SF2812 Applied linear optimization, final exam
Tuesday June 2 2020 8.00–13.00
Brief solutions

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1. (a) First prove that x optimal $\Rightarrow c'd \geq 0$:
Let x be the optimal solution for $\min c'x$ s.t. $x \in P$, then for any point $y \in P$ we have $c'x \leq c'y$. Let d be a feasible direction where $y = x + \theta d$ for some $\theta > 0$, then

$$\begin{aligned} c'x &\leq c'y = c'(x + \theta d) \\ c'x &\leq c'x + \theta c'd \\ 0 &\leq c'd \end{aligned}$$

Now we prove $c'd \geq 0 \Rightarrow x$ optimal: Let $x, y \in P$ where $y = x + \theta d$ for some $\theta > 0$. Let $c'd \geq 0$, then

$$\begin{aligned} c'y &= c'(x + \theta d) \\ c'y - c'x &= \theta c'd \geq 0 \end{aligned}$$

Therefore, for any feasible direction d we have that $c'y - c'x \geq 0$, then x must be the optimal solution for the problem $\min c'x$ s.t. $x \in P$.
Having both direction, the proof is complete for $c'd \geq 0 \iff x$ optimal.

(b) We can apply the same proof in a) and prove by contradiction that x is not unique if $c'd > 0$, which is false since we will not find another point where $c'x = c'y$ for $y \in P$, finishing the proof.

2. (a) The relaxed solution is $(0, 5/2)$.

(b) Since the only fractional variable is x_2, the branching should start there yielding 2 new nodes. The first node we should work on is $x_2 \geq 3$ since the obj. value is 5.3: it gives an infeasible node and an incumbent 1 $(1, 3)$ with obj. value of 10. Since the second node form the first branch $x_2 \leq 2$ has an obj. value of 7.6 which is better than the incumbent, the problem is not finished. This node gives an infeasible node and another fractional node $(1, 15/8)$ with obj. value 8.8875. Still being better than the upper bound thus far, we do not fathom the node and work with it: the first given node will be infeasible while the second yields a new incumbent $(1, 2)$ with obj. value 9. This last value is lower than the current upper bound, thus becoming the new upper bound. Since we have no more fractional nodes to develop, we have found the optimal integer solution, finishing the problem.

3. (a) The primal-dual system obtained is

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ 9x_1 + x_2 + x_4 &= 10 \\ 2y_1 + 9y_2 + s_1 &= 1 \\ 3y_1 + y_2 + s_2 &= 2 \\ -y_1 + s_3 &= 0 \\ y_2 + s_4 &= 0 \end{aligned}$$

Also need to include $x_y s_i = \mu, i = 1..4$.

(b) This part was declared as bonus points and optional.

4. Please refer to lesson 6.

5. (a) From the hint we can obtain the reduced costs: $y' = c_B' B^{-1} = (1/6, 1/2, 1)$. Now we need to check these values by solving the aux problem:

$$\begin{aligned} z = 1 - \max \quad & 1/6a_1 + 1/2a_2 + a_3 \\ \text{subject to} \quad & 2a_1 + 6a_2 + 10a_3 \leq 12 \\ & a \geq 0, a \in \mathbb{Z}^3 \end{aligned}$$

Solving this proble you obtain the optimal solution $a^* = (1, 0, 1)$ then $z = -1/6 < 0$ which is not optimal, then $A = (1, 0, 1)'$ enters the basis, while $(0, 0, 1)'$ exits it. Now we repeat: $y' = c_B' B^{-1} = (1/6, 1/2, 5/6)$. Then:

$$\begin{aligned} z = 1 - \max \quad & 1/6a_1 + 1/2a_2 + 5/6a_3 \\ \text{subject to} \quad & 2a_1 + 6a_2 + 10a_3 \leq 12 \\ & a \geq 0, a \in \mathbb{Z}^3 \end{aligned}$$

The solution is again $a^* = (1, 0, 1)$ but $z = 0$ then it is optimal, finishing the algorithm.

(b) Using the solution from a) we have a possible rounding: $B^{-1}b = (10/6, 15/2, 10)'$, it is not integer but we can see that by rounding this solution we have:

- 2 patterns of $(6, 0, 0)$.
- 8 patterns of $(0, 2, 0)$.
- 10 patterns of $(1, 0, 1)$.

The the optimal value is $20 > 115/6$, then we have a near optimal solution.

Good luck!