1. (a) The objective function is $f(x) = x_1^2 - 2x_1x_3 + e^{x_2} + x_2x_3 + x_3^2 - x_1 - 2x_2 - x_3$. Differentiation gives

$$
\nabla f(x) = \begin{pmatrix}
2x_1 - 2x_3 - 1 \\
e^{x_2} + x_3 - 2 \\
-2x_1 + x_2 + 2x_3 - 1
\end{pmatrix},
\nabla^2 f(x) = \begin{pmatrix}
2 & 0 & -2 \\
0 & e^{x_2} & 1 \\
-2 & 1 & 2
\end{pmatrix}.
$$

In particular, $\nabla f(\tilde{x}) = (-1 \ 0 \ -1)^T$. With $g_1(x) = -x_1^2 - x_2^2 - x_3^2 + 5$ we get $g_1(\tilde{x}) = 3$, which mean that constraint 1 is not active at $\tilde{x}$. Since $\nabla f(\tilde{x}) \neq 0$, constraint 2 must be active for $\tilde{x}$ to possibly satisfy the first-order necessary optimality conditions. These conditions require the existence of a $\tilde{\lambda}_2$ such that $\nabla f(\tilde{x}) = a\tilde{\lambda}_2$ and $a^T\tilde{x} + 2 = 0$ with $\tilde{\lambda}_2 \geq 0$.

The condition $\nabla f(\tilde{x}) = a\tilde{\lambda}_2$ takes the form

$$
\begin{pmatrix}
-1 \\
0 \\
-1
\end{pmatrix} = \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} \tilde{\lambda}_2,
$$

and it can not be fulfilled with $\tilde{\lambda}_2 = 0$. Hence, $\tilde{\lambda}_2 > 0$, and we obtain $a_1 = -1/\tilde{\lambda}_2$, $a_2 = 0$ and $a_3 = -1/\tilde{\lambda}_2$. The condition $-2/\tilde{\lambda}_2 + 2 = 0$ so that $\tilde{\lambda}_2 = 1$. Hence, $a = (-1 \ 0 \ -1)^T$.

If $a = (-1 \ 0 \ -1)^T$, then $\tilde{x}$ fulfills the first-order necessary optimality conditions together with $\tilde{\lambda} = (0 \ 1)^T$.

(b) As we only have one active linear constraint at $\tilde{x}$ we obtain

$$
\nabla^2_\lambda \mathcal{L}(\tilde{x}, \tilde{\lambda}) = \nabla^2 f(\tilde{x}) = \begin{pmatrix}
2 & 0 & -2 \\
0 & 1 & 1 \\
-2 & 1 & 2
\end{pmatrix}.
$$

Since $\tilde{\lambda}_2 > 0$, we also have that $A_+(\tilde{x}) = a^T$, where we can let $a^T = (N \ B)$ for $B = -1$ and $N = (-1 \ 0)$. We then obtain a matrix whose columns form a basis for the null space of $A_+(\tilde{x})$ as

$$
Z_+(\tilde{x}) = \begin{pmatrix}
I \\
-B^{-1}N
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix},
$$

which gives

$$
Z_+(\tilde{x})^T \nabla^2 f(\tilde{x}) Z_+(\tilde{x}) = \begin{pmatrix}
8 & -1 \\
-1 & 1
\end{pmatrix},
$$

which is a positive definite matrix. Hence, $\tilde{x}$ fulfills the second-order sufficient optimality conditions and is therefore a local minimizer.

2. We have

$$
f(x) = x_1^2 + \frac{1}{2}(x_2 - 1)^2 + x_1x_2,
$$

$$
g(x) = 2 - \frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}x_2^2 \geq 0, \ h(x) = 2x_1 + x_2,
$$

$\text{Brief solutions}$

$\text{Thursday May 28 2020 8.00–13.00}$
\[ \nabla f(x) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 - 1 \end{pmatrix}, \quad \nabla g(x) = \begin{pmatrix} -(x_1 - 1) \\ -x_2 \end{pmatrix}, \quad \nabla h(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \]
\[ \nabla^2 f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \nabla^2 g(x) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \nabla^2 h(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

(a) The point \( x^{(0)} \) is not feasible.

(b) Insertion of numerical values in the expressions above gives the first QP-problem according to

\[
\begin{align*}
\min & \quad \frac{1}{2} \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
\text{subject to} & \quad 2p_1 - 2p_2 \geq 2 \\
& \quad 2p_1 + p_2 = 0
\end{align*}
\]

This is a convex QP-problem with a globally optimal solution given by

\[
\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \lambda_1(2p_1 - 2p_2 - 2) = 0, \lambda_1 \geq 0.
\]

If (2) is not active, \( \lambda_1 = 0 \) and \( 2p_1 - 2p_2 > 2 \). Then, we have

\[
\begin{align*}
4p_1 + p_2 &= 2\lambda_2 \\
p_1 + 3p_2 &= \lambda_2 \\
2p_1 + p_2 &= 0
\end{align*}
\]

The solution is given by \( \lambda_2 = 0, p_1 = p_2 = 0 \), which is in contradiction with \( 2p_1 - 2p_2 > 2 \). Then, (2) is active and \( \lambda_1 \geq 0 \). We have

\[
\begin{align*}
4p_1 + p_2 &= 2\lambda_1 + 2\lambda_2 \\
p_1 + 3p_2 &= -2\lambda_1 + \lambda_2 \\
2p_1 - 2p_2 &= 2 \\
2p_1 + p_2 &= 0
\end{align*}
\]

The solution is given by \( p_1 = 1/3, p_2 = -2/3 \) and \( \lambda_1 = 2/3, \lambda_2 = -1/3 \). Hence,

\[ x^{(1)} = x^{(0)} + p = \begin{pmatrix} -2/3 \\ 4/3 \end{pmatrix}, \quad \lambda^{(1)} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}. \]

3. Constraint 2 are in the working set at the initial point, i.e., \( \mathcal{W} = \{2\} \). With \( H = I \) and \( c = 0 \) we obtain

\[
\begin{pmatrix} H & A^T_W \\
A_W & 0 \end{pmatrix} \begin{pmatrix} p^{(0)}_W \\ -\lambda^{(0)}_W \end{pmatrix} = - \begin{pmatrix} Hx^{(0)} + c \end{pmatrix}. \]
Insertion of numeric values gives

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
p_1^{(0)} \\
p_2^{(0)} \\
-\lambda_2^{(1)}
\end{pmatrix}
= -
\begin{pmatrix}
6 \\
2 \\
0
\end{pmatrix}
\]

We obtain

\[
p^{(0)} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}^T, \quad \lambda^{(1)} = \begin{pmatrix} 0 & -2 & 0 \end{pmatrix}^T.
\]

The maximum steplength is given by

\[
\alpha_{\text{max}} = \min_{i: a_i^T \lambda^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T \lambda^{(0)}} = \frac{1}{4},
\]

where the minimum is attained for \( i = 1 \). Consequently, \( \alpha^{(0)} = \frac{1}{4} \) so that

\[
x^{(1)} = x^{(0)} + p^{(0)} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} -4 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix},
\]

with \( \mathcal{W} = \{1, 2\} \). The solution to the corresponding quadratic program is given by

\[
\begin{pmatrix}
1 & 0 & 1 & -1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p_1^{(1)} \\
p_2^{(1)} \\
-\lambda_1^{(2)} \\
-\lambda_2^{(2)}
\end{pmatrix}
= -
\begin{pmatrix}
5 \\
1 \\
0 \\
0
\end{pmatrix}
\]

We obtain

\[
p^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(2)} = \begin{pmatrix} 3 & -2 & 0 \end{pmatrix}^T.
\]

As \( p^{(1)} = 0 \), it follows that \( x^{(2)} = x^{(1)} \) and the corresponding equality-constrained problem has been solved. However, since \( \lambda_2^{(2)} < 0 \), constraint 2 is deleted so that \( \mathcal{W} = \{1\} \). The solution to the corresponding equality-constrained quadratic program is given by

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
p_1^{(2)} \\
p_2^{(2)} \\
-\lambda_1^{(3)}
\end{pmatrix}
= -
\begin{pmatrix}
5 \\
1 \\
0
\end{pmatrix}
\]

We obtain

\[
p^{(2)} = \begin{pmatrix} -2 & 2 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T.
\]

The maximum steplength is given by

\[
\alpha_{\text{max}} = \min_{i: a_i^T x^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T \lambda^{(0)}} = \frac{3}{2},
\]
where the minimum is attained for \(i = 3\). Since \(\alpha_{\text{max}} > 1\), we let \(\alpha^{(2)} = 1\) so that
\[
x^{(3)} = x^{(2)} + p^{(2)} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.
\]

Since \(\lambda^{(3)} \geq 0\), the optimal solution has been found.

4. (See the course material.)

5. (a) Since the feasible set \(F\) is convex, to show the convexity of \((NLP)\), we only need to show the convexity of function \(x_2^2 x_1\). The Hessian of \(x_2^2 x_1\) is given by
\[
H = \begin{pmatrix}
\frac{2 x_2^2}{x_1^3} & -\frac{2 x_2^3}{x_1^4} \\
-\frac{2 x_2^3}{x_1^4} & \frac{2}{x_1^3}
\end{pmatrix}
\]
If we let \(A = 2/x_1\), \(B = -2x_2/x_1^2\) and \(C = 2x_2^2/2x_1^3\) we obtain \(A > 0\) and \(C - B^2/A = 0\) if \(x_1 > 0\). Hence, Hint 2 shows that \(H \succeq 0\) for \(x_1 > 0\), which means \(x_2^2/x_1\) is convex. Thus, \(f\) is convex on the convex feasible region, so that \((NLP)\) is a convex problem.

(b) Problem \((NLP)\) is equivalent to \((NLP')\) given by
\[
(NLP') \quad \begin{align*}
\text{minimize} & \quad x_1 + 2x_2 + x_3 \\
\text{subject to} & \quad x_3 \geq \frac{x_2^2}{x_1}, \\
& \quad x \in F,
\end{align*}
\]

since \(x_3 = x_2^2/x_1\) must hold at any optimal solution. For \(x_1 > 0\) we obtain
\[
x_3 \geq \frac{x_2^2}{x_1} \iff x_3 - \frac{x_2^2}{x_1} \geq 0 \iff \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \succeq 0,
\]

where Hint 2 has been used in the last step with \(A = x_1\), \(B = x_2\) and \(C = x_3\). Hence, \((NLP')\) is equivalent to the semidefinite program
\[
\begin{align*}
\text{minimize} & \quad x_1 + 2x_2 + x_3 \\
\text{subject to} & \quad \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \succeq 0, \\
& \quad x \in F.
\end{align*}
\]

(c) The semidefinite programs in \((5b)\) can be rewritten as
\[
(SDP') \quad \begin{align*}
\text{minimize} & \quad \text{trace}(CX) \\
\text{subject to} & \quad \text{trace}(AX) = b, \\
& \quad X = X^T \succeq 0,
\end{align*}
\]

where \(X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}\), \(C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\), \(A = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\) and \(b = 1\). The equivalence comes from the fact that the constraints in \((SDP')\) implies \(x_1 > 0\).
Otherwise, if $x_1 = 0$, we have $x_2 = \frac{1}{2}$. Then, $X = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & x_3 \end{pmatrix}$, which is not a positive semidefinite matrix for any $x_3$.

Then, the dual problem of the semidefinite program in (5b) can be formulated as

$$\begin{align*}
\text{maximize} & \quad b y \\
\text{subject to} & \quad A y \preceq C.
\end{align*}$$