SF2822 Applied nonlinear optimization, final exam
Thursday May 28 2020 8.00–13.00
Brief solutions

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1. (a) The objective function is $f(x) = x_1^2 - 2x_1x_3 + e^{x_2} + x_2x_3 + x_3^2 - x_1 - 2x_2 - x_3$. Differentiation gives

$$\nabla f(x) = \begin{pmatrix} 2x_1 - 2x_3 - 1 \\ e^{x_2} + x_3 - 2 \\ -2x_1 + x_2 + 2x_3 - 1 \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} 2 & 0 & -2 \\ 0 & e^{x_2} & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

In particular, $\nabla f(\tilde{x}) = (-1\ 0\ -1)^T$. With $g_1(x) = -x_1^2 - x_2^2 - x_3^2 + 5$ we get $g_1(\tilde{x}) = 3$, which mean that constraint 1 is not active at $\tilde{x}$. Since $\nabla f(\tilde{x}) \neq 0$, constraint 2 must be active for $\tilde{x}$ to possibly satisfy the first-order necessary optimality conditions. These conditions require the existence of a $\tilde{\lambda}_2$ such that $\nabla f(\tilde{x}) = a\tilde{\lambda}_2$ and $a^T\tilde{x} + 2 = 0$ with $\tilde{\lambda}_2 \geq 0$.
The condition $\nabla f(\tilde{x}) = a\tilde{\lambda}_2$ takes the form

$$\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tilde{\lambda}_2.$$

and it can not be fulfilled with $\tilde{\lambda}_2 = 0$. Hence, $\tilde{\lambda}_2 > 0$, and we obtain $a_1 = -1/\tilde{\lambda}_2$, $a_2 = 0$ and $a_3 = -1/\tilde{\lambda}_2$. The condition $-2/\tilde{\lambda}_2 + 2 = 0$ so that $\tilde{\lambda}_2 = 1$. Hence, $a = (-1\ \ 0\ -1)^T$.
If $a = (-1\ \ 0\ -1)^T$, then $\tilde{x}$ fulfils the first-order necessary optimality conditions together with $\tilde{\lambda} = (0\ 1)^T$.

(b) As we only have one active linear constraint at $\tilde{x}$ we obtain

$$\nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) = \nabla^2 f(\tilde{x}) = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix}.$$

Since $\tilde{\lambda}_2 > 0$, we also have that $A_+(\tilde{x}) = a^T$, where we can let $a^T = (N\ \ B)$ for $B = -1$ and $N = (-1\ \ 0)$. We then obtain a matrix whose columns form a basis for the null space of $A_+(\tilde{x})$ as

$$Z_+(\tilde{x}) = \begin{pmatrix} I \\ -B^{-1}N \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which gives

$$Z_+(\tilde{x})^T\nabla^2 f(\tilde{x})Z_+(\tilde{x}) = \begin{pmatrix} 8 & -1 \\ -1 & 1 \end{pmatrix},$$

which is a positive definite matrix. Hence, $\tilde{x}$ fulfils the second-order sufficient optimality conditions and is therefore a local minimizer.

2. We have

$$f(x) = x_1^2 + \frac{1}{2}(x_2 - 1)^2 + x_1x_2,$$
$$g(x) = 2 - \frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}x_2^2 \geq 0, \; h(x) = 2x_1 + x_2,$$

$$\nabla f(x) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 - 1 \end{pmatrix}, \ \nabla g(x) = \begin{pmatrix} -(x_1 - 1) \\ -x_2 \end{pmatrix}, \ \nabla h(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$
$$\nabla^2 f(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ \nabla^2 h(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(a) The point $x^{(0)}$ is not feasible.

(b) Insertion of numerical values in the expressions above gives the first QP-problem according to

$$\begin{array}{ll} \min & \frac{1}{2} \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ \text{subject to} & 2p_1 - 2p_2 \geq 2 \\ & 2p_1 + p_2 = 0 \end{array}$$

This is a convex QP-problem with a globally optimal solution given by

$$\begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{1}$$
$$2p_1 - 2p_2 \geq 2 \tag{2}$$
$$2p_1 + p_2 = 0 \tag{3}$$
$$\lambda_1(2p_1 - 2p_2 - 2) = 0, \lambda_1 \geq 0. \tag{4}$$

If (2) is not active. $\lambda_1 = 0$ and $2p_1 - 2p_2 > 2$. Then, we have

$$\begin{aligned} 4p_1 + p_2 &= 2\lambda_2 \\ p_1 + 3p_2 &= \lambda_2 \\ 2p_1 + p_2 &= 0 \end{aligned}$$

The solution is given by $\lambda_2 = 0$, $p_1 = p_2 = 0$, which is in contradiction with $2p_1 - 2p_2 > 2$. Then, (2) is active and $\lambda_1 \geq 0$. We have

$$\begin{aligned} 4p_1 + p_2 &= 2\lambda_1 + 2\lambda_2 \\ p_1 + 3p_2 &= -2\lambda_1 + \lambda_2 \\ 2p_1 - 2p_2 &= 2 \\ 2p_1 + p_2 &= 0 \end{aligned}$$

The solution is given by $p_1 = 1/3$, $p_2 = -2/3$ and $\lambda_1 = 2/3$, $\lambda_2 = -1/3$. Hence,

$$x^{(1)} = x^{(0)} + p = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{pmatrix}, \quad \lambda^{(1)} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

3. Constraint 2 are in the working set at the initial point, i.e., $\mathcal{W} = \{2\}$. With $H = I$ and $c = 0$ we obtain

$$\begin{pmatrix} H & A_{\mathcal{W}}^T \\ A_{\mathcal{W}} & 0 \end{pmatrix} \begin{pmatrix} p^{(0)} \\ -\lambda_{\mathcal{W}}^{(0)} \end{pmatrix} = -\begin{pmatrix} Hx^{(0)} + c \\ 0 \end{pmatrix}.$$

Insertion of numeric values gives

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(0)} \\ p_2^{(0)} \\ -\lambda_2^{(1)} \end{pmatrix} = - \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

We obtain

$$p^{(0)} = \begin{pmatrix} -4 & -4 \end{pmatrix}^T, \quad \lambda^{(1)} = \begin{pmatrix} 0 & -2 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i:a_i^T p^{(0)}<0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{1}{4},$$

where the minimium is attained for $i = 1$. Consequently, $\alpha^{(0)} = \frac{1}{4}$ so that

$$x^{(1)} = x^{(0)} + p^{(0)} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} -4 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix},$$

with $\mathcal{W} = \{1, 2\}$. The solution to the corresponding quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ -\lambda_1^{(2)} \\ -\lambda_2^{(2)} \end{pmatrix} = - \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We obtain

$$p^{(1)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(2)} = \begin{pmatrix} 3 & -2 & 0 \end{pmatrix}^T.$$

As $p^{(1)} = 0$, it follows that $x^{(2)} = x^{(1)}$ and the corresponding equality-constrained problem has been solved. However, since $\lambda_2^{(2)} < 0$, constraint 2 is deleted so that $\mathcal{W} = \{1\}$. The solution to the corresponding equality-constrained quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ -\lambda_1^{(3)} \end{pmatrix} = - \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(2)} = \begin{pmatrix} -2 & 2 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i:a_i^T p^{(2)}<0} \frac{a_i^T x^{(2)} - b_i}{-a_i^T p^{(2)}} = \frac{3}{2},$$

where the minimium is attained for $i = 3$. Since $\alpha_{\max} > 1$, we let $\alpha^{(2)} = 1$ so that

$$x^{(3)} = x^{(2)} + p^{(2)} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Since $\lambda^{(3)} \geq 0$, the optimal solution has been found.

4. (See the course material.)

5. (a) Since the feasible set F is convex, to show the convexity of (NLP), we only need to show the convexity of function $\frac{x_2^2}{x_1}$. The Hessian of $\frac{x_2^2}{x_1}$ is given by

$$H = \begin{pmatrix} \frac{2x_2^2}{x_1^3} & -\frac{2x_2}{x_1^2} \\ -\frac{2x_2}{x_1^2} & \frac{2}{x_1} \end{pmatrix}$$

If we let $A = 2/x_1$, $B = -2x_2/x_1^2$ and $C = 2x_2^2/x_1^3$ we obtain $A > 0$ and $C - B^2/A = 0$ if $x_1 > 0$. Hence, Hint 2 shows that $H \succeq 0$ for $x_1 > 0$, which means $\frac{x_2^2}{x_1}$ is convex. Thus, f is convex on the convex feasible region, so that (NLP) is a convex problem.

(b) Problem (NLP) is equivalent to (NLP') given by

$$(NLP') \quad \begin{array}{ll} \text{minimize} & x_1 + 2x_2 + x_3 \\ \text{subject to} & x_3 \geq \frac{x_2^2}{x_1}, \\ & x \in F, \end{array}$$

since $x_3 = x_2^2/x_1$ must hold at any optimal solution. For $x_1 > 0$ we obtain

$$x_3 \geq \frac{x_2^2}{x_1} \quad \Leftrightarrow x_3 - \frac{x_2^2}{x_1} \geq 0 \quad \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0,$$

where Hint 2 has been used in the last step with $A = x_1$, $B = x_2$ and $C = x_3$. Hence, (NLP') is equivalent to the semidefinite program

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_2 + x_3 \\ \text{subject to} & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0, \\ & x \in F. \end{array}$$

(c) The semidefinite programs in (5b) can be rewritten as

$$(SDP') \quad \begin{array}{ll} \underset{X}{\text{minimize}} & \text{trace}(CX) \\ \text{subject to} & \text{trace}(AX) = b, \\ & X = X^T \succeq 0, \end{array}$$

where $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = 1$. The equivalence comes from the fact that the constraints in (SDP') implies $x_1 > 0$.

Otherwise, if $x_1 = 0$, we have $x_2 = \frac{1}{2}$. Then, $X = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & x_3 \end{pmatrix}$, which is not a positive semidefinite matrix for any x_3.

Then, the dual problem of the semidefinite program in (5b) can be formulated as

$$\begin{array}{ll} \underset{y}{\text{maximize}} & by \\ \text{subject to} & Ay \preceq C. \end{array}$$