SF2822 Applied nonlinear optimization, final exam
Wednesday August 17 2022 08.00–13.00

Instructor: Jan Rolfes, tel. 08-790 74 15, in case of questions, please contact the oversight.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the nonlinear optimization problem (NLP) given by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & e^{x_1} + e^{x_2} - x_2 + x_1 x_2 + x_2^2 - 2x_2 x_3 + x_3^2 \\ \text{subject to} & 10 - x_1^3 - x_2^4 - x_3^5 \geq 0, \\ & a^\top x - 2 \geq 0, \end{array}$$

where $a \in \mathbb{R}^3$ is a given constant. Let $\tilde{x} = (0, 0, 1)^\top$.

(a) Find a such that $\tilde{x}$ satisfies the first order necessary optimality conditions to (NLP). . . . . (5p)

(b) For the value on a you found in (a), decide whether $\tilde{x}$ is a local optimal solution to (NLP). . . . . (5p)

2. Consider the nonlinear programming problem

$$(P) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g_i(x)\lambda_i - \mu &= 0, \quad i = 1, \dots, m, \end{aligned}$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. . . . . (4p)

(b) Let $x(\mu)$, $\lambda(\mu)$ be a solution to the primal-dual nonlinear equations of (2a) such that $g_i(x(\mu)) > 0$, $i = 1, \dots, m$, and $\lambda(\mu) > 0$. Show that $x(\mu)$ is a global minimizer to (P_μ) if f and $-g_i$, $i = 1, \dots, m$, are convex functions on $I\!R^n$.(2p)

(c) Derive the system of linear equations that results when the primal-dual nonlinear equations of (2a) are solved by Newton's method. (4p)

3. Consider the quadratic program (QP) defined by

$$
(QP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \geq 3, \\ & x_1 + 2x_2 \geq 3. \end{array}
$$

Solve (QP) with an active-set method, with the initial point $x^{(0)}$ given by $x^{(0)} = (0\ 4)^T$. The equality-constrained quadratic programs that arise as subproblems need not be solved in a systematic way. They may for example be solved graphically. However, the values of the generated iterates $x^{(k)}$ and corresponding Lagrange multipliers $\lambda^{(k)}$ should be calculated. . . . . (10p)

4. Consider the nonlinear optimization problem (NLP) given by

$$
(NLP) \quad \begin{array}{ll} \text{minimize} & 2(x_1 - 1)^2 + (x_2 - 2)^2 \\ \text{subject to} & 3 - x_1^2 - x_2^2 \geq 0. \end{array}
$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)} = (2\ 1)^T$ and let $\lambda^{(0)} = 0$. Perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, and you do not need to perform any linesearch. . . . . (10p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

5. Consider the nonlinear optimization problem (NLP) given by

$$
(NLP) \quad \begin{array}{ll} \text{minimize} & e^{x_3} \\ \text{subject to} & x \in F, \end{array}
$$

where F is given by $F = \{x \in I\!R^3 : x_3 \geq x_2^2/x_1,\ x_1 \geq 1\}$.

(a) Show that (NLP) is a convex optimization problem. . . . . (4p)

(b) Show that

$$
(NLP2) \quad \begin{array}{ll} \text{minimize} & x_3 \\ \text{subject to} & x \in F, \end{array}
$$

can be reformulated as a semidefinite program. . . . . (6p)

Hint: Let M be a symmetric matrix which is partitioned as

$$
M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}.
$$

Assume that $A \succ 0$. Then, $M \succeq 0$ if and only if $C - BA^{-1}B^T \succeq 0$.

Good luck!