SF2822 Applied nonlinear optimization, final exam
Wednesday August 20 2025 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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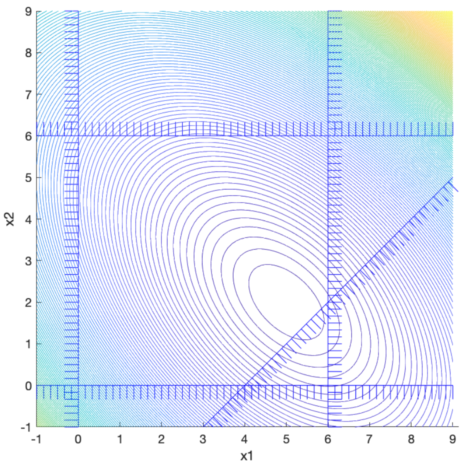
1. Consider the quadratic program (QP) defined by

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & Ax \geq b, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -12 \\ -9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -4 \end{pmatrix}.$$

The problem may be illustrated geometrically in the figure below,



(a) Solve (QP) by an active-set method. Start at $x = (6 \;\; 4)^T$ with the constraint $-x_1 \geq -6$ in the working set. You need not calculate any exact numerical values, but you may utilize the fact that the problem is two-dimensional, and make a pure geometric solution. Illustrate your iterations in the figure corresponding to Question 1a which can be found at the last sheet of the exam. Motivate each step carefully. ....(5p)

(b) Assume that b_5 is changed from -4 to -5, so that the fifth constraint reads $-x_1 + x_2 \geq -5$. Solve the corresponding quadratic program by an active-set method. Again, start at $x = (6 \;\; 4)^T$ with the constraint $-x_1 \geq -6$ in the working set. You need not calculate any exact numerical values, but you may utilize the fact that problem is two-dimensional, and make a pure geometric solution. Add the modified constraint and illustrate your iterations in the figure corresponding to Question 1b which can be found at the last sheet of the exam. Motivate each step carefully. .... (5p)

2. Consider the QP-problem (QP) defined as

$$(QP) \qquad \begin{array}{ll} \text{minimize} & \frac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} & x_1 + x_2 \geq 3. \end{array}$$

(a) For a given positive barrier parameter μ, find the corresponding optimal solution $x(\mu)$ and the corresponding multiplier estimate $\lambda(\mu)$ to the barrier-transformed problem. It is possible to obtain an analytical expression for this particular small problem. .... (5p)

(b) Show that $x(\mu)$ and $\lambda(\mu)$ which you obtained in Question 2a converge to the optimal solution and Lagrange multiplier respectively of (QP). .... (3p)

(c) Make an approximation of $\lambda(\mu) - \lambda^*$ which is valid for small positive values of μ. Is this as expected? .... (2p)

3. Consider the nonlinear program (NLP) given by

$$(NLP) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) \geq 0, \\ & x \geq 0, \\ & x \in I\!R^2, \end{array}$$

where $f : I\!R^2 \to I\!R$ and $h : I\!R^2 \to I\!R$ are twice-continuously differentiable functions, with f and $-h$ convex on $I\!R^2$.

Assume that $\widetilde{x} = (2 \; 2)^T$ is a local minimizer to (NLP) with corresponding Lagrange multiplier vector $\tilde{\lambda} = (2 \;\; 0 \;\; 0)^T$. The notation of the coursebook is used, so that with $g_1(x) = h(x)$, $g_2(x) = x_1$ and $g_3(x) = x_2$, the sign of λ is chosen such that $\mathcal{L}(x, \lambda) = f(x) - \lambda^T g(x)$.

It is known that

$$f(\widetilde{x}) = 5, \qquad \nabla f(\widetilde{x}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \nabla^2 f(\widetilde{x}) = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix},$$

$$h(\widetilde{x}) = 0, \qquad \nabla h(\widetilde{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nabla^2 h(\widetilde{x}) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

It turns out that the problem was not correctly posed, but that the correct problem is (NLP') given by

$$(NLP') \quad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) - \frac{1}{4} \geq 0, \\ & x \geq 0, \\ & x \in I\!R^2, \end{array}$$

i.e., the first constraint has been changed from $h(x) \geq 0$ to $h(x) - \frac{1}{4} \geq 0$.

(a) Give an estimate of the optimal value of (NLP') based on the knowledge of the solution of (NLP) and corresponding Lagrange multiplier vector. .... (2p)

(b) Assume that sequential quadratic programming is used for solving (NLP'). It is then natural to let $x^{(0)} = \widetilde{x}$ and let $\lambda^{(0)} = \tilde{\lambda}$. Perform one iteration, i.e. calculate $x^{(1)}$ and $\lambda^{(1)}$ given these values of $x^{(0)}$ and $\lambda^{(0)}$. We assume that no linesearch is needed. The quadratic programming problem that arises may be solved in any way, that need not be systematic. .... (8p)

4. Consider the semidefinite programming problem (P) defined as

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & G(x) \succeq 0, \end{array}$$

where $G(x) = \sum_{j=1}^{n} A_j x_j - B$ for B and A_j, $j = 1, \dots, n$, are symmetric $m \times m$-matrices. The corresponding dual problem is given by

$$(D) \quad \begin{array}{lll} \text{maximize} & \text{trace}(BY) & \\ \text{subject to} & \text{trace}(A_j Y) = c_j, & j = 1, \dots, n, \\ & Y = Y^T \succeq 0. & \end{array}$$

A barrier transformation of (P) for a fixed positive barrier parameter μ gives the problem

$$(P_\mu) \qquad \text{minimize} \quad c^T x - \mu \ln(\det(G(x))).$$

(a) Show that the first-order necessary optimality conditions for (P_μ) are equivalent to the system of nonlinear equations

$$\begin{aligned} c_j - \text{trace}(A_j Y) &= 0, \quad j = 1, \dots, n, \\ G(x)Y - \mu I &= 0, \end{aligned}$$

assuming that $G(x) \succ 0$ and $Y \succ 0$ are kept implicitly. .... (5p)

(b) Show that a solution $x(\mu)$ and $Y(\mu)$ to the system of nonlinear equations, such that $G(x(\mu)) \succ 0$ and $Y(\mu) \succ 0$, is feasible to (P) and (D) respectively with duality gap $m\mu$. .... (3p)

(c) In linear programming, when $G(x)$ and Y are diagonal, it is not an issue how the equation $G(x)Y - \mu I = 0$ is written. The linearizations of $G(x)Y - \mu I = 0$ and $YG(x) - \mu I = 0$ are then identical. Explain why this is in general not the case for semidefinite programming and how it can be handled. .... (2p)

Remark: For a symmetric matrix M we above use $M \succ 0$ and $M \succeq 0$ to denote that M is positive definite and positive semidefinite respectively. You may use the relations

$$\frac{\partial \ln(\det(G(x)))}{\partial x_j} = \text{trace}(A_j G(x)^{-1}) \quad \text{for} \quad j = 1, \dots, n,$$

without proof.

5. Consider the nonlinear optimization problems (NLP_1) and (NLP_2) defined as

$$(NLP_1) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \geq 0, \\ & x \in I\!R^n, \end{array}$$

and

$$(NLP_2) \qquad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & z - f(x) \geq 0, \\ & g(x) \geq 0, \\ & x \in I\!R^n, \ z \in I\!R, \end{array}$$

where $f : I\!R^n \to I\!R$ and $g : I\!R^n \to I\!R^m$ are twice-continuously differentiable.

Assume that $x^* \in I\!R^n$ together with $\lambda^* \in I\!R^m$ satisfy the second-order sufficient optimality conditions for (NLP_1).

Determine suitable values of z and Lagrange multipliers associated with the constraints $z - f(x) \geq 0$ and $g(x) \geq 0$ so that $x = x^*$ together with these determined values of z and the Lagrange multipliers of $z - f(x) \geq 0$ and $g(x) \geq 0$ satisfy the second-order sufficient optimality conditions for (NLP_2). Verify that these optimality conditions hold. . . . . (10p)

Good luck!

Name: .... Personal number: .... Sheet number: ....

Figure for Question 1a:

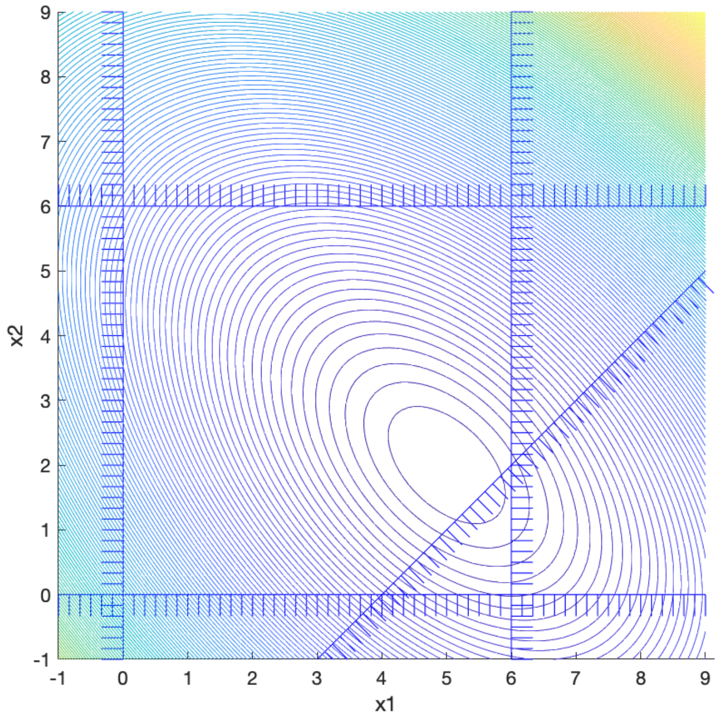


Figure for Question 1b: