SF2822 Applied nonlinear optimization, final exam
Saturday June 2 2012 9.00–14.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain thoroughly.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

---

1. Consider the nonlinear programing problem

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \; ? \; 0, \quad i = 1, \dots, 3, \\ & x \in I\!R^3, \end{array}$$

where $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R^3$ are twice continuously differentiable and each "?" is an inequality, either "$\le$" or "$\ge$". The inequalities can be of different type for the different constraints.

Assume that we have a point x^* such that

$$f(x^*) = 5, \quad \nabla f(x^*) = \begin{pmatrix} -1 & 2 & -1 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_1(x^*) = 0, \quad \nabla g_1(x^*) = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T, \quad \nabla^2 g_1(x^*) = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$g_2(x^*) = 3, \quad \nabla g_2(x^*) = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}^T, \quad \nabla^2 g_2(x^*) = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$g_3(x^*) = 0, \quad \nabla g_3(x^*) = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T, \quad \nabla^2 g_3(x^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Is it possible to replace "?" by "$\le$" or "$\ge$" so that x^* becomes a local minimizer to (NLP)? .... (10p)

2. Consider the quadratic program (QP) given by

$$(QP)\qquad \begin{array}{ll} \min & \frac{1}{2}x^THx \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ -5 \\ -6 \\ 5 \end{pmatrix}.$$

Solve (QP) using an active-set method. Start at the point $x = (5 \;\; 2)^T$ and let the constraint $-x_1 \geq -5$ be active in the first iteration. You may use the fact that the problem is two-dimensional and for example illustrate the iterations in a figure. Motivate each step carefully. .... (10p)

3. Consider the nonlinear optimization problem (NLP) given by

$$(NLP)\qquad \begin{array}{ll} \text{minimize} & 4(x_1-2)^2 + (x_2-1)^2 \\ \text{subject to} & 1 - x_1^2 - x_2^2 \geq 0. \end{array}$$

We want to solve (NLP) by sequential quadratic programming. Let $x^{(0)} = (2\ 1)^T$ and let $\lambda^{(0)} = 0$. Perform one iteration, i.e., calculate $x^{(1)}$ and $\lambda^{(1)}$. You may solve the subproblem in an arbitrary way that need not be systematic, e.g. graphically, and you do not need to perform any linesearch. .... (10p)

Remark: In accordance to the notation of the textbook, the sign of λ is chosen such that $\mathcal{L}(x,\lambda) = f(x) - \lambda^T g(x)$.

4. Consider the nonlinear programming problem

$$(P)\qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu)\qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} \nabla f(x) - \nabla g(x)\lambda &= 0, \\ g_i(x)\lambda_i - \mu &= 0, \quad i = 1, \dots, m, \end{aligned}$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. .... (4p)

(b) Let $x(\mu)$, $\lambda(\mu)$ be a solution to the primal-dual nonlinear equations of (**??**) such that $g_i(x(\mu)) > 0$, $i = 1, \dots, m$, and $\lambda(\mu) > 0$. Show that $x(\mu)$ is a global minimizer to (P_μ) if f and $-g_i$, $i = 1, \dots, m$, are convex functions on $I\!R^n$. (2p)

(c) Derive the system of linear equations that results when the primal-dual nonlinear equations of (**??**) are solved by Newton's method. ....(4p)

5. Consider the nonlinear optimization problem (NLP) given by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in F, \end{array}$$

where $f(x) = x_2^2/x_1$, and F is a convex polytope such that $F \subseteq \{x \in I\!R^2 : x_1 \geq 1\}$.

(a) Show that (NLP) is a convex optimization problem. ....(3p)

(b) Show that (NLP) can be reformulated as a semidefinite program. ....(7p)
Hint 1: Introduce a new variable x_3 plus a constraint $x_3 \geq f(x)$.
Hint 2: Let M be a symmetric matrix which is partitioned as

$$M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}.$$

Assume that $A \succ 0$. Then, $M \succeq 0$ if and only if $C - BA^{-1}B^T \succeq 0$.

Good luck!