SF2812 Applied linear optimization, final exam
Thursday March 11 2021 8.00–13.00
Brief solutions

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1. (a) From Dual Simplex we know that $y = c_B' B^{-1} = [0 \;\; 1]$, now we need to know the values of s, which can be checked using the dual formulation $s = c' - A'y = [2 \;\; 0 \;\; 0 \;\; 1]$.

(b) We can check that the basis given is infeasible since $x_3 = -3$, then, we can iterate using the dual basis since it is dual feasible. Since we have 3 nonnegative variables, we can add another and assume we have a degenerate basis, for example $\mathcal{B}_d = \{y_1, y_2, s_1, s_4\}$. Checking the reduced costs we have that $\bar{c}_5 = 3$ which will enter the basis, and s_1 should exit (it can also be s_4 but we will use Bland's Rule).

On the next iteration we have that all reduced costs are negative, obtaining our optimal dual basis $\mathcal{B}_d = \{y_1, y_2, s_3, s_4\}$. We can check that $x = b_B B_d^{-1} = [2 \;\; 2 \;\; 0 \;\; 0]$ and that the optimal primal basis is $\mathcal{B} = \{x_1, x_2\}$.

2. (a) From the output we can see that $x = [2 \;\; 0 \;\; 8]$ (in standard form we have $x = [2 \;\; 0 \;\; 8 \;\; 24 \;\; 0 \;\; 0]$). Also, we have the dual prices from each constraint $y = [0 \;\; 1 \;\; 1]$, and finally we have that reduced costs where $\bar{c}_2 = -.5$ which is related to $s_2 \geq 0$. We have everything to reconstruct the dual optimal basis: $y = [0 \;\; 1 \;\; 1], s = [0 \;\; 1/2 \;\; 0]$

(b) We can check that constraint 1 has a dual price of 0, which means the slack variable is nonnegative. Since the current value of $x_4 = 24$, we can correctly say that the optimal basis will not change if $-24 \leq \delta$.

(c) To obtain (and prove this bound) we need to solve the following equation:

$$B^{-1}\tilde{b} = \begin{pmatrix} 0 & -1/2 & 3/2 \\ 0 & 2 & -4 \\ 1 & 2 & -8 \end{pmatrix} \begin{pmatrix} 48 + \delta \\ 20 \\ 8 \end{pmatrix} \geq 0$$

Where we obtain $-24 \leq \delta$. Since there is no upper bound, we assume infinite.

(d) We need to check the reduced costs $\bar{c} = c_N - \tilde{c_B}' B^{-1} N \leq 0$, where $\tilde{c_B}' = c_B + e_1 \beta$. We obtain the following:

$$\frac{5}{4}\beta \geq -\frac{1}{2}$$
$$\frac{\beta}{2} \leq 1$$
$$\frac{3}{2}\beta \geq -1$$

Then $-\frac{2}{3} \leq \beta \leq 2$.

(e) We need to check first if the current basis is valid:

$$B^{-1}\tilde{b} = \begin{pmatrix} 0 & -1/2 & 3/2 \\ 0 & 2 & -4 \\ 1 & 2 & -8 \end{pmatrix} \begin{pmatrix} 48 \\ 20 + \delta \\ 8 \end{pmatrix} \geq 0$$

Obtaining:

$$\begin{aligned} 2-\frac{\delta}{2} &\geq 0 \\ 2\delta+8 &\geq 0 \\ 2\delta+24 &\geq 0 \end{aligned}$$

Then $-4 \leq \delta \leq 4$. Since the new value means $\delta = -1$, then the current basis stays optimal. Now, checking the dual value of the second constraint, which is $y_2 = 1$, the new optimal value is $28 + \delta y_2 = 28 - 1 = 27$.

3. (a) Currently, the lower bound is -151/13 and the upper bound is -11.

(b) Since the upper and lower bound are not the same, we do not have enough information to say the current incumbent (upper bound) is optimal.

(c) If we start from the bound $x_2 \geq 3$, then we obtain the suggested point. From $x_2 \leq 2$, we have $x_1 = 1/3, x_2 = 2, z = -8.\bar{6}$. No update on the upper bound because of the former node, but on the lower bound we fathom the latter node, thus obtaining the new lower bound -11.

(d) Since both upper and lower bound are equal, we have enough evidence to say the suggested point is the optimal solution.

4. The suggested initial extreme points $v_1 = (1\ \ 1\ {-1}\ {-1})^T$ and $v_2 = (1\ \ 1\ \ 1\ \ 1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^Tv_1\ c^Tv_2) = (-8\ \ -2)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} 7 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -8 \\ -2 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{11}{4} \end{pmatrix}.$$

By forming $c^T - y_1 A = (-3/2\ \ -5/4\ \ -1/2\ \ 1/2)$ we obtain the subproblem

$$\begin{array}{lll} \frac{11}{4}+ & \text{minimize} & -\frac{3}{2}x_1 - \frac{5}{4}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ & \text{subject to} & -1 \leq x_j \leq 1, \quad j = 1, \ldots, 4. \end{array}$$

An optimal extreme point to the subproblem is given by $v_3 = (1\ \ 1\ \ 1\ \ -1)^T$ with optimal value -1. Hence, α_3 should enter the basis. The corresponding column in the master problem is given by

$$\begin{pmatrix} Av_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The change to the basic variables is given by

$$\begin{pmatrix} 7 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Finding the maximum step η for which $\alpha + \eta p \geq 0$ gives

$$\begin{pmatrix} \frac{3}{8} \\ \frac{5}{8} \end{pmatrix} + \eta \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $\eta = 3/4$ so that α_1 leaves the basis.

Hence, the new basis corresponds to v_2 and v_3 so that

$$B = \begin{pmatrix} Av_2 & Av_3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}.$$

The basic variables are given by

$$\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_2\ c^T v_3) = (-2\ \ -6)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

By forming $c^T - y_1 A = (-1\ \ -1\ \ -1\ \ \ 0)$ we obtain the subproblem

$$\begin{array}{lll} 3+ & \text{minimize} & -x_1 - x_2 - x_3 \\ & \text{subject to} & -1 \leq x_j \leq 1, \quad j = 1, \dots, 4. \end{array}$$

Both v_2 and v_3 are optimal extreme points to the subproblem, so the optimal value of the subproblem is 0. Hence, the master problem has been solved. The solution to the original problem is given by

$$v_2\alpha_2 + v_3\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \frac{1}{4} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \frac{3}{4} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix}.$$

The optimal value is –5.

5. Please refer to lesson 11.