SF2822 Applied nonlinear optimization, final exam
Wednesday August 12 2020 8.00–13.00
Brief solutions

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1. Constraint 3 is in the working set at the initial point, i.e., $\mathcal{W} = \{3\}$.

Iteration 1: We get $\mathcal{W}^{(0)} = \{3\}$, and solution to the corresponding quadratic program is given by

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(0)} \\ p_2^{(0)} \\ p_3^{(0)} \\ -\lambda_3^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \\ 0 \end{pmatrix}.$$

This system of equations give that $p^{(0)} = (\frac{1}{2} \; -1 \; 0)^T$ and $\lambda_3^{(1)} = -4$. The maximal step length is given by

$$\alpha_{\max}^{(0)} = \min_{i: a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{2}{3} < 1, \quad \text{for } i = 1.$$

Therefore we obtain $\alpha^{(0)} = \frac{2}{3}$ and $x^{(1)} = x^{(0)} + \alpha^{(0)} p^{(0)} = (\frac{1}{3} \; -\frac{2}{3} \; 0)^T$, and $\mathcal{W}^{(1)} = \{1, 3\}$.

Iteration 2: The solution to the corresponding quadratic program is given by

$$\begin{pmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ p_3^{(1)} \\ -\lambda_1^{(2)} \\ -\lambda_3^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 4 \\ 0 \\ 0 \end{pmatrix}.$$

By solving this system of equations, we get $p^{(1)} = (-\frac{1}{12} \; -\frac{1}{12} \; 0)^T$, $\lambda_1^{(2)} = \frac{1}{2}$ and $\lambda_3^{(2)} = -4$. Maximal step length is given by

$$\alpha_{\max}^{(1)} = \min_{i: a_i^T p^{(1)} < 0} \frac{a_i^T x^{(1)} - b_i}{-a_i^T p^{(1)}} = 16 > 1, \quad \text{for } i = 2.$$

Therefore we obtain $\alpha^{(1)} = 1$ and $x^{(2)} = x^{(1)} + \alpha^{(1)} p^{(1)} = (\frac{1}{4} \; -\frac{3}{4} \; 0)^T$.

Iteration 3: As $\lambda_3^{(2)} = -4 < 0$, we delete constraint 3 and obtain $\mathcal{W}^{(2)} = \{1\}$. The solution to the corresponding quadratic program is then given by

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ p_3^{(2)} \\ -\lambda_1^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 4 \\ 0 \end{pmatrix}.$$

Then, we have $p^{(2)} = (0 \; 0 \; 2)^T$ and $\lambda_1^{(3)} = \frac{1}{2}$. Since $a_i^T p^{(2)} \geq 0, i = 1, 2, 3$, we get $\alpha^{(2)} = 1$ and $x^{(3)} = x^{(2)} + \alpha^{(2)} p^{(2)} = (\frac{1}{4} \; -\frac{3}{4} \; 2)^T$. As $\lambda^{(3)} \geq 0$ we are done.

The optimal solution is $x = (\frac{1}{4} \; -\frac{3}{4} \; 2)^T$.

2. (See the course material.)

3. (a) Write the problem on the following form

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & c_1(x) \geq 0, \\ & c_2(x) \geq 0, \end{array}$$

where c_i either is equal to g_i or $-g_i$. The task is for $i = 1$ and 2 to find out which choice of sign that fulfils the first order of optimality conditions in x^*. These conditions gives that there will be *non-negative* λ_1 and λ_2 such that $\nabla f(x^*) = \lambda_1 \nabla c_1(x^*) + \lambda_2 \nabla c_2(x^*)$, i.e. such that

$$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix} = \pm \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \lambda_1 \pm \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \lambda_2.$$

The non-negative condition on λ makes that $\lambda = (3\ 6)^T$ and that the sign will be chosen as $-$ respectively $+$, i.e. $g_1(x) \leq 0$ and $g_2(x) \geq 0$, which corresponds to $(P2)$.

(b) Both constraints active give the Jacobian corresponding to the active constraints as

$$\begin{pmatrix} -2 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

A matrix whose columns form a base for the null-space to this matrix is $Z = (-1\ \ 1\ -1)^T$. Further the Hessian of the Lagrange function in (x^*, λ) is

$$\nabla^2_{xx} \mathcal{L}(x^*, \lambda) = \begin{pmatrix} 0 & -6 & 0 \\ -6 & 6 & 6 \\ 0 & 6 & 0 \end{pmatrix},$$

which makes that $Z^T \nabla^2_{xx} \mathcal{L}(x^*, \lambda) Z = 6$. This is a positive definite matrix (of dimension 1×1). As also $\lambda > 0$ is fulfilled, the second order of sufficient optimality conditions are fulfilled and x^* is with that a local minimum of $(P2)$.

4. (a) To take the derivative gives that

$$f(x) = e^{x_1+x_2} + x_1^2 + x_1 x_2 + 2x_2^2 + x_1,$$

$$\nabla f(x) = \begin{pmatrix} e^{x_1+x_2} + 2x_1 + x_2 + 1 \\ e^{x_1+x_2} + x_1 + 4x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} e^{x_1+x_2} + 2 & e^{x_1+x_2} + 1 \\ e^{x_1+x_2} + 1 & e^{x_1+x_2} + 4 \end{pmatrix},$$

$$g_1(x) = 10 - x_1^2 - x_2^2, \ \nabla g_1(x) = -2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \nabla^2 g_1(x) = -2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g_2(x) = x_2 - 1, \ \nabla g_2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \nabla^2 g_2(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\nabla^2_{xx}\mathcal{L}(x,\lambda) = \begin{pmatrix} e^{x_1+x_2}+2+2\lambda_1 & e^{x_1+x_2}+1 \\ e^{x_1+x_2}+1 & e^{x_1+x_2}+4+2\lambda_1 \end{pmatrix},$$

The SQP-subproblem then become

$$(QP) \quad \begin{array}{ll} \min & \frac{5}{2}p_1^2 + 2p_1p_2 + \frac{7}{2}p_2^2 + 7p_2 \\ \text{subject to} & 4p_1 - 4p_2 \geq -2, \\ & p_2 \geq -1. \end{array}$$

The original problem is convex with strict convex objective function, therefore (QP) is well defined.

(b) We can write (QP) on the form

$$(QP) \quad \begin{array}{ll} \min & \frac{1}{2}p^THp + c^Tp, \\ \text{subject to} & Ap \geq b, \end{array}$$

where $H = \nabla^2_{xx}\mathcal{L}(x,\lambda)$, $A = A(x)$, $c = \nabla f(x)$ and $b = -g(x)$. One iteration in an primal-dual interior point method for (QP) takes the form

$$\begin{pmatrix} H & -A^T \\ \Lambda A & \operatorname{diag}(Ap-b) \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta\lambda \end{pmatrix} = -\begin{pmatrix} Hp + c - A^T\lambda \\ \Lambda(Ap-b) - \mu e \end{pmatrix},$$

where $e = (1\ 1\ \ldots 1)^T$ and $\Lambda = \operatorname{diag}(\lambda)$. With included numerical values we obtain that

$$\begin{pmatrix} 5 & 2 & -4 & 0 \\ 2 & 7 & 4 & -1 \\ 4 & -4 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \\ \Delta\lambda_1 \\ \Delta\lambda_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ -1 \\ -1 \end{pmatrix}.$$

(c) One iteration in a primal-dual interior-point method for (P) takes the form

$$\begin{pmatrix} \nabla^2_{xx}\mathcal{L}(x,\lambda) & -A(x)^T \\ \Lambda A(x) & \operatorname{diag}(g(x)) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} = -\begin{pmatrix} \nabla f(x) - A(x)^T\lambda \\ \Lambda g(x) - \mu e \end{pmatrix},$$

In (4b) we have $H = \nabla^2_{xx}\mathcal{L}(x,\lambda)$, $A = A(x)$, $c = \nabla f(x)$ and $b = -g(x)$. If we let $p = 0$ in (4b) and use the same λ in (4b) and here, the system of equations become identical.

5. (a) To add a multiple of the unit matrix to another matrix do shift its eigenvalues with that multiple. With that the optimal value of (P) is given by $\lambda_{\max}(A)$, the largest eigenvalue of A.

(b) We obtain the dual

$$(D) \quad \begin{array}{ll} \max & \operatorname{trace}(AZ) \\ \text{subject to} & \operatorname{trace}(Z) = 1, \\ & Z = Z^T \succeq 0. \end{array}$$

If we specially assume that $Z = vv^T$ for some vector v we obtain

$$\begin{array}{ll} \max & \operatorname{trace}(Avv^T) \\ \text{subject to} & \operatorname{trace}(vv^T) = 1, \end{array}$$

which gives a lower bound of the optimal value of (D). We can equivalently write

$$\begin{array}{ll} \max & v^T A v \\ \text{subject to} & v^T v = 1. \end{array}$$

This problem have an optimal solution $\lambda_{\max}(A)$, and an optimal solution corresponding to the normalized eigenvector $\hat{v}$. As the optimal value of this problem agree with the optimal of (P) we also obtain the optimal value of (D) which is $\lambda_{\max}(A)$, and an optimal solution $\hat{v}\hat{v}^T$, where $\hat{v}$ is the corresponding normalized eigenvector.

(c) The matrix A do have eigenvalue 1 and 3, and $(\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}})^T$ is a normalized eigenvector corresponding to the largest eigenvalue. With that (P) and (D) have optimal value 3, which also is the optimal solution of (P). The optimal solution of (D) is given by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$