wave or particle mature of light, already discussed in the 17th century by Huygens and Newton, was still under discussion. There were valid arguments for both wave (Huygens' favorite that explains diffraction and Young's double slit experiment) and particle (Newton's favorite that could explain light's straight propagation and refraction, among other things) nature of light. Because Newton was such a prominent figure in physics his opinion in favor of the particle nature of light had a lasting impact. Planck's theory for black-body radiation was published 1901 followed by Einstein's interpretation of light as quantized electromagnetic radiation in 1905 that made a new quantum mechanical description of light necessary. The name photon was introduced in 1928 by Arthur Compton, it is derived from the Greek word $\phi\omega\sigma$ for light. Dirac gave the first quantum mechanical solution for the interaction between atoms and light fields in 1927 and shortly after, Fermi gave a complete review of quantum electrodynamics in 1932.

In quantum electrodynamics, the electromagnetic field is quantized and the photon is the smallest quantum of light. A complete derivation of the quantization of the light field can be found in several textbooks [Loudon1983, Fox2006]. We focus on photon statistics where the quantum description of light is required to fully classify different states of light and give a brief introduction to the quantization of the electromagnetic field. In simplified terms, we can replace the field amplitudes in the classical electrodynamics description with bosonic creation $\hat{a}^+$ and annihilation operators $\hat{a}$ that can add or remove a photon. Each mode of the light field (k, λ) can be described independently by a harmonic oscillator:

we have an energy ladder with constant energy spacings whatever the number of photons already present. The consecutive operation of the annihilation and creation operator is equal to the operation of a new quantum mechanical operator n, the photon number operator which gives the number of photons in one mode: $n = \hat{a}^{+}\hat{a}$. The quantum mechanical approach allows to sum up all modes as independent quantum mechanical harmonic oscillators. The Hamiltonian of the electromagnetic field $\hat{H}_{em}$ for an arbitrary number of modes is given by:

$$\hat{H}_{\text{em}} = \sum_{k}\sum_{\lambda}\frac{1}{2}\hbar\omega_k\left(\hat{a}_{k\lambda}\hat{a}^{\dagger}_{k\lambda} + \hat{a}^{\dagger}_{k\lambda}\hat{a}_{k\lambda}\right)$$

Where for each mode k we simply take the number of photons with wavelength λ to obtain the total amount of energy. If we consider only the fundamental mode, we can rewrite the Hamiltonian with the help of the photon number operator:

$$\hat{H}_{\text{em}} = \hbar\omega\left(\hat{n} + \frac{1}{2}\right)$$

Just like in the quantum mechanical harmonic oscillator, we have a ground state with finite energy, called the vacuum state given by the term 1/2. The photon number operator n gives the number of photons with energy $\hbar\omega$ in the fundamental mode. The mean photon number of a mode $< n >$ is an important figure for the characterization of the light states. Together with the photon number variance $(\Delta n)^2$, we will identify different light states and use them to categorize light sources.

3.0.1 Number of Photons in a given Volume

To give an intuitive introduction to photon statistics, we consider a simple experiment taken from [Fox2006]. A light source emits photons that are detected by an ideal detector as shown in the figure below. The detector is sensitive down to the single photon level and produces a short electrical pulse in response to every single photon impinging the detector. Counting electronics records the number of electrical pulses and their arrival times within a defined time interval T. We will go deeper with single photon detection technology later.

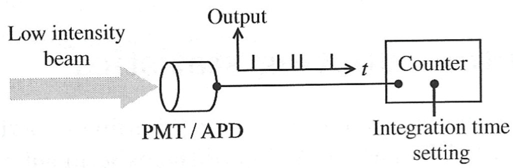

Figure 3.1: A single photon detector turns a stream of photons into electrical pulses that can be recorded and analyzed.

We start with a perfectly monochromatic light beam of frequency ω and constant intensity I and calculate the mean number of photons $< n(T) >$ passing through a cross section A of the beam in a given time interval T, set by the counting electronics:

$$< n(T) >= IAT/(\hbar\omega) = PT/(\hbar\omega)$$

where P is the power of the light beam, this equation represents the average properties of the beam and any light source will have fluctuations at short time scales. These fluctuations or differences in statistics of the photon numbers are used to characterize different light sources. Coming back to our example, we can calculate the constant photon flux Φ, i.e. the mean number of photons in a given cross-section of a beam of light with photon energy 1.0 eV and average power of 1 nW:

$$\Phi = (< n(T) >)/T = P/(\hbar\omega) = 10^{-9}/(1.6.10^{-19}) = 6.2.10^9 photons/s$$

Light travels at approximately 3.10^8 m/s, the photons are quite spread in space during this one second: We have a beam segment with a length of 3.10^8 m that contains $6.2.10^9$ photons. If we now reduce the volume we are interested, we only consider one meter instead of $3.10^8 m$ we have on average 20.67 photons in that volume. Since photons are the smallest quantum of light and are discrete, 20.67 mean photons does not make sense physically, instead there will be fluctuations in the number of photons and these fluctuations in mean photon numbers will increase the smaller we make the volume we are looking at. Similarly, one can also look at shorter and shorter time windows. Let us look at a segment of 1.5 m which on average contains 31 photons. We will divide the 1.5 m in 31 sub-segments. There are different options how the photons will be distributed within these sub-segments, here are three possible outcomes:

0 0 2 1 0 4 0 3 1 0 1 0 2 0 2 0 1 0 0 2 1 0 3 0 0 5 0 1 1 0 1
0 1 0 0 2 1 0 2 1 4 0 2 1 0 1 0 0 1 1 0 1 2 1 1 0 2 1 3 0 2 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

Even though in all 3 cases the mean photon number $< n >$ for each sub-segment is 1, the distribution function for each case is fundamentally different.

In the first row it is super-Poissonian, in the second row it is Poissonian, and in the third row it is sub-Poissonian. We will now define these distributions.

3.1 Coherent state

A coherent state, or a Glauber state, describes the electromagnetic wave of a laser mode. In 1963, Roy Glauber provided a complete quantum mechanical description of these light states and got a Nobel prize for it. The coherent state is an eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}_i \left|\alpha_i\right\rangle = \alpha_i \left|\alpha_i\right\rangle$$

Being an eigenstate of $\hat{a}$, a coherent state *remains unchanged by the annihilation of a photon.* Additionally, since the vacuum state can be written as an eigenstate of the annihilation operator with $\alpha = 0$, all coherent states have the same minimal uncertainty as the vacuum state. Linear superposition of these states allows for an expression of the coherent state in the basis of the photon number operator n :

$$\left|\alpha_i\right\rangle = e^{-\frac{1}{2}|\alpha_i|^2} \sum_{n=0}^{\infty} \frac{\alpha_i^2}{\sqrt{n!}} \left|n_i\right\rangle$$

We can calculate the probability distribution P for n photons in a given mode i.

$$\begin{aligned} P_{coherent}(n) &= \left|\left\langle n \mid \alpha_i\right\rangle\right|^2 \\ &= \left|\exp\left(-\tfrac{1}{2}\left|\alpha_i\right|^2\right) \cdot \sum_m \frac{\alpha_i^m}{\sqrt{m!}} \left\langle n \mid m\right\rangle\right|^2 \\ &= \exp\left(-\left|\alpha_i\right|^2\right) \cdot \frac{\left|\alpha_i\right|^{2n}}{n!} \end{aligned}$$

This is the characteristic *Poisson statistics* used to describe coherent light states. The expected value of the photon number in one mode of a coherent state is therefore:

$$\langle n\rangle = \langle\alpha|\,\hat{n}\,|\alpha\rangle = |\alpha|^2$$

The variance of a coherent state is given as:

$$(\Delta n)^2_{\text{Glauber}} = \left\langle n^2\right\rangle - \langle n\rangle^2 = |\alpha|^2 = \langle n\rangle$$

For a coherent state the maximum probability to find n photons in a mode is at the expected value $<n>$. The probability distribution of the photon number obeys a Poisson distribution. Any light state with larger (smaller) variance is called super (sub)- Poissonian light. The photon number distribution for a Glauber state is plotted below for 2 different mean photon numbers. We see that for a mean photon number of 1, we have as many 0 as 1 photon states and a rapidly decreasing probability of states with higher photon numbers. for a mean photon number of 5, the probability distribution is wider.

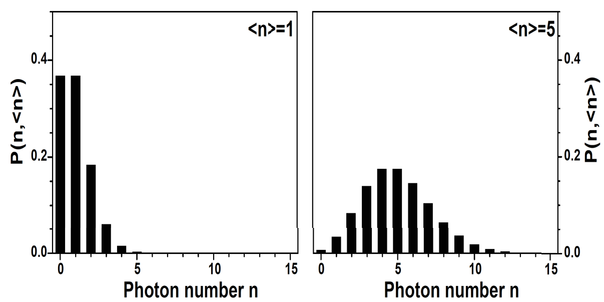

Figure 3.2: Photon number distribution for a coherent state with average photon number of 1 (left) and 5 (right).

3.2 Fock state

The Fock or photon number state, introduced by Vladimir Fock, results directly from the quantization of the electromagnetic field, since the Fock state is the eigenstate of the photon number operator n_i:

$$\hat{n}_i |n_i\rangle = n_i |n_i\rangle$$

The eigenvalue n_i of the photon number operator describes the number of photons in a specific mode i. The probability $P_{Fock(n)}$ to find n_i photons in one mode is either 1 for $n = n_i$ or 0 for $n \neq n_i$, this is a very simnple and very clean state! A special characteristic of the Fock state: the photon number is precisely determined. Thus, the probability distribution of the photon number follows a δ-distribution. The expected value of the photon number in a Fock state is equal to the number of photons in the state:

$$\langle n\rangle = \langle n_i| \hat{n} |n_i\rangle = n_i$$

For the Fock state the variance is therefore zero, there is no variance only one number state is populated:

$$(\Delta n)^2_{\text{Fock}} = \left\langle n^2 \right\rangle - \langle n \rangle^2 = 0$$

The Fock state fulfills the inequality $\Delta n < \sqrt{n}$, showing a variance smaller than the coherent state. Such sub-Poisson statistics cannot be described by classical electromagnetic theory; thus such light is classified as non-classical light. The figure below shows the photon number distribution for two Fock states. Light emitters with a Fock state $n = 1$ are called single photon sources, since they can only emit one single photon at a time.

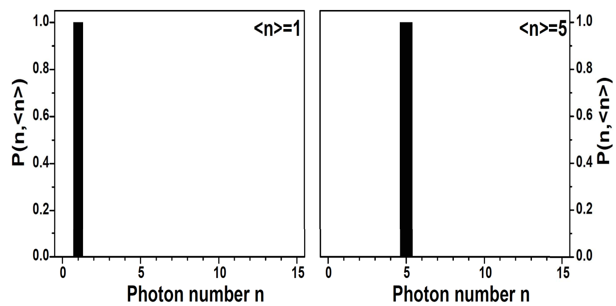

Figure 3.3: Photon number distribution for a Fock state with average photon number of 1 (left) and 5 (right). It can't get simpler than that: a Fock state with average photon number 1 only has single photon states and a Fock state with average photon number 5 only has 5 photon states.

3.3 Thermal state

The thermal state which is well described by the black-body radiation, it is an incoherent mixture of different photon number states. A quantum mechanical description of the thermal state takes advantage of the density matrix notation. The thermal state, being a quantum mechanical mixed state, can be written as the sum over all possible photon number states weighted with their occurrence probability:

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n(n) \left| n \right\rangle \left\langle n \right|$$

where ρ is the density matrix operator and $P_n(n)$ gives the probability of finding n photons in a certain mode i of the thermal state. This is identical to the probability of having a certain photon number state occupied. $P(n)$ can be expressed as a function of n and $<n>$ for a single mode:

$$P(n, \langle n \rangle) = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}}$$

$P(n, <n>)$ has the form of a Bose-Einstein distribution; the state with maximum probability is always the vacuum state with $n = 0$. The variance for a thermal state is given by:

$$(\Delta n)^2_{\text{Thermal}} = \langle n \rangle^2 + \langle n \rangle$$

It follows that fluctuations in photon number are typically larger than the mean photon number. Therefore, thermal light states are also called chaotic light. Since $\Delta n > \sqrt{n}$ one often describes *thermal state statistics as super-Poissonian statistics*. The thermal state distribution for 2 different mean photon numbers is plotted below. Finding zero photons in the mode always has the highest probability.

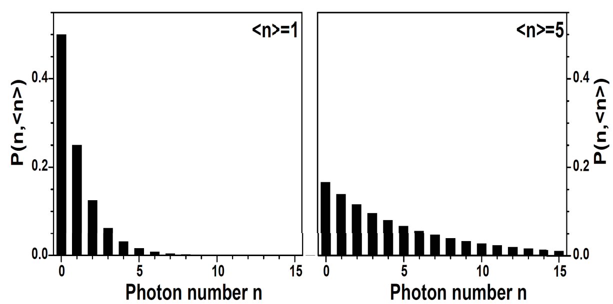

Figure 3.4: Photon number distribution for a thermal state with average photon number of 1 (left) and 5 (right).

3.4 Measuring light statistics

We have now seen the three possible photon statistics. But important questions remain: how do we measure photon statistics? How does one check in the lab that a light source is a single photon light source? We have seen that different light states are defined by their underlying photon statistics, photon probability distribution function and in the fluctuations of the photon numbers. With the help of the second-order correlation function $g^{(2)}(\tau)$ introduced by Glauber in 1963, we can classify the different light states. First, we introduce the classical second-order intensity correlation function:

$$g^{(2)}_{class.}(\tau) = \frac{\langle I(t) I(t+\tau) \rangle}{\langle I(t) \rangle^2} = \frac{\langle E^*(t) E^*(t+\tau) E(t+\tau) E(t) \rangle}{\langle E^*(t) E(t) \rangle^2}$$

with $I|E(t)|$ and $I(t+)$ being the averaged intensities of the mode at a given time. Based on this definition the $g^{(2)}(\tau)$ function describes the correlation between two temporally separated intensity signals with time difference $\tau = t_2 - t_1$ from one light source. Using the transformation formalism from classical field quantities into equivalent quantum mechanical operators using the second quantization, we can rewrite the electric field $E(t)$ of a mode k with the help of annihilation $\hat{a}$ and creation $\hat{a}^+$ operators:

$$\hat{E}_k(t) = \hat{E}_k^{(+)}(t) + \hat{E}_k^{(-)}(t)$$

with

$$\begin{aligned} \hat{E}_k^{(+)}(t) &\propto \hat{a}_k \cdot \exp\left(-i\left(\omega_k t - \vec{k}\cdot\vec{r}\right)\right) \\ \hat{E}_k^{(-)}(t) &\propto (\hat{a}_k)^\dagger \cdot \exp\left(+i\left(\omega_k t - \vec{k}\cdot\vec{r}\right)\right) \end{aligned}$$

representing the 'positive' and 'negative' ωk frequency parts of the mode. For a single mode we can rewrite the $g^{(2)}(\tau)$ function using the commutator relation.

$$\begin{aligned} g^{(2)}_{QM}(\tau) &= \frac{\left\langle \hat{E}_k^{(-)}(t)\,\hat{E}_k^{(-)}(t+\tau)\,\hat{E}_k^{(+)}(t+\tau)\,\hat{E}_k^{(+)}(t)\right\rangle}{\left\langle \hat{E}_k^{(-)}(t)\,\hat{E}_k^{(+)}(t)\right\rangle^2} \\ &\overset{(\tau\to 0)}{=} \frac{\left\langle (\hat{a}_k)^\dagger(\hat{a}_k)^\dagger\hat{a}_k\hat{a}_k\right\rangle}{\left\langle(\hat{a}_k)^\dagger\hat{a}_k\right\rangle^2} = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2} \quad . \end{aligned}$$

Of particular interest is $g^{(2)}(0)$, since it represents the conditional probability how likely is it to detect a second photon at the same time one photon was already detected, it is a measure of the temporal photon coincidences, required to distinguish between different light states. Using the second factorial moment and the variance we can simplify the equation:

$$\begin{aligned} g^{(2)}_{QM}(0) &= \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} \\ &= \frac{(\Delta n)^2 + \langle n\rangle^2 - \langle n\rangle}{\langle n\rangle^2} \\ &= 1 + \frac{(\Delta n)^2 - \langle n\rangle}{\langle n\rangle^2} \quad . \end{aligned}$$

With the given variance of the different light states, we can now calculate the $g^{(2)}(0)$ value for the three different light states:

$$(\Delta n)^2_{thermal} = \langle n^2 \rangle + \langle n \rangle \quad \Rightarrow \quad g^{(2)}_{QM}(0) = 2$$

$$(\Delta n)^2_{coherent} = \langle n \rangle \quad \Rightarrow \quad g^{(2)}_{QM}(0) = 1$$

$$(\Delta n)^2_{Fock} = 0 \quad \Rightarrow \quad g^{(2)}_{QM}(0) = 1 - \frac{1}{n} \quad (n \geq 1)$$

$$\Rightarrow \quad g^{(2)}_{QM}(0) = 0 \quad (n = 0)$$

Since in the Glauber state photon emission is completely uncorrelated, the $g^{(2)}(\tau)$ function is unity for all delay times τ , given an infinite coherence time of the state. As we can see from the equations above, the thermal state has a higher probability to emit more than one photon at the same time. However, this happens only in time periods shorter than the coherence time, which is typically very short for thermal/chaotic light. This effect is called *photon bunching.* In contrast, Fock states give $g_F ock^{(}2))(0) < 1$, leading to a reduced probability to emit two photons at the same time. This effect is called *photon antibunching.*

The graph below depicts the $g^{(2)}(\tau)$ function for three light states: thermal, coherent, and Fock state with a photon number of $n = 1$. If in this state a single photon is annihilated (e.g. detected), there is no photon left and no second photon can be detected. Fock states are therefore called non-classical light states and the first demonstration of photon antibunching by Kimble et al in 1977 was proof of the non-classical nature of light. Photon sources with $n = 1$ are single photon sources and important for the realization of different applications in photonic quantum technologies such as quantum key distribution.

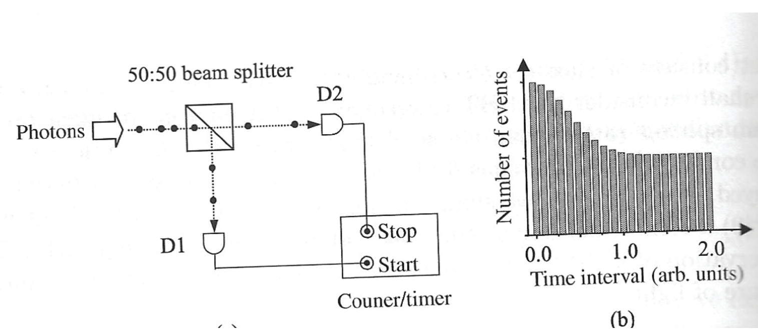

Figure 3.5: To identify a single photon emitter, a Hanbury-Brown Twiss setup (left) can be used where a 50-50 beamsplitter divides the incoming photon flux into two beams. Single photon detectors are placed at both outputs and correlations between the two detectors are measured. A histogram of the time intervals between the detection events can then be plotted (right).

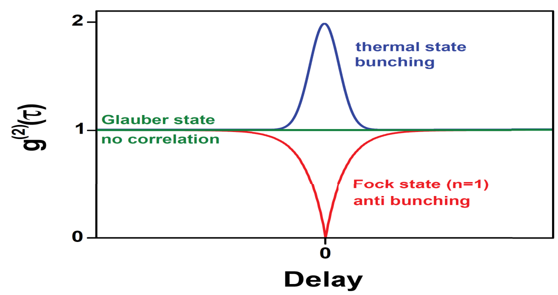

Figure 3.6: Schematic of the $g^{(2)}$ function for the three possible light statistics. The value around zero delay carries crucial information.