SF2822 Applied nonlinear optimization, final exam
Wednesday August 16 2023 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a nonlinear programming problem (NLP) defined by

$$(NLP) \quad \begin{array}{ll} \text{minimize} & e^{x_1} + x_1x_2 + x_2^2 - 2x_2x_3 + x_3^2 - 3x_1 - x_2 - x_3 \\ \text{subject to} & -x_1^2 - x_2^2 - x_3^2 + 4 \geq 0, \\ & a^Tx + 3 \geq 0, \end{array}$$

where $a \in I\!R^3$ is a given constant vector. Let $\widetilde{x} = (0\ 1\ 1)^T$.

(a) Determine a such that $\widetilde{x}$ fulfils the first-order necessary optimality conditions for (NLP). .... (6p)

(b) For the value on a which you determined in (1a), determine if $\widetilde{x}$ is a local minimizer to (NLP). .... (4p)

2. Consider the nonlinear optimization problem (NLP) defined as

$$(NLP) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \\ \text{subject to} & x_1 + x_2 + x_2^2 + 2 = 0. \end{array}$$

You have obtained a printout from an SQP solver for this problem. The initial point is $x = (0\ 0)^T$ and $\lambda = 0$. Six iterations, without linesearch, have been performed. The printout reads:

It	x_1	x_2	λ	$\Vert\nabla f(x) - \nabla g(x)\lambda\Vert$	$\Vert g(x)\Vert$
0	0	0	0	0	2
1	-1	-1	-1	2	1
2	-1.25	-0.25	-1.25	0.3750	0.5625
3	-1.7250	-0.4250	-1.7250	0.1663	0.0306
4	-1.7610	-0.3889	-1.7610	0.0026	0.0013
5	-1.7622	-0.3895	-1.7622	$1.5 \cdot 10^{-6}$	$4.0 \cdot 10^{-7}$
6	-1.7622	-0.3895	-1.7622	$2.8 \cdot 10^{-13}$	$9.2 \cdot 10^{-14}$

(a) Formulate the first QP subproblem. Verify that the solution to this QP subproblem is given by the printout above. .... (6p)

(b) How would the iterates change if the constraint in (NLP) would be changed to $x_1 + x_2 + x_2^2 + 2 \leq 0$? .... (2p)

(c) For the original problem (NLP), show that in this case the iterates converge to a global minimizer. (You need not verify the numerical values.) ...... (2p)

Note: According to the convention of the book we define the Lagrangian $\mathcal{L}(x, \lambda)$ as $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$, where $f(x)$ the objective function and $g(x)$ is the constraint function.

3. Consider the strictly convex bound-constrained quadratic program (QP) given by

$$(QP) \qquad \begin{array}{ll} \underset{x \in I\!R^3}{\text{minimize}} & \frac{1}{2} x^T H x + c^T x \\ \text{subject to} & x \geq 0, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Solve (QP) using an active-set method. Start at the point $x = (0\ 0\ 1)^T$ and let the constraints $x_1 \geq 0$ and $x_2 \geq 0$ be active in the first iteration. The linear equations that arise may be solved in any way that need not be systematic. Motivate each step carefully. .... (10p)

Hint: You may find the following relationship helpful:

$$\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 1 \end{pmatrix}.$$

4. Consider the nonlinear programming problem

$$(P) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are continuously differentiable.

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(g_i(x)).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} \nabla f(x) - \nabla g(x) \lambda &= 0, \\ g_i(x) \lambda_i - \mu &= 0, \quad i = 1, \ldots, m, \end{aligned}$$

assuming that $g(x) > 0$ and $\lambda > 0$ is kept implicitly. .... (4p)

(b) Let $x(\mu)$, $\lambda(\mu)$ be a solution to the primal-dual nonlinear equations of (4a) such that $g_i(x(\mu)) > 0$, $i = 1, \dots, m$, and $\lambda(\mu) > 0$. Show that $x(\mu)$ is a global minimizer to (P_μ) if f and $-g_i$, $i = 1, \dots, m$, are convex functions on $I\!R^n$. (2p)

(c) Derive the system of linear equations that results when the primal-dual nonlinear equations of (4a) are solved by Newton's method. .... (4p)

5. Consider the optimization problem

$$(NLP) \quad \begin{array}{ll} \underset{x \in I\!R^n}{\text{minimize}} & \frac{1}{2}\sum_{i \in \mathcal{U}} (p_i^T x - u_i)_+^2 + \frac{1}{2}\sum_{i \in \mathcal{L}} (l_i - p_i^T x)_+^2, \\ \text{subject to} & x \geq 0, \end{array}$$

where $\mathcal{L}$ and $\mathcal{U}$ are nonintersecting index sets such that $\mathcal{L} \bigcup \mathcal{U} = \{1, \dots, m\}$, and the subscript $''+''$ denotes the positive part, i.e., $x_+ = \max(x, 0)$. The constants u_i, $i \in \mathcal{U}$, and l_i, $i \in \mathcal{L}$, are known as well as the constant vectors p_i, $i = 1, \dots, m$. This means that we pay a quadratic penalty cost for violating lower bounds l_i, $i \in \mathcal{L}$, and upper bounds u_i, $i \in \mathcal{U}$, respectively.

The formulation (NLP) is straightforward, but a drawback is that the objective function is not twice-continuously differentiable. Your task is to show that we may obtain a smooth problem by introducing additional variables and constraints.

(a) Show that the objective function of (NLP) has continuous gradient but discontinuities in the Hessian. .... (2p)

(b) Show that (NLP) is equivalent to the quadratic programming problem

$$(QP) \quad \begin{array}{ll} \underset{x \in I\!R^n, y \in I\!R^m}{\text{minimize}} & \frac{1}{2}\sum_{i \in \mathcal{U}} y_i^2 + \frac{1}{2}\sum_{i \in \mathcal{L}} y_i^2, \\ \text{subject to} & y_i \geq p_i^T x - u_i, \ i \in \mathcal{U}, \\ & y_i \geq l_i - p_i^T x, \ i \in \mathcal{L}, \\ & x \geq 0. \end{array}$$

Do so by showing minimization over y in (QP) for a given x gives $y_i = (p_i^T x - u_i)_+$, $i \in \mathcal{U}$, and $y_i = (l_i - p_i^T x)_+$, $i \in \mathcal{L}$. .... (4p)

(c) For a given positive barrier parameter μ, formulate the perturbed first-order optimality conditions that are to be solved approximately if a primal-dual interior method is applied to (QP). .... (4p)

Note: The motivation for considering this reformulation is that we obtain a smooth problem. The increased dimensionality introduced by the y variables can be eliminated in the linear equations that are solved in a primal-dual interior method.

Good luck!