Dynamics and Motion control
Lecture 2
Modelling and analysis of dynamics as a basis for control design and simulation

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1. Introduction

2. Mathematical descriptions of models

3. Dynamic analysis

4. Basic modeling

5. Linearization

6. Models of typical components and phenomena in mechatronic systems.

7. Example: Hydraulic actuator

8. Example: Brushless DC-Motor
Why models?

Some examples

• Model based feedback and feed forward control design
• Model based state estimation, non measurable states
• Model based failure diagnostics
• Model based Hardware In the Loop, HIL simulation
• Simulation for various purposes,
• Machine dynamics simulation.
• State machine models for simulation of logic algorithms

How good is your model?
How good does it have to be?
How do you measure the quality of a model?

Dynamics and Motion Control
Model characteristics

- **Physical properties**
  - *mechanical*
  - *electrical*
  - fluid mechanics
  - thermal etc.

- **System properties**
  - time variance vs *invariance*
  - *single* vs multivariable
  - *linear vs nonlinear*

- **Modelling strategy**
  - kinematic (motion without forces) / *dynamic*
  - (interaction of forces and motion) / static
  - *lumped* / distributed parameters
  - *continuous / discrete / state machines*
Model details complexity

- Real system
  - Detailed nonlinear model
  - Detailed linear model
  - Linear model

- Analysis
- Design
- Verification
- Parameter calculation
- Parameter identification through experiments
- Dynamics and Motion Control
- Lineariation
Lecture outline

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## Model types

### Continuous time

**Differential equations (time)**

$$\dot{y} = -ay + bu$$

**State space models (time)**

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

**Transfer functions (frequency)**

$$y(s) = \frac{b}{s + a} u(s)$$

### Discrete time

**Difference equations (time)**

$$y[n] = ay[n - 1] + bu[n]$$

**State space models (time)**

\[
\begin{align*}
x[n + 1] &= \Phi x[n] + \Gamma u[n] \\
y[n] &= Cx[n] + Du[n]
\end{align*}
\]

**Transfer functions (frequency)**

$$y(z) = \frac{b}{z + a} u(z)$$

---

Block diagrams for good physical insight

Dynamics and Motion Control
Example: state space model

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Force balance for the rolling mass,

\[ m\ddot{y} = \sum F_e = F - ky - d\dot{y} \]

Select states

\[ x_1 = y, \quad x_2 = \dot{y} \]

Model the derivatives of the state

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = \frac{1}{m}(F - kx_1 - dx_2) \]

Write in matrix form

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F \]
Differential equations are modeled by using integrators

\[ \dot{y} = \frac{1}{m} (F - ky - d\dot{y}) \]

The signals have real units, force, position etc. for increased understanding. It is a specification that you also can simulate.
Simple to extend to nonlinear behavior

Model of a nonlinear spring

\[ \ddot{y} = \frac{1}{m} (F - ky^2 - d\dot{y}) \]
Transfer function models

The Laplace transform of a time serie \( u(t) \) is defined as:

\[
L\{u(t)\} = \int_{0}^{\infty} u(t)e^{-st} \, dt
\]

A transfer function \( G(s) \), is the ratio of the output Laplace transform with the input Laplace transform.

\[
G(s) = \frac{L\{y(t)\}}{L\{u(t)\}}
\]

\[
Y(s) = G(s)U(s)
\]

Two important special cases: derivative and integration. If the initial conditions are zero, \( u(0)=0 \), then:

\[
L\left\{ \frac{d^n u}{dt^n} \right\} = s^n
\]

\[
L\left\{ \int_{0}^{\infty} u(t) \, dt \right\} = \frac{1}{s}
\]
Initial and final value theorems

**Final value theorem:**

\[ f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \]

Final value for a step input is

\[ f(\infty) = \lim_{s \to 0} sF(s) \cdot \frac{1}{s} = \lim_{s \to 0} F(s) \]

Example, final value for

\[ G(s) = \frac{1}{s + a} \]

with step input is \( \frac{1}{a} \)

**Initial value theorem**

\[ f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \]

For all \( G(s) \) with higher order denominator as numerator is the initial value for a step input zero.
Example: Transfer function

The transfer function can be calculated from the state space model.
You have to take a matrix inverse.
OK numerically in Matlab and symbolically in Maple

\[ L\{ \dot{x} = Ax + Bu \} \]
\[ sx - Ax = Bu \]
\[ x = (sI - A)^{-1} Bu \]
\[ y = Cx \]
\[ y = C(sI - A)^{-1} Bu \]
\[ G(s) = \frac{y}{u} = C(sI - A)^{-1} B \]

Direct calculation from the differential equation is OK for low order models

\[ L\{ m\ddot{y} = F - d\dot{y} - ky \} \]
\[ ms^2 y = F - dsy - ky \]
\[ \left( ms^2 + ds + k \right) y = F \]
\[ G(s) = \frac{1}{ms^2 + ds + k} \]
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Planes and tools

• Frequency domain
  • \( G(s) = G(j\omega) \), \( \omega \) is the frequency
• Complex pole-zero plane
  • Solve for \( s \) in numerator and denominator polynomials
• Time domain
  • The response \( y = G(s)u \) for different \( u \), e.g., step, ramp, etc.
\[ G(s) = \frac{N(s)}{D(s)} \]

Zeros: set \( N(s) = 0 \) and solve for \( s \).

Poles: set \( D(s) = 0 \) and solve for \( s \).

Poles and zeros can be plotted in the complex plane, the real part vs. the imaginary part.

The absolute value of \( s \), \( |s| = \sqrt{a^2 + b^2} \)

Represents a frequency rad/s:

- In time domain how fast a response to an input is.
- In the frequency plane (Bode) it represents a change in amplitude and phase.
- \(|s|\) is often called \( \omega_0 \)
The poles are the eigenvalues of the A matrix calculated by:

\[
\text{det}(sI - A)^{-1}
\]

The zeros depends on the output, that is: the C matrix
Different C matrix gives different zeros
Example: mass and spring

\[
x_1 = \text{position}
\]
\[
x_2 = \text{velocity}
\]
\[
u = \text{force}
\]

\[
\dot{x} = \begin{bmatrix}
0 & 1 \\
k/m & 0
\end{bmatrix}x + \begin{bmatrix}
0 \\
1
\end{bmatrix}F
\]

if:
\[
y = \begin{bmatrix}1 & 0\end{bmatrix}x
\]
then:
\[
G(s) = \frac{1}{ms^2 + k}
\]

if:
\[
y = \begin{bmatrix}0 & 1\end{bmatrix}
\]
then:
\[
G(s) = \frac{s}{ms^2 + k}
\]
Frequency domain response

For any transferfunction $G(s)$ with the input

$$u(t) = 1.0 \sin(\omega t)$$

will give the output

$$y(t) = a \sin(\omega t + \phi)$$

With the gain,

$$a = |G(j\omega)|$$

and the phase

$$\phi = \tan^{-1}\left(\frac{\text{imag}G(j\omega)}{\text{real}G(j\omega)}\right)$$
Integrator and derrivator

Derrivation

\[ G(j\omega) = j\omega \]
\[ |j\omega| = \omega \]
\[ \arg j\omega = \text{atan}(\omega/0) = \pi/2 \]

Integration

\[ G(j\omega) = \frac{1}{j\omega} \]
\[ |j\omega| = \frac{1}{\omega} \]
\[ \arg j\omega = \text{atan}(-\omega/0) = -\pi/2 \]

Poles and zeros:

**Integrator:**
- pole: \( s = 0 \)
- zero: none

**Derivate:**
- pole: none
- zero: \( s = 0 \)

Step response for an integrator

\[ G(s) = s, 10s, \frac{1}{s}, \frac{10}{s} \]

What is the step response for a Derrivator?
**First order polynomial**

Two ways of writing:

\[ G(s) = \frac{k}{s + a} \]

Good for frequency domain,

\[ G(s) = \frac{k}{\tau s + 1} \]

Good for time domain

**Characteristics are:**

- Pole
- Dc-gain
- Time constant
- Cut-off frequency
- Phase lag at high freq.

**Example:**

\[ G_1 = \frac{1}{s + 1} \]

\[ G_2 = \frac{5}{s + 10} \]
Second order TF in complex plane

Model:

\[ G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \]

Un-damped resonance frequency:

\[ |s| = \sqrt{(\zeta\omega_0)^2 + (1 - \zeta^2)\omega_0^2} = \omega_0 \]

Poles, complex conjugate when: \( \zeta < 1 \)

\[ s = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1} \omega_0 = -\zeta\omega_0 \pm j\sqrt{1 - \zeta^2} \omega_0 \]

Dynamics and Motion Control
Second order TF in time and frequency domains

Two models with same frequency but different damping

Low pass characteristics, 180 degree phase shift at high frequencies
Overshot, What is the damping ratio?

Dynamics and Motion Control
A second order model \( G_1(s) = \frac{100}{s^2 + 6s + 100} \) is superpositioned with a first order model \( G_2(s) = \frac{a}{s + a} \), such that \( G(s) = G_1(s)G_2(s) \).

In the left figure, \( a=2 \) and in the right figure, \( a=20 \).

The TF with the slowest pole dominates the step response.
Influence of a real zero

\[
G(s) = \frac{\left(\frac{s}{\omega_0 a} + 1\right)\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}
\]

\(a = [0.5, 1, 2, 4]\)

Dynamics and Motion Control
Higher order models: pole/zero in bode

\[ G_1(s) = \frac{s + 10}{(s + 1)(s + 10)}, \quad G_2(s) = \frac{s + 100}{(s + 1)(s + 10)}, \quad G_3(s) = \frac{s + 1}{(s + 10)(s + 100)} \]
Pole access

The number of poles and zeros equals the order of the denominator and nominator respectively.

For a TF we define the number of poles and zeros as $n_D$ and $n_N$

$$G(s) = \frac{N(s)}{D(s)}$$

The pole access is defined as: $n_A = n_D - n_N$

• TF with $n_A > -1$ are called proper.
• If $n_A = 0$, is the model output constant at high frequencies, a step response will give a nonzero initial value.
• If $n_A > 0$, is the model output zero at high frequencies, a step response has zero initial value
• If $n_A < 0$, is the model not proper, the gain at high frequencies is infinite, it is not possible to make a step response for such a model

Dynamics and Motion Control
Example: pole access

\[ G_1(s) = \frac{s + 1}{s + 10}, \quad n_A = 0 \]
\[ G_2(s) = \frac{s + 5}{(s + 10)(s + 1)}, \quad n_A = 1 \]
\[ G_3(s) = \frac{(s + 1)(s + 50)}{(s + 10)}, \quad n_A = -1 \]

Relationship between initial value and Dc-gain in frequency and time domain for models with different pole access. OBS! No step response for \( G_3(s) \)

\[ \text{dB} = 20 \log_{10}(\text{mag}) \]
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Modelling from physical properties

Mechatronic system design *Janscheck*

- section 2.3-2.3.4 (except the parts with Lagrange and Hamilton)
- section 2.3.8

- **Lumped models**
- **Descriptions of basic elements**
- **Energy storage and dissipative energy**
- **Mechanical, translational and rotation**
  - mass, inertia, damping, friction, stiffness
- **Electric**
  - Resistors, inductors, capacitors

Dynamics and Motion Control
Distributed vs. lumped parameters models

• A spring has a distributed mass, it gives a force when compressed or extended
  • The model is a partial differential equation with mass distribution
  • If the spring is first compressed and then released it starts to oscillate with zero speed at the fixed end.

• Modeling the spring as a massless spring and a point mass gives a lumped model with two elements.
  • The spring can now be modeled using ordinary differential equations with an equivalent mass $m$, and spring stiffness $k_f$. 
Different concepts of modelling

Mechatronic System

Lumped parameters
Energy conservation laws

Scalar energy functions

- Lagrange formalism
- Hamilton's equations

Port-Hamiltonian formalism

Multi-ports

- Bond graphs
- Kirchhoff networks
  - Power coupled
  - Power decoupled
- Signal-coupled networks

(hybrid) differential algebraic equations (DAE) system

(hybrid) state space model (ODE)

Dynamics and Motion Control
Trough and across variables

Mechanical components

$v_i$ velocity at node $i$

$f$ force through the component

$V = v_1 - v_2$ velocity across the component

Electrical components

$v_i$ voltage at node $i$

$i$ current through the component

$U = v_1 - v_2$ voltage across the component

How do you measure the variables?

Dynamics and Motion Control
## Elements that can store energy

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Physical element</th>
<th>Constitutive relation</th>
<th>Stored energy</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Inductance Diagram" /></td>
<td>Inductance</td>
<td>$L \frac{di}{dt} = U = v_1 - v_2$</td>
<td>$E = \frac{1}{2} Li^2$</td>
</tr>
<tr>
<td><img src="image" alt="Capacitance Diagram" /></td>
<td>Capacitance</td>
<td>$C \frac{dU}{dt} = i, U = v_1 - v_2$</td>
<td>$E = \frac{1}{2} Cu^2$</td>
</tr>
<tr>
<td><img src="image" alt="Translational Spring Diagram" /></td>
<td>Translational Spring</td>
<td>$f = ky, y = \int V dt$ [ \frac{1}{k} \frac{df}{dt} = V ]</td>
<td>$E = \frac{1}{2} ky^2$</td>
</tr>
<tr>
<td><img src="image" alt="Translational Mass Diagram" /></td>
<td>Translational Mass</td>
<td>$m \frac{dV}{dt} = f, V = v_1 = v_2$</td>
<td>$E = \frac{1}{2} mV^2$</td>
</tr>
</tbody>
</table>
### Rotational mechanical elements

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Physical element</th>
<th>Constitutive relation</th>
<th>Stored energy</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Rotational Spring Diagram" /></td>
<td>Rotational Spring</td>
<td>( T = k\varphi, \dot{\varphi} = \int \Omega , dt ) ( \dot{T} = k\phi )</td>
<td>( E = \frac{1}{2}k\varphi^2 )</td>
</tr>
<tr>
<td><img src="image2" alt="Rotational Mass Diagram" /></td>
<td>Rotational mass</td>
<td>( J \frac{d\Omega}{dt} = T, \Omega = \omega_1 = \omega_2 )</td>
<td>( E = \frac{1}{2}m\Omega^2 )</td>
</tr>
</tbody>
</table>
## Elements that only dissipates energy

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Physical element</th>
<th>Constitutive equation</th>
<th>Stored energy</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Resistance" /></td>
<td>Resistance</td>
<td>$U = Ri$</td>
<td>——</td>
</tr>
<tr>
<td><img src="image" alt="Damping" /></td>
<td>Damping (friction)</td>
<td>$f = dV$</td>
<td>——</td>
</tr>
<tr>
<td><img src="image" alt="Dry friction" /></td>
<td>Dry friction</td>
<td>$f = \mu mg$</td>
<td>——</td>
</tr>
</tbody>
</table>
Connecting basic elements

- Mechanical properties -> Newton's laws
- Electrical properties -> Kirchhoff's laws
- Parallel and series equations
  - Node and loop equations
- The principles of impedance and mobility
- The order of the differential equations equals the number of energy storage elements
Connection of elements

### Series connection

- **The velocity/voltage across the component**
  \[
  V_1 = v_1 - v_2 \\
  V_2 = v_2 - v_3 \\
  \sum V = 0
  \]
  \[
  V = V_1 + V_2 \\
  V = (v_1 - v_2) + (v_2 - v_3) \\
  V = v_1 - v_3
  \]

- **The force/current through the component**
  \[
  \text{The same:} \\
  f_1 = f_2
  \]

### Parallel connection

- **Trough C_1**
  \[
  f_1
  \]

- **Trough C_2**
  \[
  f_2
  \]

- **The same:**
  \[
  V_1 = V_2 = v_1 - v_2 \\
  \sum f = 0 \\
  f = f_1 + f_2
  \]
## Connection of components

<table>
<thead>
<tr>
<th>Mechanical:</th>
<th>Parallel Node equation</th>
<th>Series Loop equation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m \frac{dv}{dt} = \sum f$</td>
<td>$\frac{df}{dt} = k_i(v_{i_1} - v_{i_2}) - \cdots - k_n(v_{n_1} - v_{n_2})$</td>
</tr>
<tr>
<td></td>
<td>force balance equation or Newton's 2nd law</td>
<td>compatibility equation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Electrical:</th>
<th>$C \frac{dU}{dt} = \sum i$</th>
<th>$L \frac{di}{dt} = R_1(u_{1_1} - u_{1_2}) - \cdots - R_n(u_{n_1} - u_{n_2})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Kirchhoff's current law</td>
<td>Kirchhoff's voltage law</td>
</tr>
</tbody>
</table>

**Dynamics and Motion Control**
State space modelling steps

- Make a lumped sketch of the elements (for mechanical modeling)
- Make a free-body figure (mechanical) or circuit diagram (electrical)
- Give notation to parameters, node and loop variables
- Write the constitutive equations
  - Gives the states of the model
- Write the node and loop equations
- Eliminate unwanted variables
- Write the equations in matrix form
Example: Electric circuit

Constitutive equations:
\[ L \frac{dI}{dt} = U_L \]
\[ C \frac{dU_C}{dt} = I_C \]
\[ U_o = R I_R \]

Loop equations:
\[ U_C = U_R = U_o \]
\[ U_L + U_o = U_i \]

Node equation:
\[ I_L = I_C + I_R \]

State and output:
\[ x_1 = I_L \quad x_2 = U_C \]
\[ y = \begin{bmatrix} U_o \\ I_L \end{bmatrix} \]

Input:
\[ U_i \]

Eliminate:
\[ U_L \quad I_C \quad I_R \quad U_R \quad U_o \]

Model:
\[ \dot{x}_1 = \frac{1}{L} (U_i - x_2) \]
\[ \dot{x}_2 = \frac{1}{C} (x_1 - \frac{1}{R} x_2) \]

Matrix form:
\[ \dot{x} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U_i \]
\[ y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x \]
Example: Mechanical

Constitutive equations

\[ \begin{align*}
\frac{1}{k} \frac{df_k}{dt} &= V_k \\
m \frac{dV_1}{dt} &= f_m \\
f_d &= dV_1
\end{align*} \]

Loop equations

\[ \begin{align*}
V &= V_k + V_1 \\
V_k &= v_i - v_1 \\
V_1 &= v_1
\end{align*} \]

Node equation

\[ f_m = f_k - f_d \]

State and outputs

\[ \begin{align*}
x_1 &= f_k, \quad x_2 = v_1 \\
y &= \begin{bmatrix} v_1 \\ f_m \end{bmatrix}
\end{align*} \]

Input

\[ v_i \]

Eliminate

\[ \begin{align*}
V_k \\
f_m \\
V_1 \\
V \\
f_d
\end{align*} \]

Model

\[ \begin{align*}
\dot{x}_1 &= k(v_i - x_2) \\
\dot{x}_2 &= \frac{1}{m}(x_1 - dx_2)
\end{align*} \]

Matrix form

\[ \dot{x} = \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} k \\ 0 \end{bmatrix} v_i \]

\[ y = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix} x \]

Dynamics and Motion Control
Compare the mechanical and electric systems

Mechanical: \[
\begin{bmatrix}
\frac{1}{m} & k & d
\end{bmatrix}
\]

Electrical: \[
\begin{bmatrix}
\frac{1}{C} & \frac{1}{L} & \frac{1}{R}
\end{bmatrix}
\]

The mechanical system:

the mass and damper are in parallell!
Sometimes the position is needed as state or output of the model

$$y$$ is the position that corresponds to the velocity.
Select the states as position and velocity

$$x_1 = y_1, \quad x_2 = v_1$$

$$m \dot{v}_2 = f_k - f_d$$
$$f_k = k(y_i - y_1)$$
$$f_d = dv_1$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{k}{m}(y_i - x_1) - \frac{d}{m}x_2$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} y_i$$

$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$
Using impedance and mobility as modelling tools

For electric circuits \( U = Zi \) where \( Z \) is the impedance.

For mechanical systems \( V = Mf \) where \( M \) is the mobility.

Equivalent impedance and mobility for series connections
\[
Z_e = Z_1 + Z_2 + \cdots + Z_n
\]

Equivalent impedance and mobility for parallel connections
\[
\frac{1}{Z_e} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}
\]

Two elements in parallel
\[
Z_e = \frac{Z_1 Z_2}{Z_1 + Z_2}
\]

### Electrical

\[
Z_C = \frac{1}{Cs}
\]
\[
Z_L = Ls
\]
\[
Z_R = R
\]

### Mechanical

\[
M_m = \frac{1}{ms}
\]
\[
M_k = \frac{1}{ks}
\]
\[
M_d = \frac{1}{d}
\]

Dynamics and Motion Control
Divisions, getting other outputs

Voltage and velocity divisions

\[ V = V_1 + V_2 \]

\[ f = \frac{1}{Z_1 + Z_2} V \]
\[ V_1 = Z_1 f \]
\[ V_2 = Z_2 f \]

Current and force divisions

\[ V = \frac{Z_1 Z_2}{Z_1 + Z_2} f \]
\[ f_1 = \frac{1}{Z_1} V \]
\[ f_2 = \frac{1}{Z_2} V \]

Dynamics and Motion Control
The equivalent impedance from \( i \) to \( U_i \).

\[ U_i = Z_e i \]

\( C \) and \( R \) in parallel, \( Z_p = \frac{R / Cs}{1/Cs + R} = \frac{R}{RCs + 1} \)

\( L \) in serie \( Z_e = Ls + Z_p = \frac{LRCs^2 + Ls + R}{RCs + 1} \)

The output impedance from \( U_i \) to \( U_o \).

\[ U_o = Z_o U_i \]

\[ Z_o = \frac{Z_p}{Z_p + Z_L} \]

\[ Z_o = \frac{R}{RCLs^2 + Ls + R} \] What is the order of the model?

State space model

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U_i \\
y &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x
\end{align*}
\] What is the dc-gain?
The equivalent mobility from $f$ to $V$.

$V = v_i = M_e f$

$M_e = M_k + \frac{M_m M_d}{M_m + M_d}$

$M_e = \frac{s}{k} + \frac{1/(dms)}{1/(ms) + 1/d}$

$M_e = \frac{ms^2 + ds + k}{(ms + d)k}$

The velocity at node 1, $v_1$, using velocity division

\[
v_1 = \frac{\frac{M_m M_d}{M_m + M_d}}{M_k + \frac{M_m M_d}{M_m + M_d}} v_i
\]

\[
v_1 = \frac{k}{ms^2 + ds + k} v_i
\]

Transferfunction from $v_i$ to $v_1$ by:

\[
v_1(s) = C(sI - A)^{-1}B v_i(s)
\]

Calculated in Maple we get:

\[
v_1 = \frac{k}{ms^2 + ds + k} v_i
\]
Example where state space technique is simpler

Using node and loop equations

\[ m_1 \ddot{v}_1 = f - f_{d1} \]
\[ m_2 \ddot{v}_2 = f_{d1} - f_{d2} \]
\[ f_{d1} = d_1(v_1 - v_2) \]
\[ f_{d2} = d_2v_2 \]

\[ m_1 \ddot{v}_1 = f - d_1(v_1 - v_2) \]
\[ m_2 \ddot{v}_2 = d_1v_1 - (d_1 + d_2)v_2 \]

\[ x_1 = v_1, \quad x_2 = v_2 \]

\[ \dot{x} = \begin{bmatrix} \frac{-d_1}{m_1} & \frac{d_1}{m_1} \\ \frac{d_1}{m_2} & \frac{-d_1 + d_2}{m_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{m_1} \\ 0 \end{bmatrix} f \]

Dynamics and Motion Control
Same example with mobility technique

\[ V_3 = V_4 = V_{34} = v_2 \]  \hspace{2cm} \text{Parallel ->}  \hspace{2cm} M_{34} = \frac{M_{d1}M_{m1}}{M_{d1} + M_{m1}}

\[ V_2 = v_1 - v_2 \neq V_{34} \]  \hspace{2cm} \text{Series ->}  \hspace{2cm} M_{234} = M_2 + M_{34}

\[ V_{234} = V_2 + V_{34} = (v_1 - v_2) + v_2 = v_1 \]

\[ V_{234} = V_2 + V_{34} = (v_1 - v_2) + v_2 = v_1 \]

\[ V_1 = v_1 = V_{234} \]  \hspace{2cm} \text{Parallel ->}  \hspace{2cm} M_{1234} = \frac{M_1M_{234}}{M_1 + M_{234}}

Dynamics and Motion Control
Two systems with parallel connections, $V_1=V_2$

- **First order model to velocity**
  \[ v = \frac{1}{ms + d} f_i \]

- **Second order model to position**
  \[ y = \frac{1}{ms^2 + ds} f_i \]

- **Second order model to velocity**
  \[ v = \frac{s}{ms^2 + k} f_i \]

- **Second order model to position**
  \[ y = \frac{1}{ms^2 + k} f_i \]

Draw the step response for each model
1. Introduction
2. Mathematical descriptions of models
3. Dynamic analysis
4. Basic modeling
5. **Linearization**
6. Models of typical components and phenomena in mechatronic systems.
7. Example: Hydraulic actuator
8. Example: Brushless DC-Motor
Linearization

- A nonlinear model \( \dot{x} = f(x, u) \quad y = g(x, u) \) can be linearized around some operating point \( \{x_Q, u_Q\} \) by considering a neighbourhood around the operating point and approximating the nonlinear model with a truncated Taylor series.

Set \( x = x_Q + \Delta x \), \( u = u_Q + \Delta u \) and \( y = v_Q + \Delta v \), then

\[
\begin{align*}
\dot{x} & \approx f(x_Q, u_Q) + \frac{\partial f}{\partial x} \bigg|_{x = x_Q} \Delta x + \frac{\partial f}{\partial u} \bigg|_{u = u_Q} \Delta u \\
y & \approx g(x_Q, u_Q) + \frac{\partial g}{\partial x} \bigg|_{x = x_Q} \Delta x + \frac{\partial g}{\partial u} \bigg|_{u = u_Q} \Delta u \\
\end{align*}
\]

\[
\begin{align*}
\Delta x &= A \Delta x + B \Delta u \\
\Delta y &= C \Delta x + D \Delta u \\
\end{align*}
\]

\[
\begin{align*}
A &= \frac{\partial f}{\partial x} \bigg|_{x = x_Q} \\
B &= \frac{\partial f}{\partial u} \bigg|_{u = u_Q} \\
C &= \frac{\partial g}{\partial x} \bigg|_{x = x_Q} \\
D &= \frac{\partial g}{\partial u} \bigg|_{u = u_Q} \\
\end{align*}
\]
Example: pendulum

Differential equation

\[ J\ddot{\phi} = \sum M_y = -mg l \sin \phi \]
\[ J = ml^2 \]

Nonlinear state space model

\[ x_1 = \phi, \quad x_2 = \dot{\phi} \]
\[ x = f(x_1, x_2) \]
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -\frac{g}{l} \sin x_1 \]

Equilibrium point and Linearization

\( (\dot{x}_2 = 0) \Rightarrow x_1Q = 0 \)
\[ A = \frac{\partial f}{\partial x} \bigg|_{x = x_2} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos 0 & 0 \end{bmatrix} \]

Linear model

\[ x = x_1Q + \Delta x = \Delta x \]
\[ \dot{x}_1 = \Delta x_2 \]
\[ \dot{x}_2 = -\frac{g}{l} \Delta x_1 \]
Example: nonlinear spring, $f=ky^2$

**Nonlinear model**

$x_1 = y$, $x_2 = \dot{y}$

$\dot{x} = f(x)$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{1}{m}(mg - kx_1^2)$

**Differential eq.**

$m\ddot{y} = mg - ky^2$

**Equilibrium point**

$\dot{x} = 0$

$\Rightarrow$

$x_{1Q} = \sqrt{\frac{mg}{k}}$

$x_{2Q} = 0$

**Linearization**

$A = \frac{\partial f}{\partial x} \bigg|_{x=x_Q} = \begin{bmatrix} 0 & \frac{1}{2} \frac{mg}{k} \sqrt{m} \\ -2 & 0 \end{bmatrix}$

**Linearized model**

$\Delta \dot{x} = A \Delta x$

$y = \Delta x_1 + x_{1Q}$

See Simulink model
Lecture outline

1. Introduction
2. Mathematical descriptions of models
3. Dynamic analysis
4. Basic modeling
5. Linearization
6. Models of typical components and phenomena mechatronic systems
7. Example: Hydraulic actuator
8. Example: Brushless DC-Motor
**DC motor with permanent magnets in the stator**

**Electric part:**
The rotor winding has an inductance, a resistance and a back-emf voltage proportional to rotor velocity.

\[
U_i = U_L + U_r + E
\]

\[
U_i = L \frac{di}{dt} + Ri + K_{emf} \dot{\phi}
\]

**Mechanical part:**
A torque \( T_m \) between rotor and stator is proportional to rotor current.
The rotor inertia, \( J_R \)
A load \( T_L \) on the outgoing shaft.

\[
J_r \dot{\phi} = T_m - T_L - T_f
\]

\[
T_m = k_T i
\]
Gearbox model

\[ \varphi_l = n \varphi_m \]
\[ n > 1 \]

\[ J_r \] is the motors rotor inertia, \( J_g \) the gearbox inertia calculated on the motor side, \( J_l \) the inertia of a load connected to the gearbox output and \( n \) the gear ratio.

\[
(J_r + J_g) \ddot{\varphi}_r = T_m - T_g
\]

\[
J_l \ddot{\varphi}_l = n T_g - T_l
\]

solve for \( T_g \) in one eq, and put in the other.

\[
T_g = \frac{J_l \ddot{\varphi}_l}{n} + \frac{T_l}{n} = \frac{J_l \ddot{\varphi}_r}{n^2} + \frac{T_l}{n}
\]

\[
\left( J_r + J_g + \frac{J_l}{n^2} \right) \ddot{\varphi}_r = T_m - \frac{T_l}{n}
\]

Compare with an electrical transformer

Dynamics and Motion Control
Nonlinear friction model

**Standard Coulomb model**

\[ F_f(v) = F_c \text{sgn}(v) + dv \]

- \( F_f(v) \) is discontinuous and not unique, numerical problems.
- Friction force can be higher than applied force.

**Dynamic zero velocity model**

\[ \frac{dF}{dt} = \sigma_0 \left( 1 - \frac{F}{F_c} \text{sgn}(v) \right) v \]

- High stiffness, \( \sigma_0 = [10^3 \ldots 10^5] \) numerical problems.

**Kamop stick-slip model**

\[ -d_v < v < d_v \]
\[ v = 0 \]
\[ F_f = F_{applied} \]
\[ -d_v > v > d_v \]
\[ F_f = f(v) \]

- Small velocity error around zero velocity.

Dynamics and Motion Control
Implementation of Karnop’s friction model

$v =$ velocity
$F_a =$ applied torque
$F_f =$ friction torque
$F_c =$ Coulomb friction level
$d =$ velocity proportional friction
$d_v =$ velocity deadband

\[ F_f = F_c \text{sgn}(v) + dv \]

Dynamics and Motion Control
Coulumb friction

Dc-motor simulation with torque input and Coulumb friction

\[ T_{\text{applied}} = 1.5e^{-3}\sin(0.5t) \]

\[ T_c = 0.4e^{-3} \]

Blue line is applied torque
Green line is friction torque

Typical velocity with Coulomb friction

\[ J\ddot{\phi} = T_{\text{applied}} - T_{\text{friction}} \]
Higher order dynamics in moving machine parts

- All material has finite stiffness
- Lumped models with mass, spring and damper
  - Multi Body Systems, MBS
- Resonance and anti resonance frequencies,
- Gives phase lag which can make feedback systems instable
- For a general theory on MBS see any textbook in Robotics or for an introduction, Jansheck chapter 4.
- Reading material Jansheck section 4.4 – 4.7.5
- Which frequencies can affect a feedback system in a negative way
Two basic types of MBS systems

Machines where parts can move with relative motion in different coordinate systems

Machines where the relative motion is because of flexible (not stiff) parts. Same coordinate system.

Dynamics and Motion Control
General nonlinear model of MBS systems

Based on Newton Euler can a general matrix based equations of motion be written as the nonlinear model

\[ M(q,t)\ddot{q} + g(q,\dot{q},t) = f(q,\dot{q},t) \]

Where:

\[ q \in \mathbb{R}^{N_{DOF}} \] are \( N_{DOF} \) the minimal number of generalized coordinates

\[ M \in \mathbb{R}^{N_{DOF} \times N_{DOF}} \] is the mass matrix

\[ g \in \mathbb{R}^{N_{DOF}} \] generalized spring, damping, Coriolis forces

\[ f \in \mathbb{R}^{N_{DOF}} \] generalized external forces
Linearized model of MBS

Linearizing around a stable position \( q_{*0} \) gives that \( q(t) = q_{*0} + y(t) \) and the equations of motion as

\[
M\ddot{y} + (B + G)\dot{y} + (K + N)y = f(t)
\]

Where all matrices are \( N_{DOF} \times N_{DOF} \)

\[
M = M^T, M\ddot{y} \text{ are the inertial forces}
\]

\[
B = B^T, By \text{ are the damping forces}
\]

\[
G = -G^T, Ny \text{ are the gyroscopic forces}
\]

\[
K = K^T, Ky \text{ are the spring forces}
\]

\[
N = -N^T, Ny \text{ are the non-conservative forces}
\]

N is always zero for our models
Structured modeling of MBS with flexible linkage

\[
M = \text{diag}(m_1 \ m_2 \ m_3 \ m_4 \ \cdots \ m_N)
\]

\[
K = \begin{bmatrix}
k_1 + k_2 & -k_2 \\
-k_2 & k_2 + k_3 & -k_3 \\
-k_3 & k_3 + k_4 & -k_4 & \ddots \\
& & & -k_N \\
& & -k_N & k_N + k_{N+1}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
b_1 + b_2 & -b_2 \\
-b_2 & b_2 + b_3 & -b_3 \\
& -b_3 & b_3 + b_4 & -b_4 & \ddots \\
& & & -b_N \\
& & -b_N & b_N + b_{N+1}
\end{bmatrix}
\]

Define a state vector

\[
x_i = (y_i \ \dot{y}_i)^T
\]

Gives the state space model

\[
x = Ax + Bf = \begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}B \end{bmatrix}x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}f
\]
**Dynamic of the MBS**

**Poles** are simply calculated as the eigenvalues of the $A$ matrix.

The **zeros** and therefore also the frequency response depends on which mass is actuated and which mass is measured. That is, on which row in the $B$ matrix and which column in the $C$ matrix.

**Example: 2 mass**

- **Bode Diagram**
- **Observe phase difference**

- From $F_a$ to $y_1$
  - Four poles and two zeros

- From $F_a$ to $y_2$
  - Four poles and no zeros

*Dynamics and Motion Control*
**Dc motor with load and week shaft**

Stiff shaft model (with gear ratio 1.0)

\[ T_m = k_T i \]

\[ \varphi_1 = \varphi_2 \]

\[ T = \varphi_1 \]

\[ s_i = b_1 \]

\[ b_3 \]

\[ J_m \]

\[ J_1 \]

\[ \varphi_1 = \frac{b_s}{s(s+a_s)} i \]

Where:

\[ b_s = \frac{k_T}{J_m+J_1} \]

\[ a_s = \frac{b_1 + b_2}{J_m+J_1} \]

Shaft with torisational spring and damper model

\[ T_m = k_T i \]

\[ k_2 \]

\[ J_m \]

\[ J_1 \]

\[ b_1 \]

\[ b_2 \]

\[ b_3 \]

\[ M = \text{diag}(J_m, J_1) \]

\[ K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \]

\[ k_1 = k_3 = 0 \]

\[ B = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix} \]

\[ \varphi_1 = \frac{b_s}{s(s+a_s)} \left( \frac{s^2/\omega_a^2 + 2\zeta_a s/\omega_a + 1}{s^2/\omega_a^2 + 2\zeta_0 s/\omega_0 + 1} \right) i \]
Dc motor with weak shaft

\[ G_w(s) = \frac{b_w}{s(s + a_w)} \frac{s^2 / \omega_d^2 + 2\zeta_a s / \omega_d + 1}{s^2 / \omega_0^2 + 2\zeta_0 s / \omega_0 + 1} \]

\[ G_s(s) = \frac{b_s}{s(s + a_s)} \]

If \( |a_s| \) is sufficiently smaller than \( \omega_0 \)

Then:

\[ G_w(s) \approx \frac{b_s}{s(s + a_s)} \frac{s^2 / \omega_d^2 + 2\zeta_a s / \omega_d + 1}{s^2 / \omega_0^2 + 2\zeta_0 s / \omega_0 + 1} \]
Example: Identify MBS model

Simplest approach is to make a step response to velocity and measure time constant, $1/a$ and resonance frequency $\omega_0$.

Model parameters:

$m_1 = 1$, $m_2 = 2$

$k_2 = 1000$

$b_1 = 4$, $b_2 = 1$, $b_3 = 1$

$a \approx 1/0.6 = 1.67$

$\omega_0 \approx \frac{2\pi}{T_0} = 42$
Antiresonance frequency

$$\omega_a = \omega_0 \sqrt{\frac{m_1}{m_1 + m_2}} = 24.2$$

Gives the parametric model

$$G = \frac{0.2 \cdot 1.67 \frac{(s/24.2)^2 + 1}{s + 1.67} \frac{(s/42)^2 + 1}{s}}$$

Compare step response

Red lines original model
Blue lines identified model
Model of: Backlash or play (glapp på svenska)

Simple model:

\[
\begin{align*}
    z_2 &= z_1 - z_t & \text{for} & \quad \dot{z}_1 > 0 \\
    z_2 &= z_1 + z_t & \text{for} & \quad \dot{z}_1 < 0
\end{align*}
\]

Spring loaded model:

\[
\begin{align*}
    z_2 &= z_1 - z_t & \text{for} & \quad z_1 \geq 0 \\
    z_2 &= z_1 + z_t & \text{for} & \quad z_1 \leq 0 \\
    z_2 &= 0 & \text{for} & \quad -z_t < z_1 < z_t
\end{align*}
\]

Dynamics and Motion Control
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7. **Example: Hydraulic actuator**
8. Example: Brushless DC-Motor
Hydraulic systems

- Pressure difference is the *across* variable
- Volume flow is the *through* variable
- Node and loop equations
- Fluid capacitance and fluid resistance
- Volume and pressure sources -> Pumps
- Flow and pressure control valves -> servo valves
- Fluid to mechanical transformers -> cylinders and motors
- Modeling example: flow controlled hydraulic cylinder
Hydraulic components

**Across type**

\[ C_f \frac{dP}{dt} = Q \]

\( C_f = \text{fluid capacitance} \)

**Through type**

\[ I_f \frac{dQ}{dt} = P \]

\( I_f = \text{fluid inerstance} \)

**Disipative type**

\[ R_f Q = P \]

\( R_f = \text{fluid resistance} \)

**Not so important!**

\[ p = \rho gh = \frac{\rho g}{A} V \]

\[ C_f = \frac{A}{\rho g} \]

\[ C_f \frac{dp}{dt} = \frac{dV}{dt} = Q \]

\( V = \text{liquid volume} \quad [m^3] \)

\( A = \text{cross sectional area} \quad [m^2] \)

\( Q = \text{volume flow} \quad \left[\frac{m^3}{s}\right] \)

\( p = \text{pressure} \quad \left[Pa, \frac{N}{m^2}\right] \)

For a circular pipe

\[ I_f = \frac{\rho l}{A} \]

\( \rho = \text{density} \quad \left[\frac{kg}{m^3}\right] \)

\( l = \text{length of pipe} \quad [m] \)

\( A = \text{cross sectional area of pipe} \quad [m^2] \)

\[ Q = R_f \sqrt{(p_i - p_o)} \]
Compressibility of hydraulic oil

Density increase, (volume decrease) of hydraulic oil is more than 100 times larger then that of steel. So it can not be neglected.

Bulk modulus, \[ \beta = -V \left( \frac{\partial p}{\partial V} \right) = \rho \left( \frac{\partial p}{\partial \rho} \right) \approx 2 \cdot 10^6 \left[ \frac{N}{m^2} \right] \]

density, \( \rho = \frac{m}{V} \left[ \frac{kg}{m^3} \right] \)

Mass flow into a constant volume, \( \dot{m} = Q \rho = \frac{d}{dt} (V_0 \rho) = V_0 \frac{d \rho}{dt} \)

From definition, \( d \rho = \frac{\rho}{\beta} d p \)

Hence, \( Q = V_0 \frac{d p}{\beta \frac{d t}{dt}} \)

With, \( C_f = \frac{V_0}{\beta} \)

Constitutive equation

Dynamics and Motion Control
There are a lot of hydraulic details in a system but we will concentrate on a few components that are important for the dynamics.

- Cylinder
- Flow control valve
- Variable displacement pump
4-way 3 position directional valve (closed center)

From pump

To tank

Spool to the right

Spool to the left

Load pressure

$\Delta P = P_1 - P_2 > 0$

Load pressure

$\Delta P = P_1 - P_2 < 0$

Standard液压阀组件草图

Dynamics and Motion Control
Spool valve model

- Spool assumptions
  - No leakage, equal cylinder actuator areas
  - Sharp edged, steady flow
  - Opening area proportional to $x_v$
  - Return pressure is zero
  - Symmetrical

Orifice model for sharp edged orifice:

$Q$, flow  
$C_d$, Discharge constant  
$A_o$, effective opening area  
$\rho$, density  
$\Delta p$, pressure drop over orifice  
$R_v$, a constant given by valve data sheet

\[ Q = C_d A_o \sqrt{\frac{2 \Delta p}{\rho}} \left[ \frac{m^3}{s} \right] \]

set:

\[ Q_1 = R_v \sqrt{p_s - p_1 x_v} \quad x_v > 0 \]
\[ Q_1 = R_v \sqrt{p_2 - p_t x_v} \]
\[ Q_1 = R_v \sqrt{p_1 - p_t x_v} \quad x_v < 0 \]
\[ Q_1 = R_v \sqrt{p_s - p_2 x_v} \]
The complete model

Constitutive equations:

\[ m \dot{v} = f_m \]
\[ C_f \dot{p}_1 = Q_1 \]
\[ C_f \dot{p}_2 = Q_2 \]
\[ Q_{1v} = R_v \sqrt{p_s - p_1 x_v} \]
\[ Q_{2v} = R_v \sqrt{p_2 x_v} \]

Node eq.

\[ f_m = p_1 A - p_2 A - f_f - f_e \]
\[ Q_1 = Q_{1v} - Q_c \]
\[ Q_2 = -Q_{2v} + Q_c \]

The valve dynamics, spool mass and solenoid must be modeled. Physical model is difficult, flow forces on spool.

A second order model from valve input signal to spool position is usually sufficient. Parameters from valve data sheets.

\[ x_v = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} u \]

Volume flow due to piston velocity.

\[ Q_c = Av \]

Dynamics and Motion Control
Linearizing the model

Linearize around an operating point \( p_1Q, p_2Q \) and \( x_vQ \), assume \( p = f_f = 0 \)

\[
0 = p_1QA - p_2QA - f_e \Rightarrow p_1Q = p_2Q - \frac{f_e}{A}
\]

\[
0 = R_v\sqrt{p_s - p_1Q}x_v - Av \Rightarrow \left( \frac{Av}{C_vx_v} \right)^2 = p_s - p_1Q
\]

\[
0 = -R_v\sqrt{p_2Q}x_v + Av \Rightarrow \left( \frac{Av}{C_vx_v} \right)^2 = p_2Q
\]

Define, \( R_i, K_i \)

\[
Q_1 = K_1\Delta x_v + R_i\Delta p_i
\]

\[
Q_2 = K_2\Delta x_v + R_2\Delta p_2
\]

where:

\[
p_1 = p_1Q + \Delta p_1
\]

\[
p_2 = p_2Q + \Delta p_2
\]

\[
x_v = x_vQ + \Delta x_v
\]

\[
R_1 = \left| \frac{\partial Q_1}{\partial p_1} \right|_{p_1=p_1Q, x_v=x_vQ} = -\frac{R_vx_vQ}{2\sqrt{p_s - p_1Q}} = -\frac{1}{\sqrt{2}} \frac{R_vx_vQ}{\sqrt{p_s - \frac{f_e}{A}}}
\]

\[
R_2 = \left| \frac{\partial Q_2}{\partial p_2} \right|_{p_2=p_2Q, x_v=x_vQ} = -\frac{R_vx_vQ}{2\sqrt{p_2Q}} = -\frac{1}{\sqrt{2}} \frac{R_vx_vQ}{\sqrt{p_s - \frac{f_e}{A}}}
\]

\[
K_1 = \left| \frac{\partial Q_1}{\partial x_v} \right|_{p_1=p_1Q, x_v=x_vQ} = R_v\sqrt{p_s - p_1Q} = R_v\sqrt{\frac{p_s}{2} - \frac{f_e}{2A}}
\]

\[
K_2 = \left| \frac{\partial Q_2}{\partial x_v} \right|_{p_2=p_2Q, x_v=x_vQ} = -R_v\sqrt{p_2Q} = -R_v\sqrt{\frac{p_s}{2} - \frac{f_e}{2A}}
\]
Linear model

Select states:
\[ x_1 = x_v, x_2 = v_v, x_3 = v, x_4 = p_1, x_5 = p_2 \]

\[ d, \text{linear friction coefficient} \]

\[
\dot{x} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-\omega_v & -2\zeta\omega_v & 0 & 0 & 0 \\
0 & 0 & -\frac{d}{m} & \frac{A}{m} & -\frac{A}{m} \\
\frac{K_1}{C_f} & 0 & -\frac{A}{C_f} & \frac{R_1}{C_f} & 0 \\
\frac{K_2}{C_f} & 0 & \frac{A}{C_f} & 0 & \frac{R_2}{C_f}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} u
\]

Step response

\[ p_s = 20 [\text{MPa}] \]
\[ m = 100 [\text{kg}] \]
\[ A_c = \pi 0.025^2 [\text{m}^2] \]
1. Introduction

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4. Basic modeling

5. Linearization

6. Models of typical components and phenomena in mechatronic systems.

7. Example: Hydraulic actuator

8. **Example: Brushless DC-Motor**
3-phase electric motors

- Asynchronous machines have windings in both stator and rotor
- Permanent magnet 3-phase motors have only winding in stator
  - Also called Synchronous motors (rpm synchronous to electric field rotation)
- Two types
  - Brushless DC motor BLDC or Trapezoidal motor
  - Permanent Magnet Synchronous Machine PMSM or Sinusoidal motor
- Advantage over DC-motor
  - cooling -> higher currents and/or smaller size
- Disadvantage over DC-motor
  - More advanced control -> electronic commutation (software)
Electromechanical design

8-pole motor (4 magnets)

2-pole motor (1 magnet)

http://www.stefanv.com/rcstuff/qf200212.html
Back EMF depends on motor design

Trapezoidal EMF

Sinewave EMF

Different Modeling and control strategies are used for the two kinds
BLDC control structure (trapezoidal)

BLDC speed control

- $n^*$
- $n_{act}$
- PWM
- 3 Phase Inverter
- BLDC

Commutation Sequence
Hall Sensors

BLDC speed and current cascaded control

- $n^*$
- $n_{act}$
- $i^*$
- $i_{act}$
- PWM
- 3 Phase Inverter
- BLDC

Commutation Sequence
D/A
Hall Sensors

Dynamics and Motion Control
Phase currents are sampled synchronously to PWM signals
Commutation of trapezoidal motor (BLDC)

Dynamics and Motion Control
BLDC model structure

Modeling steps:
1. Set up the differential equations for the phase currents
2. Model the shape of the EMF and flux
3. Calculate the electric torque
4. Model the commutation logic based on hall sensors or position
5. Model the inverter
4 and 5 can be modeled in one state machine (state flow)

Dynamics and Motion Control
1. Differential equations for phase currents

\[
\begin{bmatrix}
U_{ab} \\
U_{bc} \\
U_{ca}
\end{bmatrix} = \left( R + L \frac{d}{dt} \right) \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix} + \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
e_a \\
e_b \\
e_c
\end{bmatrix}
\]

\[
i_c = -i_a - i_b
\]

\[
\begin{bmatrix}
U_{ab} \\
U_{bc}
\end{bmatrix} = \left( R + L \frac{d}{dt} \right) \begin{bmatrix}
1 & -1 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
i_a \\
i_b
\end{bmatrix} + \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
e_a \\
e_b
\end{bmatrix}
\]

\[
d \frac{d}{dt} i_a = \frac{1}{L} \left( -R i_a + \frac{2}{3} (U_{ab} - E_{ab}) + \frac{1}{3} (U_{bc} - E_{bc}) \right)
\]

\[
d \frac{d}{dt} i_b = \frac{1}{L} \left( -R i_b + \frac{1}{3} (U_{ab} - E_{ab}) + \frac{1}{3} (U_{bc} - E_{bc}) \right)
\]

\[
z = Ri + L \frac{d}{dt} i
\]

Same \( R \) and \( L \) in each phase

Phase to phase voltages

Dynamics and Motion Control
2. Back EMF model

- $e_a = f(\theta)K_e\omega_e$
- $e_b = f\left(\theta - \frac{2\pi}{3}\right)K_e\omega_e$
- $e_c = f\left(\theta - \frac{4\pi}{3}\right)K_e\omega_e$

\[
f = \begin{cases} 
1, & 0 \leq \theta < \frac{2\pi}{3} \\
1 - \frac{6}{\pi}\left(\theta - \frac{2\pi}{3}\right), & \frac{2\pi}{3} \leq \theta < \pi \\
-1, & -1, \pi \leq \theta < \frac{5\pi}{3} \\
\frac{6}{\pi}\left(\theta - \frac{5\pi}{3}\right), & -1, \frac{5\pi}{3} \leq 2\pi
\end{cases}
\]

A simple way to simulate is to, take $\cos(f(\theta))$ and saturate it between -0.5…0.5 and then multiply it with 2. Which is how the plot above has been done.
3. Electric torque

Same shape as the EMF

\[ T_e = K_t \left( f(\theta) i_a + f(\theta - \frac{2\pi}{3}) i_b + f(\theta - \frac{4\pi}{3}) i_c \right) \]
4. Commutation logic

One way to find the correct commutation sequence is to calculate the phase to phase EMF, \( E_{xy} = e_x - e_y \). Maximum magnetic torque is achieved when the phase currents are flow in the same direction, for example for \( E_{ab} \) should \( i_a > 0 \) and \( i_b < 0 \). Which is achieved with \( U_{ab} = V_s \).

See xxx for proof.

Energized phases ->
- \( U_{ca} = -V_s \)
- \( U_{bc} = V_s \)
- \( U_{ab} = -V_s \)
- \( U_{ca} = V_s \)
- \( U_{bc} = -V_s \)
- \( U_{ab} = V_s \)

Closed transistors ->
5. Model the inverter

Each energized state must be modeled separately
Let’s start with state $Q_1 Q_4$ when $U_{ab} = V_s$
What is then $U_{bc}$?

Redraw the motor inverter system for easier analysis

Is phase C connected to plus or ground?
It depends on the direction of the current in C from previous state
For positive direction (rotation) was previous state $U_{bc} = -V_s$, $Q_5 Q_4$ closed
Gives $i_c > 0$
5. Model the inverter

Now can we calculate $U_{bc}$

$U_{bc} = 0$ short circuit between B and C

After some time the current in C will become zero, what happens then?
5. Model the inverter

Equivalent circuit when \( i_c = 0 \)

There are two loops, one directly from B to C and one via A

Loop BC, \(-e_b - Zi_b + e_c + U_{bc} = 0\)

Loop BAC, \(V_s - e_a - Zi_a + e_c + U_{bc} = 0\)

\(i_c = 0\), gives \(i_b = -i_a\)

hence:

\[ U_{bc} = \frac{1}{2}(-V_s + e_a + e_b - 2e_c) = \frac{1}{2}(-V_s + E_{ac} + E_{bc}) \]

This must be done for all six states in both directions of rotation but, only for \(U_{ab}\) and \(U_{bc}\) since \(U_{ca}\) is not needed in the differential equations.
5. Model the inverter

\[ i_n > 0 \text{ or } i_n < 0 \]
depends on state

\[ U_{ab} = \ldots \]
\[ U_{bc} = \ldots \]

\[ i_n = 0 \]

Transition from previous state

Position > \( xx \)

\[ i_n > 0 \text{ or } i_n < 0 \]
depends on state

\[ U_{ab} = \ldots \]
\[ U_{bc} = \ldots \]

\[ i_n = 0 \]

Dynamics and Motion Control
Voltage step input

Voltage step input

Phase to phase voltage $U_{ab}$

Phase to phase voltage $U_{bc}$

Dynamics and Motion Control
Step response cont.

Currents

Torque ripple because of the commutation

Dynamics and Motion Control