SF2822 Applied nonlinear optimization, final exam
Wednesday August 17 2022 8.00–13.00
Brief solutions

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1. (a) The objective function $f(x) = e^{x_1} + e^{x_2} - x_2 + x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2$ yields the following gradient $\nabla f(x)$ and Hessian H_f

$$\nabla f(x) = \begin{pmatrix} e^{x_1} + x_2 \\ e^{x_2} - 1 + x_1 + 2x_2 - 2x_3 \\ -2x_2 + 2x_3 \end{pmatrix}, \quad H_f = \begin{pmatrix} e^{x_1} & 1 & 0 \\ 1 & e^{x_2} + 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

In particular, we obtain $\nabla f(\tilde{x}) = (1, -2, 2)^\top$. Moreover, since $g_1(\tilde{x}) = 10 - 0^3 - 0^4 - 1^5 = 9 > 0$, we have that Constraint 1 is not active in $\tilde{x}$. As $\nabla f(\tilde{x}) \neq 0$, Constrain 2 needs to be active with non-negative Lagrange multipliers for $\tilde{x}$ in order to satisfy the first order necessary optimality constraints, i.e., we obtain: $a\lambda_2 = \nabla f(\tilde{x})$ with $\lambda_2 \geq 0$ and $a^\top \tilde{x} = 2$.
Since $a\lambda_2 = \nabla f(\tilde{x})$ can not be fulfilled with $\lambda_2 = 0$, we obtain $a = \frac{1}{\lambda_2}\nabla f(\tilde{x})$. Hence, we obtain $2 = a^\top \tilde{x} = \frac{1}{\lambda_2}\nabla f(\tilde{x})^\top \tilde{x} = 2\lambda_2$. Thus, we obtain $\lambda_2 = 1$ and $a = \nabla f(\tilde{x}) = (1, -2, 2)^\top$.

(b) We check the second-order necessary optimality conditions. Since the only active constraint is linear, we have that $H_{g_2} = 0$ and thus

$$\nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) = \nabla^2 f(\tilde{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

In addition, we have that $A_A(\tilde{x}) = a^\top$, where we choose the first entry as the invertible basis, i.e. $B = 1$, $N = (-2, 2)$. Thus, we obtain the following basis of null$A_A(\tilde{x})$:

$$Z_A(\tilde{x}) = \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which leads to

$$Z_A(\tilde{x})^\top \nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) Z_A(\tilde{x}) = \begin{pmatrix} 11 & -8 \\ -8 & 6 \end{pmatrix}.$$

Since the determinant of $Z_A(\tilde{x})^\top \nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) Z_A(\tilde{x})$ is negative and $Z_A(\tilde{x})^\top \nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) Z_A(\tilde{x})$ is a 2×2 matrix, we observe that $Z_A(\tilde{x})^\top \nabla^2_{xx}\mathcal{L}(\tilde{x}, \tilde{\lambda}) Z_A(\tilde{x}) \not\succeq 0$. Hence, $\tilde{x}$ does not satisfy the second-order necessary optimality conditions and is therefore not a local minimum.

2. (See the course material.)

3. No constraints are active at the initial point. Hence, the working set is empty, i.e., $\mathcal{W} = \emptyset$. Since $H = I$ and $c = (1, 1)^\top$, we obtain $p^{(0)} = -(Hx^{(0)} + c) = -x^{(0)} - c = (-1, -5)^\top$. The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{1}{7},$$

where the minimum is attained for $i = 1$. Consequently, $\alpha^{(0)} = 1/7$ so that

$$x^{(1)} = x^{(0)} + \alpha^{(0)} p^{(0)} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \frac{1}{7} \begin{pmatrix} -1 \\ -5 \end{pmatrix} = \begin{pmatrix} -1/7 \\ 23/7 \end{pmatrix},$$

with $\mathcal{W} = \{1\}$. The solution to the corresponding equality-constrained quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(1)} \\ p_2^{(1)} \\ -\lambda_1^{(2)} \end{pmatrix} = - \begin{pmatrix} 6/7 \\ 30/7 \\ 0 \end{pmatrix}$$

We obtain

$$p^{(1)} = \begin{pmatrix} \frac{54}{35} & -\frac{108}{35} \end{pmatrix}^T.$$

The maximum steplength is given by

$$\alpha_{\max} = \min_{i: a_i^T p^{(0)} < 0} \frac{a_i^T x^{(0)} - b_i}{-a_i^T p^{(0)}} = \frac{20}{27},$$

where the minimum is attained for $i = 2$. Consequently, $\alpha^{(1)} = 20/27$ so that

$$x^{(2)} = x^{(1)} + \alpha^{(1)} p^{(1)} = \begin{pmatrix} -1/7 \\ 23/7 \end{pmatrix} + \frac{20}{27} \begin{pmatrix} 54/35 \\ -108/35 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with $\mathcal{W} = \{1, 2\}$. The solution to the corresponding equality-constrained quadratic program is given by

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_1^{(2)} \\ p_2^{(2)} \\ -\lambda_1^{(3)} \\ -\lambda_2^{(3)} \end{pmatrix} = - \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

We obtain

$$p^{(2)} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T, \quad \lambda^{(3)} = \begin{pmatrix} 2/3 & 2/3 \end{pmatrix}^T.$$

As $p^{(2)} = 0$ and $\lambda^{(3)} \geq 0$, the optimal solution has been found. Hence, $x^{(2)}$ is optimal.

4. We have

$$f(x) = 2(x_1 - 1)^2 + (x_2 - 2)^2 \qquad g(x) = 3 - x_1^2 - x_2^2 \geq 0,$$

$$\nabla f(x) = \begin{pmatrix} 4(x_1 - 1) \\ 2(x_2 - 2) \end{pmatrix}, \qquad \nabla g(x) = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix},$$

$$\nabla^2 f(x) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \nabla^2 g(x) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The first QP-subproblem becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}p^T\nabla^2_{xx}\mathcal{L}(x^{(0)},\lambda^{(0)})p + \nabla f(x^{(0)})^Tp \\ \text{subject to} & \nabla g(x^{(0)})^Tp \geq -g(x^{(0)}), \end{array}$$

Insertion of numerical values gives

$$\begin{array}{ll} \text{minimize} & 2p_1^2 + p_2^2 + 4p_1 - 2p_2 \\ \text{subject to} & -4p_1 - 2p_2 \geq 2. \end{array}$$

We now utilize the fact that the subproblem is of dimension two with only one constraint. The subproblem is convex, since it is a quadratic program with positive definite Hessian. Moreover, its unconstrained minimizer $p = (-1, 1)^\top$ is feasible. Evaluating the gradient at the optimal point of the quadratic program gives

$$\begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ -\lambda^+ \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix},$$

so that $\lambda = 0$. Consequently, we obtain

$$x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda^{(1)} = 0.$$

5. (a) The Hessian of the objective function is given by

$$\nabla^2 f(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{x_3} \end{pmatrix}.$$

We observe, that $\nabla^2 f(x) \succeq 0$ since $e^{x_3} > 0$ for every $x_3 \in \mathbb{R}$ implying the convexity of f. It remains to be shown, that F is convex, which due to the fact that $g_2(x) = x_1 - 1$ is linear, boils down to the question whether

$$g_1(x) = x_3 - x_2^2/x_1$$

is concave. Consider the Hessian:

$$\nabla^2 g_1(x) = \begin{pmatrix} -2x_2^2/x_1^3 & 2x_2/x_1 & 0 \\ 2x_2/x_1^2 & -2/x_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In order to show that $-\nabla^2 g_1(x) \succeq 0$, we apply the hint. First, we observe that $2/x_1 > 0$ as $x_1 \geq 1 > 0$, which leads to the following argument

$$-\nabla^2 g_1(x) \succeq 0 \Leftrightarrow \frac{2x_2^2}{x_1^3} - (-\frac{2x_2}{x_1^2})\frac{x_1}{2}(-\frac{2x_2}{x_1^2}) \geq 0.$$

However, the latter statement holds, as we have

$$\frac{2x_2^2}{x_1^3} - (-\frac{2x_2}{x_1^2})\frac{x_1}{2}(-\frac{2x_2}{x_1^2}) = \frac{2x_2^2}{x_1^3} - \frac{2x_2}{x_1^2}\frac{x_2}{x_1} = 0$$

and thus we have that (NLP) is a convex program.

(b) We consider

$$(NLP2) \qquad \begin{array}{ll} \text{minimize} & x_3 \\ \text{subject to} & x_3 \geq \frac{x_2^2}{x_1}, \\ & x_1 \geq 1. \end{array}$$

For $x_1 > 0$ we obtain

$$x_3 \geq \frac{x_2^2}{x_1} \quad \Leftrightarrow x_3 - \frac{x_2^2}{x_1} \geq 0 \quad \Leftrightarrow \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0,$$

where Hint 2 has been used in the last step with $A = x_1$, $B = x_2$ and $C = x_3$. Hence, $(NLP2)$ is equivalent to the semidefinite program

$$\begin{array}{ll} \text{minimize} & x_3 \\ \text{subject to} & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0, \\ & x_1 \geq 1. \end{array}$$