SF2822 Applied nonlinear optimization, final exam
Thursday June 2 2022 08.00–13.00

Instructor: Jan Rolfes, tel. 08-790 74 15, in case of questions, please contact the oversight.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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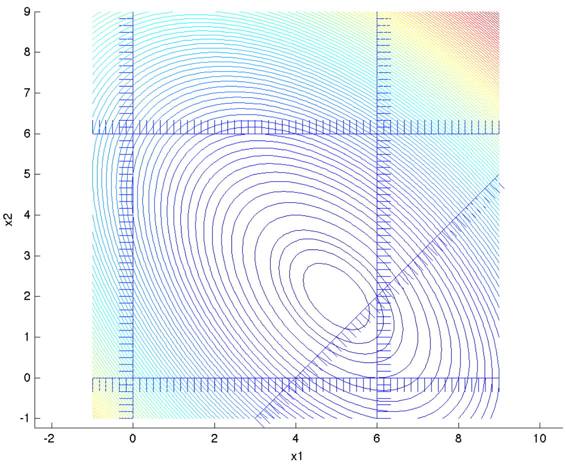
1. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -12 \\ -9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -4 \end{pmatrix}.$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (6 \ \ 5)^T$ with the constraint $-x_1 \geq -6$ in the working set. You need not calculate every exact numerical value, but instead you may utilize the fact that the problem is two-dimensional, and replace, if possible, some calculations with geometric arguments. Illustrate your iterations in the figure corresponding to Question 1a which can be found at the last sheet of the exam. Motivate each step carefully. ....(4p)

(b) Assume that b_3 is changed from -6 to -4, so that the third constraint reads $-x_1 \geq -4$. Solve the corresponding quadratic program by an active-set method. Start at $x = (2 \;\; 6)^T$ with no constraint in the working set. You need not calculate every exact numerical value, but instead you may utilize the fact that the problem is two-dimensional, and replace, if possible, some calculations with geometric arguments. Add the modified constraint and illustrate your iterations in the figure corresponding to Question 1b which can be found at the last sheet of the exam. Motivate each step carefully. ....(6p)

2. Consider the NLP problem (P) defined as

$$
(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \geq 0, \quad i = 1, \dots, m, \\ & x \in I\!R^n, \end{array}
$$

where f and g are twice-continuously differentiable.

A regular point with respect to the constraints is a point x^* such that $\nabla g_i(x^*)$, $i \in \{k : g_k(x^*) = 0\}$, are linearly independent.

(a) Formulate the second-order necessary optimality conditions for a regular point x^* to be a local minimizer for (P). ....(5p)

(b) For the special case when $g(x) = Ax - b$, prove the first-order necessary optimality conditions for a regular point x^* to be a local minimizer of (P). ..(5p)

3. Consider the QP-problem (QP) defined as

$$
(QP) \qquad \begin{array}{ll} \text{minimize} & -x_1 x_2 \\ \text{subject to} & x_1 + x_2 = 2. \end{array}
$$

(a) For a given positive penalty parameter $\mu < 2$, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the quadratic-penalty-transformed problem. It is possible to obtain an analytical expression for this small problem. ....(5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in Question 3a converge to the optimal solution and Lagrange multiplier respectively of (QP), when $\mu \to 0$. ....(3p)

(c) For μ small and positive, use your results of Question 3b to give an estimate of $x(\mu) - x^*$ in terms of μ, where x^* denotes the optimal solution to (QP). Is this as expected? ....(2p)

4. Consider the nonlinear program (NLP) given by

$$
(NLP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 3)^2 \\ \text{subject to} & 1 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \geq 0. \end{array}
$$

Assume that one wants to solve (NLP) by a sequential quadratic programming method for the initial point $x^{(0)} = (0\ 1)^T$ and $\lambda^{(0)} = 2$.

(a) Your friend JR claims that there is no need to perform any iterations. He claims that $x = (2, 3)^T$ must be a global minimizer to (NLP), since (NLP) is a convex optimization problem and $\nabla f(x^{(0)}) = 0$. Explain why he is wrong. .... (2p)

(b) Perform one iteration by sequential quadratic programming for solving (NLP) for the given $x^{(0)}$ and $\lambda^{(0)}$, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, and you do not need to perform any linesearch. .... (8p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

5. Let $C \in \mathcal{S}^n$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \dots \lambda_n$. Consider the following (semidefinite) optimization programs (SDP) given by

$$\mu_1 = \max_Y \left\{ \text{trace}(CYY^T) : Y^T Y = I_k, Y \in \mathbb{R}^{n \times k} \right\},$$

$$\mu_2 = \max_X \left\{ \text{trace}(CX) : \text{trace}(X) = k, I_n - X \succeq 0, X \succeq 0 \right\}.$$

(a) Show that $\lambda_1 + \ldots + \lambda_k \leq \mu_1$. .... (4p)

(b) Show that $\lambda_1 + \ldots + \lambda_k \leq \mu_2$. .... (2p)

(c) Dualize the relaxation of μ_2, where the constraint $I_n - X \succeq 0$ is dropped. .... (4p)

Hint: For (a) and (b) you may try a matrix Y, whose columns consist of eigenvectors of C.

Good luck!

Name: . . . . Personal number: . . . . Sheet number: . . . .

Figure for Question 1a:

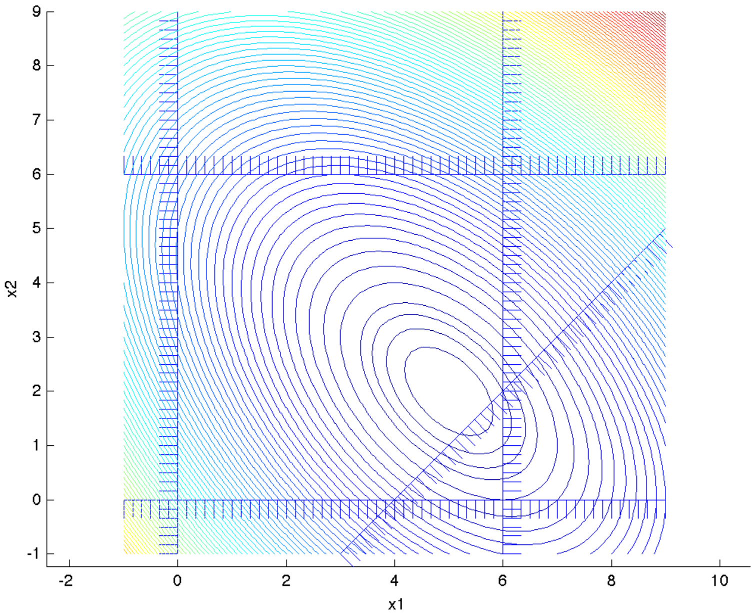


Figure for Question 1b: