SF2812 Applied linear optimization, final exam
Thursday March 11 2021 8.00–13.00

Lecturer: Gianpiero Canessa (tel 072 935 6288).
Examiner: Anders Forsgren.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

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1. Consider the linear program

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}^T.$$

(a) Show that the basis given by $\mathcal{B} = \{x_2, x_3\}$ gives a corresponding dual basic solution which is feasible to the dual problem associated with (LP). .....(3p)

(b) Solve (LP) by the dual simplex method, starting from the basis given in Question 1a. ....(7p)

2. Let (LP) and its dual (DLP) be defined as

$$(LP) \quad \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b, \\ & x \geq 0, \end{array} \quad \text{and} \quad (DLP) \quad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & A^T y \geq c, \\ & y \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 8 & 6 & 1 \\ 4 & 2 & 1.5 \\ 2 & 1.5 & 0.5 \end{pmatrix}, \quad b = \begin{pmatrix} 48 \\ 20 \\ 8 \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} 6 & 3 & 2 \end{pmatrix}^T.$$

(a) A person named GC has used GAMS to model and solve this problem. GC has been told that he can solve either (LP) or (DLP) for finding the optimal solutions to (LP) and (DLP). He has chosen to solve (LP). The GAMS input file can be found at the end of the exam, and a partial GAMS output file reads:

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               S O L V E      S U M M A R Y

     MODEL   example            OBJECTIVE  cost
     TYPE    LP                 DIRECTION  MAXIMIZE
     SOLVER  CPLEX              FROM LINE  22

Optimal solution found
Objective:           28.000000


                           LOWER          LEVEL          UPPER         MARGINAL

---- EQU of                  .              .              .             1.0000
---- EQU r1                -INF          24.0000        48.0000           .
---- EQU r2                -INF          20.0000        20.0000          1.0000
---- EQU r3                -INF           8.0000         8.0000          1.0000

  of  objective function
  r1  constraint number 1
  r2  constraint number 2
  r3  constraint number 3

                           LOWER          LEVEL          UPPER         MARGINAL

---- VAR x1                  .             2.0000        +INF             .
---- VAR x2                  .              .            +INF          -0.5000
---- VAR x3                  .             8.0000        +INF             .
---- VAR cost              -INF          28.0000        +INF             .
```

The only catch is that GC does not know how to extract the optimal solutions from the GAMS output. Help GC obtain the optimal solutions to (LP) and (DLP) from the GAMS output file. ....(2p)

(b) GC claims that if b_1 is changed to $48+\delta$, the optimal value is unchanged. Show that GC is right. Do so without solving any system of linear equations. . (2p)

(c) Give bounds on δ for which the answer of Question 2b is valid. The system of linear equations that arises need not be solved in a systematic way. ..... (2p)

(d) GC is wondering what would happen if c_1 is changed to $6+\beta$. Obtain the bounds in which the optimal basis is unchanged. The system of linear equations that arises need not be solved in a systematic way. ....(2p)

(e) GC has recently received the news that the value of b_2 is now 19. Explain what will happen to the current solution value and what would be the consequences in the optimal basis, if there are any. .... (2p)

Hint: Use this as the optimal basis inverted matrix:

$$B^{-1} = \begin{pmatrix} 0 & -1/2 & 3/2 \\ 0 & 2 & -4 \\ 1 & 2 & -8 \end{pmatrix},$$

3. Consider the integer program (IP) defined by

$$(IP)\qquad \begin{array}{ll} \text{minimize} & 4x_1 - 5x_2 \\ \text{subject to} & 3x_1 + 2x_2 \geq 5, \\ & -2x_1 + 3x_2 \leq 7 \\ & x \geq 0, \quad x \text{ integer.} \end{array}$$

A person named GC has solved (IP) for you as a relaxed continuous problem and obtained the following: $x_1 = 1/13, x_2 = 31/13$ giving us the relaxed solution value $x_{LP} = -151/13$, but we need the integer solution. AF suggests to GC to use an approximation of the solution: $\hat{x}_1 = 1, \hat{x}_2 = 3$ then $z_{IP} = -11$.

(a) State the current lower and upper bound on the optimal value of the problem. .... (2p)

(b) Based on the information above, can you conclude if the suggested point $\hat{x}_1 = 1, \hat{x}_2 = 3$ is the optimal solution? .... (2p)

(c) GC suggests to start branching the problem from variable x_2. Obtain the new node that you obtain from this branching and check the solution (use any method that you like, no need to be systematic). Update the upper and lower bounds. .... (4p)

(d) Based on the new information, can you conclude if the suggested point $\hat{x}_1 = 1, \hat{x}_2 = 3$ is the optimal solution? .... (2p)

4. Consider the linear program (LP) given by

$$(LP)\qquad \begin{array}{ll} \text{minimize} & -3x_1 - 2x_2 + x_3 + 2x_4 \\ \text{subject to} & 2x_1 + x_2 - 2x_3 - 2x_4 = 2, \\ & -1 \leq x_j \leq 1, \quad j = 1, \dots, 4. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $2x_1 + x_2 - 2x_3 - 2x_4 = 2$ the complicating constraint. Use the extreme points $v_1 = (1\ \ 1\ -1\ -1)^T$ and $v_2 = (1\ \ 1\ \ 1\ \ 1)^T$ for obtaining an initial basic feasible solution to the master problem.

The subproblem(s) that arise may be solved in any way, that need not be systematic. .... (10p)

5. Consider the integer program (IP) defined by

$$(IP)\qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0, \quad x \text{ integer.} \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D)\qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \geq 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \geq d, x \geq 0 \text{ integer}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integer requirement, i.e.,

$$
(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0. \end{array}
$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. ....(10p)