Dynamics and Motion control
Lecture 2
Modelling and analysis of dynamics as a basis for control design and simulation

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Lecture outline

1. **Introduction**
2. Mathematical descriptions of models
3. Dynamic analysis
4. Basic modeling
5. Linearization
6. Models of typical components and phenomena in mechatronic systems.
7. Example: Hydraulic actuator
8. Example: Brushless DC-Motor

Why models ?

Some examples

- Model based feedback and feed forward control design
- Model based state estimation, non measurable states
- Model based failure diagnostics
- Model based Hardware In the Loop, HIL simulation
- Simulation for various purposes,
- Machine dynamics simulation.
- State machine models for simulation of logic algorithms

How good is your model?
How good does it have to be?
How do you measure the quality of a model?

Model characteristics

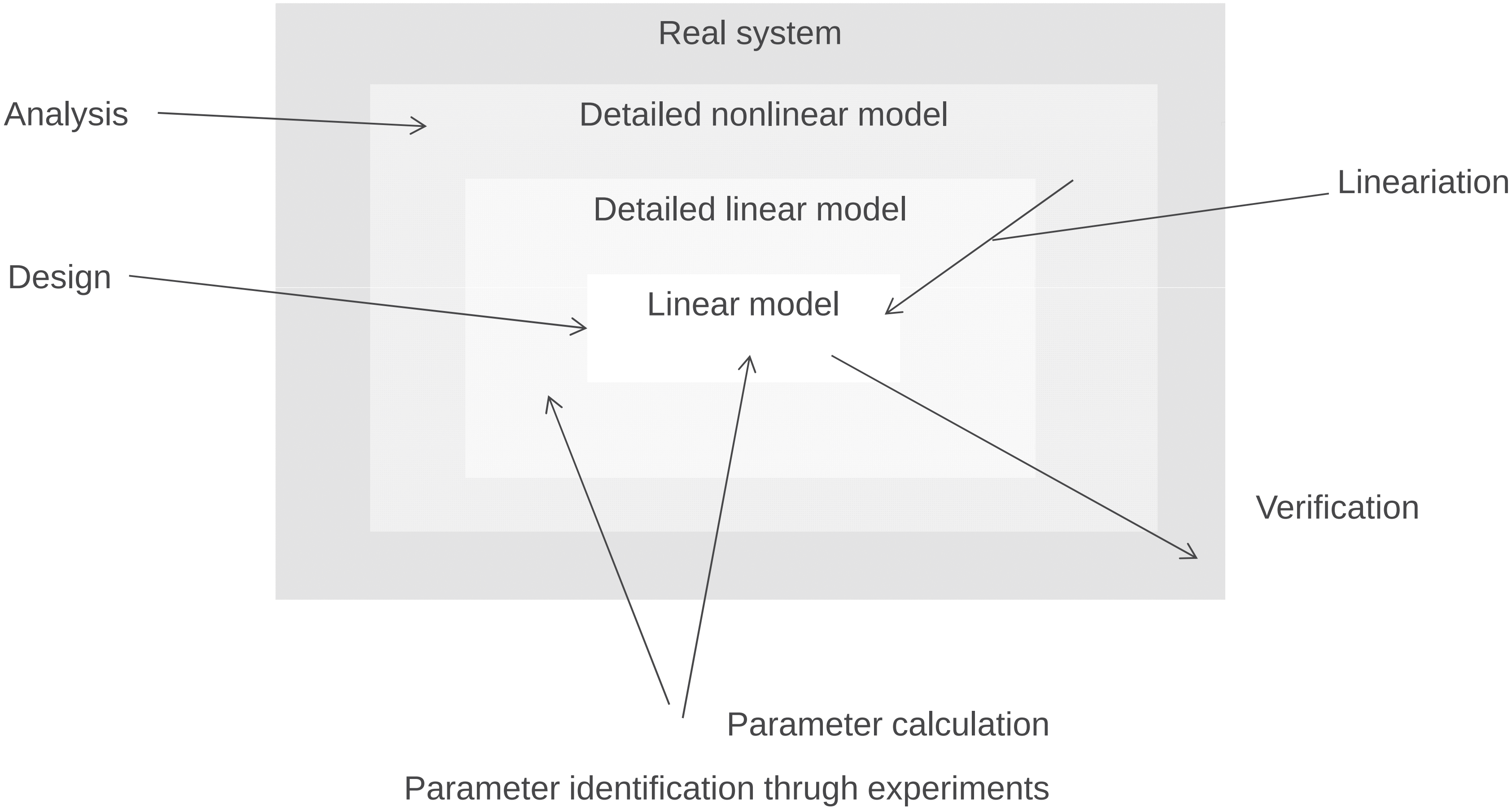
- Physical properties
  - *mechanical*
  - *electrical*
  - fluid mechanics
  - thermal etc.
- System properties
  - time variance vs *invariance*
  - *single* vs multivariable
  - *linear vs nonlinear*
- Modelling strategy
  - kinematic (motion without forces) /
    dynamic
  - (interaction of forces and motion) / static
  - *lumped* / distributed parameters
  - *continuous / discrete /state machines*

$$\dot{y}(t) = -ay(t) + bu(t)$$

$$\dot{y}(t) = -a(t)y(t) + b(t)u(t)$$

$$\dot{y}(t) = -a\sin y(t) + bu(t)$$

Model details complexity



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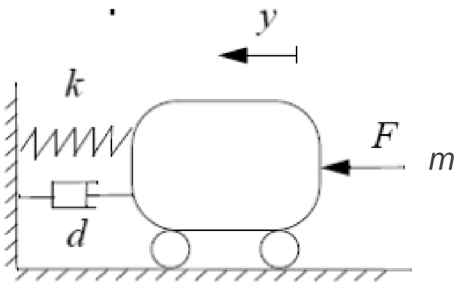
Model types

Continuous time	Discrete time
Differential equations (time)	Difference equations (time)
$\dot{y} = -ay + bu$	$y[n] = ay[n-1] + bu[n]$
State space models (time)	State space models (time)
$\dot{x} = Ax + Bu$ $y = Cx + Du$	$x[n+1] = \Phi x[n] + \Gamma u[n]$ $y[n] = Cx[n] + Du[n]$
Transfer functions (frequency)	Transfer functions (frequency)
$y(s) = \frac{b}{s+a} u(s)$	$y(z) = \frac{b}{z+a} u(z)$

Block diagrams for good physical insight

Example: state space model



$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Force balance for the rolling mass,

$$m\ddot{y} = \sum F_e = F - ky - d\dot{y}$$

Select states

$$x_1 = y, \quad x_2 = \dot{y}$$

Model the derivatives of the state

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(F - kx_1 - dx_2)$$

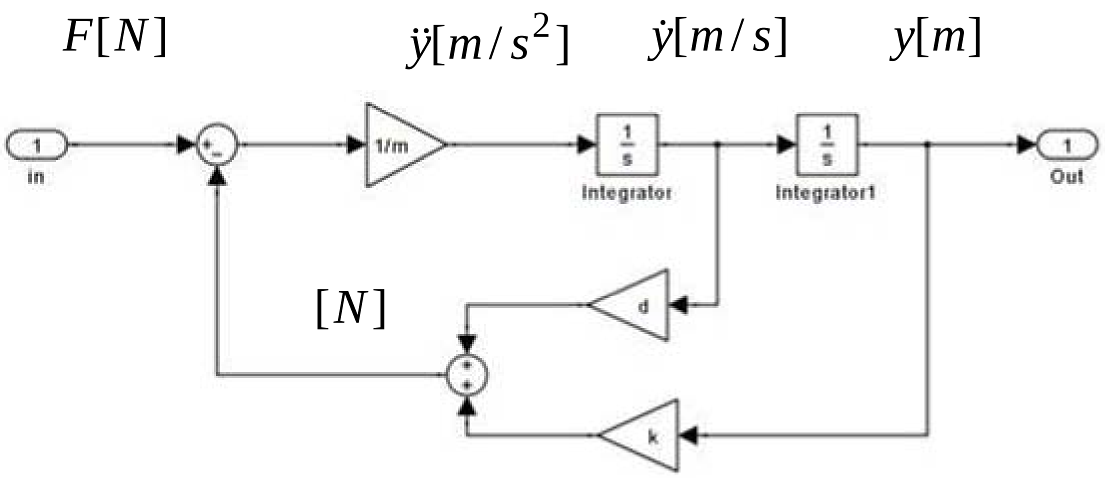
Write in matrix form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

Example: Block diagram

Differential equations are modeled by using integrators $\frac{1}{s}$

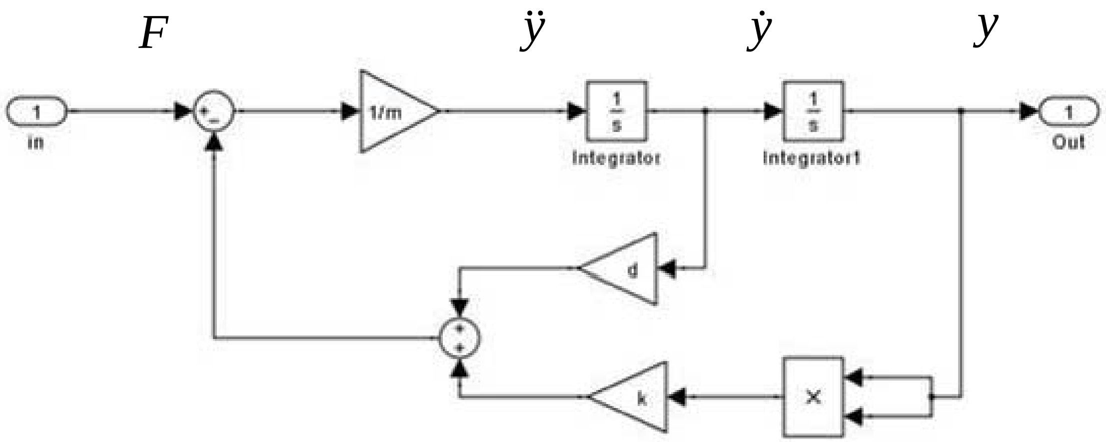
$$\ddot{y} = \frac{1}{m}(F - ky - d\dot{y})$$



The signals have real units, force, position etc. for increased understanding. It is a specification that you also can simulate.

Model of a nonlinear spring

$$\ddot{y} = \frac{1}{m}(F - ky^2 - d\dot{y})$$



Transfer function models

The Laplace transform of a time serie *u(t)* is defined as:

$$L\{u(t)\} = \int_0^\infty u(t)e^{-st}dt$$

A transfer function *G(s)*, is the ratio of the output Laplace transform with the input Laplace transform.

$$G(s) = \frac{L\{y(t)\}}{L\{u(t)\}}$$

$$Y(s) = G(s)U(s)$$

Two important special cases: derivative and integration. If the intitial conditions are zero, *u(0)=0*, then:

$$L\left\{\frac{d^n u}{dt^n}\right\} = s^n$$

$$L\left\{\int_0^\infty u(t)dt\right\} = \frac{1}{s}$$

Initial and final value theorems

Final value theorem:

$$f(\infty) = \lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[sF(s)]$$

Final value for a step input is

$$f(\infty) = \lim_{s\to 0}\left[sF(s)\frac{1}{s}\right] = \lim_{s\to 0}[F(s)]$$

Example,
final value for

$$G(s) = \frac{1}{s+a}$$

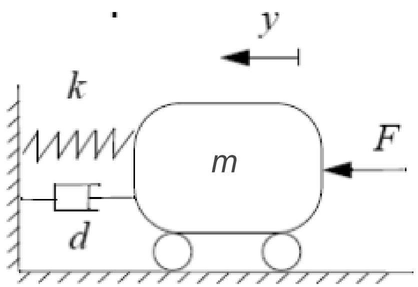
with step input is *1/a*

Initial value theorem

$$f(0) = \lim_{t\to 0}[f(t)] = \lim_{s\to\infty}[sF(s)]$$

For all G(s) with higher order denominator as numerator is the initial value for a step input zero.

Example: Transfer function



The transfer function can be calculated from the state space model.

You have to take a matrix inverse.

OK numerically in Matlab and symbolically in Maple

$$L\{\dot{x} = Ax + Bu\}$$
$$sx - Ax = Bu$$
$$x = (sI - A)^{-1}Bu$$
$$y = Cx$$
$$y = C(sI - A)^{-1}Bu$$
$$G(s) = \frac{y}{u} = C(sI - A)^{-1}B$$

Direct calculation from the differential equation is OK for low order models

$$L\{m\ddot{y} = F - d\dot{y} - ky\}$$
$$ms^2 y = F - dsy - ky$$
$$\left(ms^2 + ds + k\right)y = F$$
$$G(s) = \frac{1}{\left(ms^2 + ds + k\right)}$$

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Planes and tools

- Frequency domain
  - $G(s) = G(j\omega)$, ω is the frequency
- Complex pole-zero plane
  - Solve for s in numerator and denominator polynomials
- Time domain
  - The response $y = G(s)u$ for different u, e.g., step, ramp, etc.

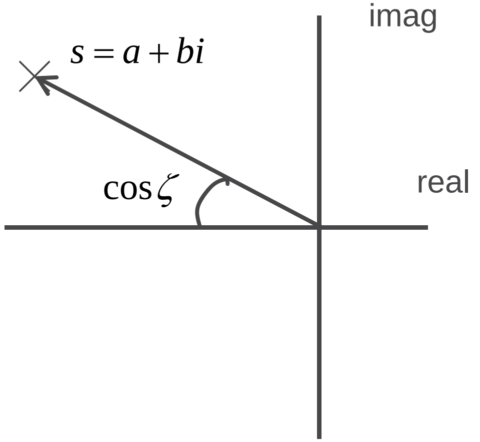
Complex plane: poles and zeros TF

$$G(s) = \frac{N(s)}{D(s)}$$

Zeros: set *N(s)=0* and solve for *s*.

Poles: set *D(s)=0* and solve for *s*.

Poles and zeros can be plotted in the complex plane, the real part vs. the imaginary part



The absolute value of s, $|s| = \sqrt{a^2 + b^2}$

Represents a frequency rad/s:

- In time domain how fast a response to an input is.
- In the frequency plane (Bode) it represents a change in amplitude and phase
- $|s|$ is often called ω_0

Complex plane: poles and zeros state space

The poles are the eigenvalues of the A matrix calculated by:

$$\det(sI - A)^{-1}$$

The zeros depends on the output, that is: the C matrix
Different C matrix gives different zeros
Example: mass and spring

$$x_1 = \text{position}$$
$$x_2 = \text{velocity}$$
$$u = \text{force}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

if :

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

then :

$$G(s) = \frac{1}{ms^2 + k}$$

if :

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

then :

$$G(s) = \frac{s}{ms^2 + k}$$

Frequency domain response

For any transferfunction *G(s)*
with the input

$$u(t) = 1.0\sin(\omega t)$$

will give the output

$$y(t) = a\sin(\omega t + \phi)$$

With the gain,

$$a = |G(j\omega)|$$

and the phase

$$\phi = \tan^{-1}\left(\frac{\text{imag}G(j\omega)}{\text{real}G(j\omega)}\right)$$

Integrator and derrivator

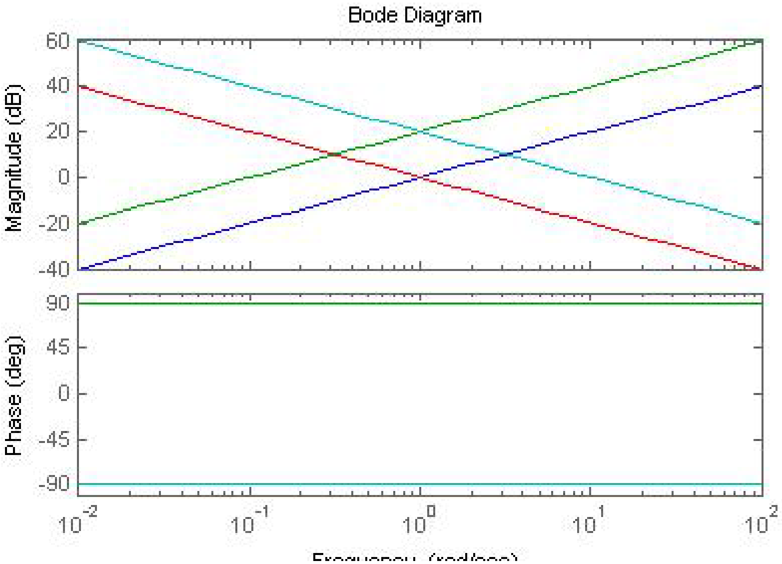
Derrivation

$$G(j\omega) = j\omega$$
$$|j\omega| = \omega$$
$$\arg\, j\omega = \text{atan}(\omega/0) = \pi/2$$

Integration

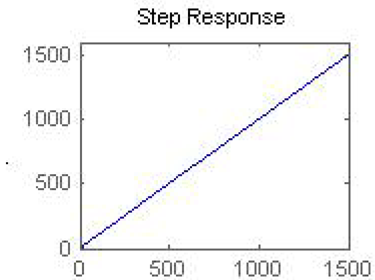
$$G(j\omega) = 1/(j\omega)$$
$$|j\omega| = 1/\omega$$
$$\arg\, j\omega = \text{atan}(-\omega/0) = -\pi/2$$

Poles and zeros:
Integrator*:*
pole: s=0
zero: none
derivate*:*
pole: none
zero: s=0



$$G(s) = s, 10s, \frac{1}{s}, \frac{10}{s}$$

Step response for an integrator



What is the step respone for a Derrivator ?

First order polynomial

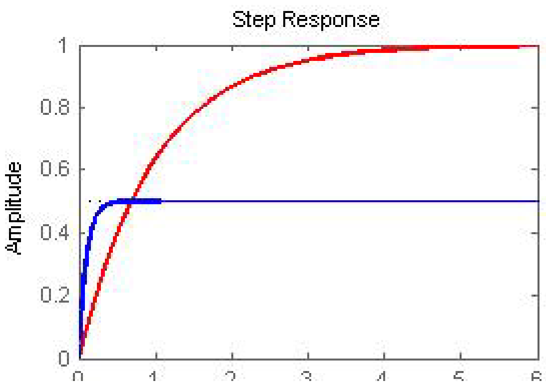
Two ways of writing:

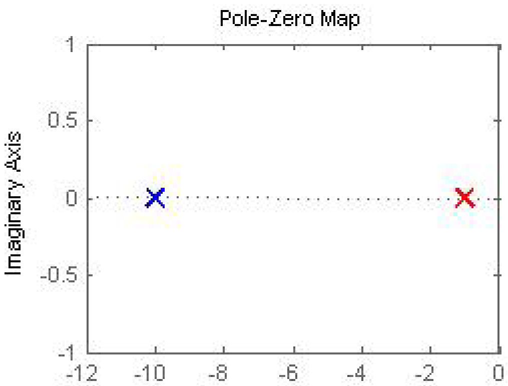
$$G(s) = \frac{k}{s+a}$$ Good for frequency domain,

$$G(s) = \frac{k}{\tau s+1}$$ Good for time domain

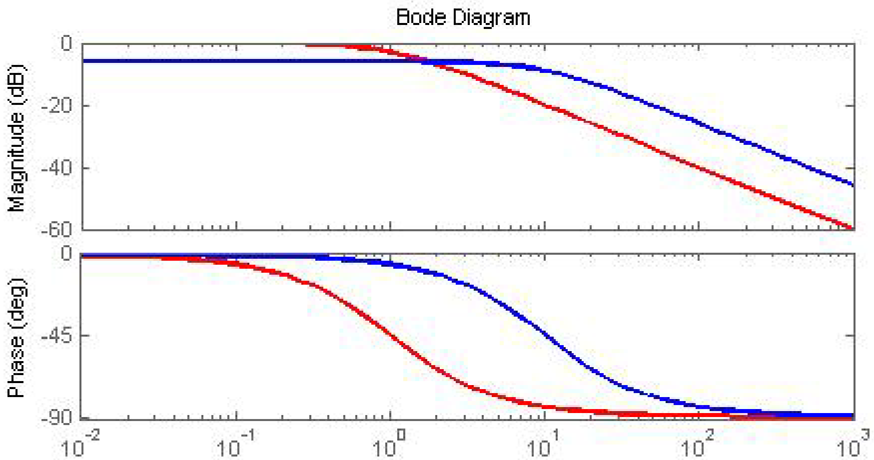
Characteristics are:

Pole
Dc-gain
Time constant
Cut-off frequency
Phase lag at high freq.





Example: $$G_1 = \frac{1}{s+1}$$ $$G_2 = \frac{5}{s+10}$$



Second order TF in complex plane

Model:

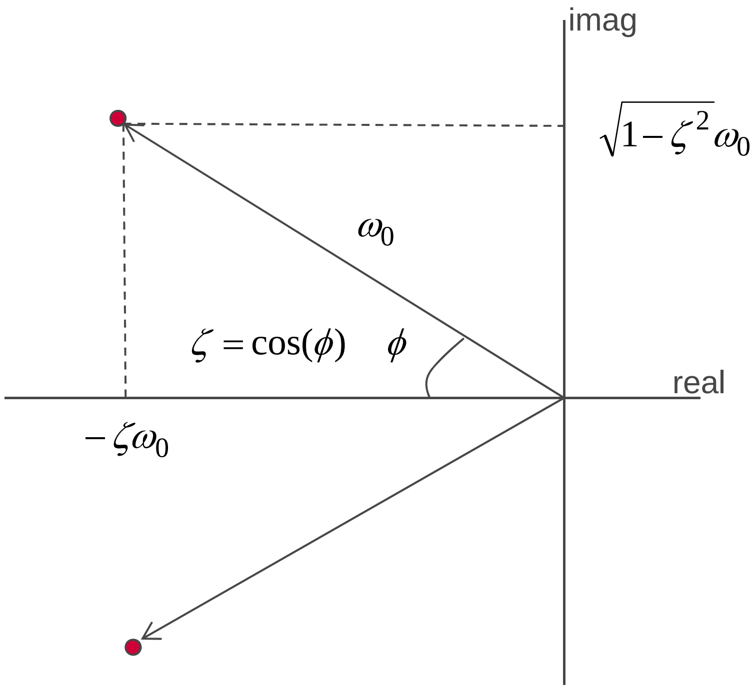
$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega s + \omega_0^2}$$

Un-damped resonance frequency:

$$|s| = \sqrt{(\zeta\omega_0)^2 + (1-\zeta^2)\omega_0^2} = \omega_0$$

Poles, complex conjugate when: $\zeta < 1$

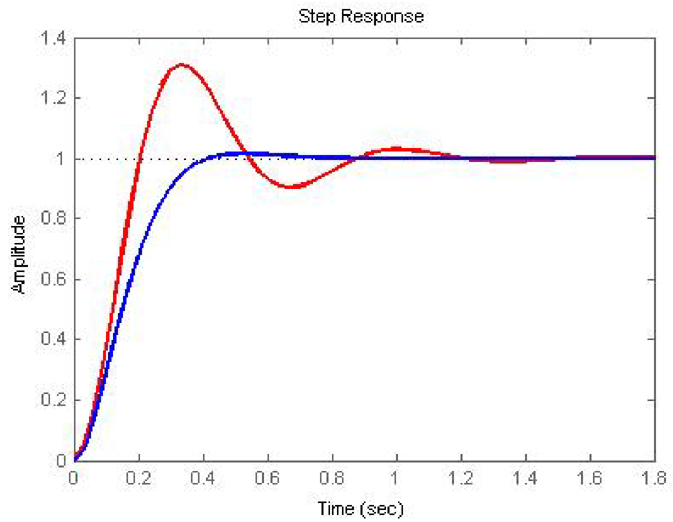
$$s = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1}\,\omega_0 = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$

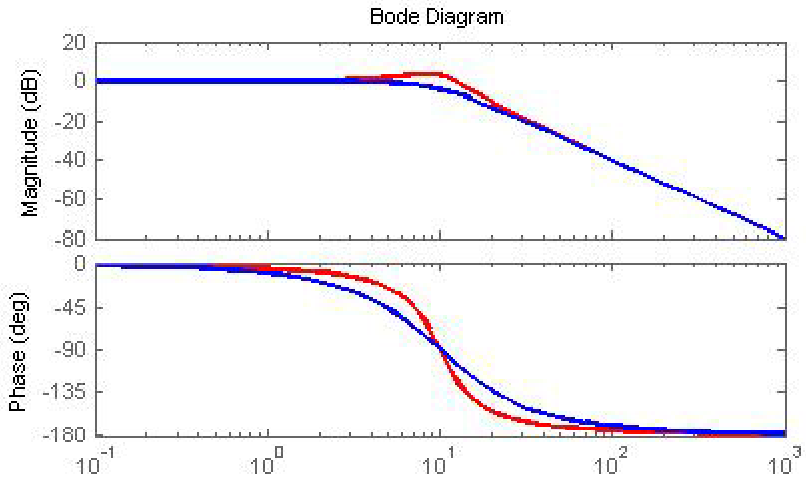


Damped resonance frequency, what you would measure in time domain

Second order TF in time and frequency domains

Two models with same frequency but different damping





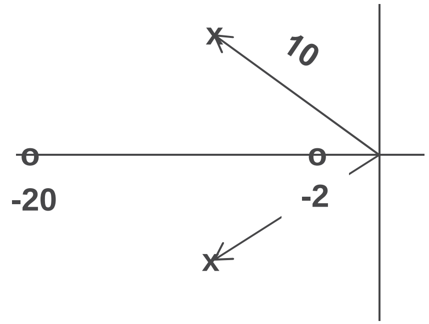
Low pass characteristics, 180 degree phase shift at high frequencies
Overshot, What is the damping ratio ?

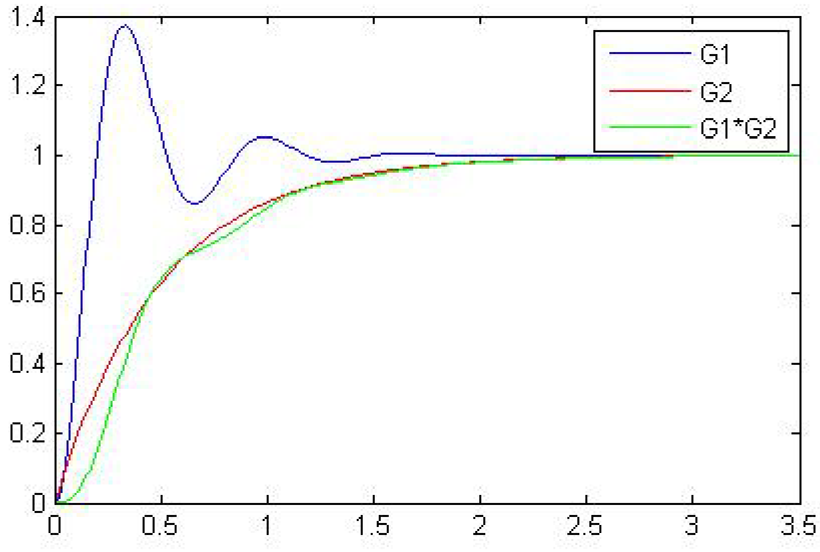
Superposition and dominant dynamics (poles)

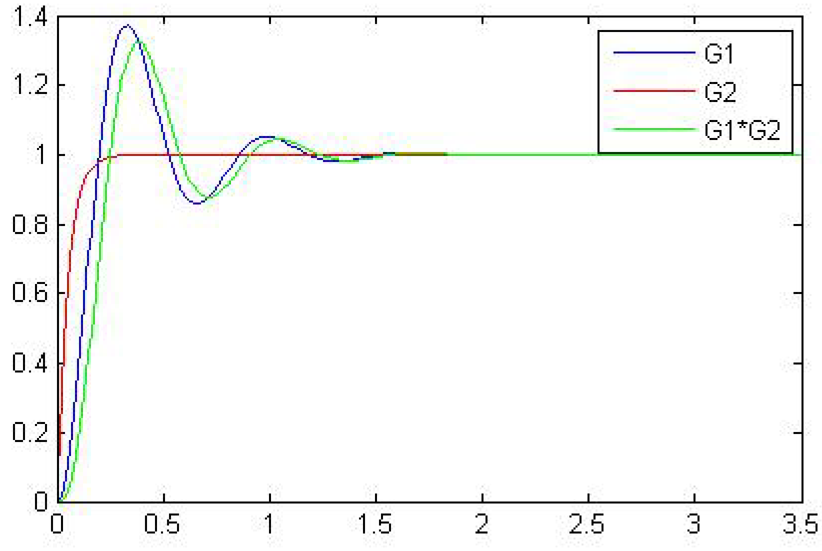
A second order model $G_1(s) = \dfrac{100}{s^2 + 6s + 100}$ is superpositioned

with a first order model $G_2(s) = \dfrac{a}{s + a}$, such that $G_s(s) = G_1(s)G_2(s)$

In left figure is *a=2* and in right figure *a=20*



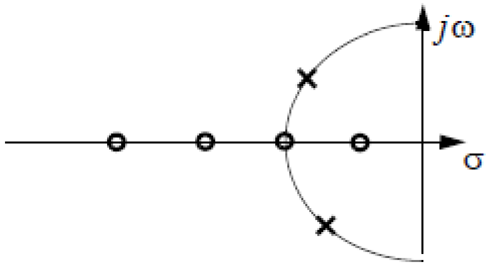


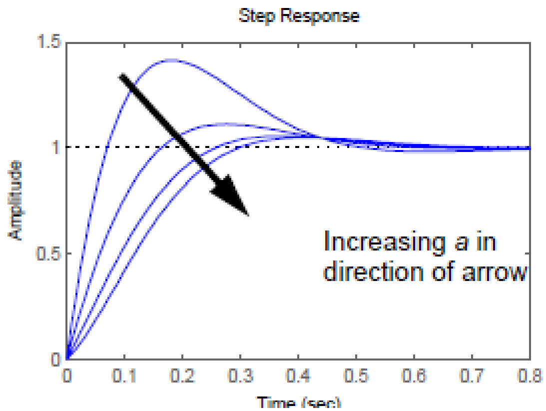


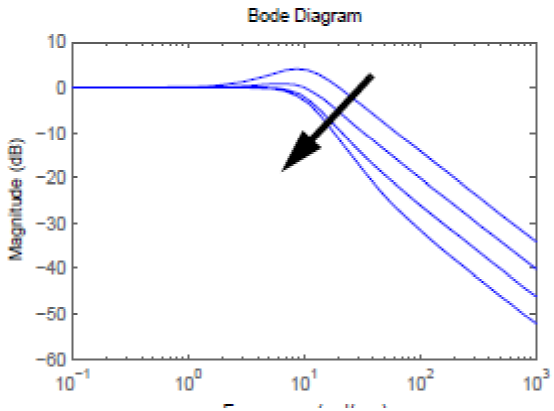
The TF with the slowest pole dominates the step response

Influence of a real zero

$$G(s) = \frac{\left(\frac{s}{\omega_0 a} + 1\right)\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \quad a = [0.5, 1, 2, 4]$$

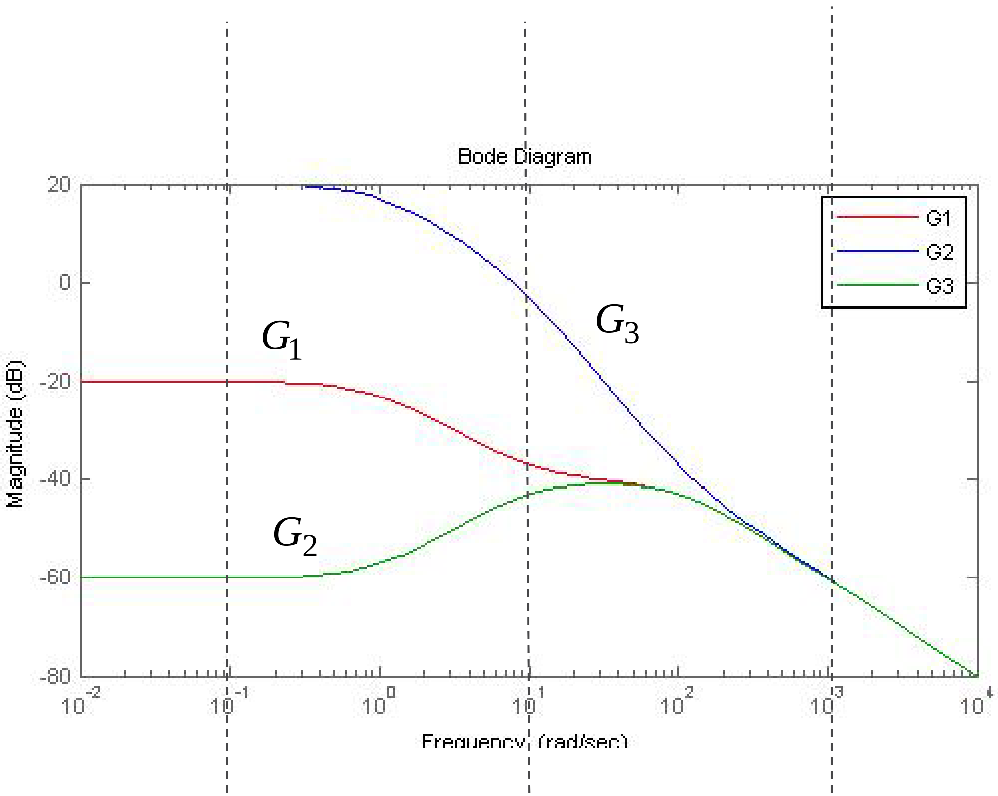






Higher order models: pole/zero in bode

$$G_1(s) = \frac{s+10}{(s+1)(s+100)},\ G_2(s) = \frac{s+100}{(s+1)(s+10)},\ G_3(s) = \frac{s+1}{(s+10)(s+100)}$$



Pole access

The number of poles and zeros equals the order of the denominator and nominator respectively.
For a TF we define the number of poles and zeros as n_D and n_N

$$G(s) = \frac{N(s)}{D(s)}$$

The pole access is defind as: $n_A = n_D - n_N$

- TF with n_A > -1 are called proper.
- If $n_A = 0$, is the model output constant at high frequencies, a step response will give a nonzero initial value.
- If $n_A > 0$, is the model output zero at high frequencies, a step response has zero initial value
- If $n_A < 0$, is the model not proper, the gain at high frequencies is infinite, it is not possible to make a step response for such a model

Example: pole access

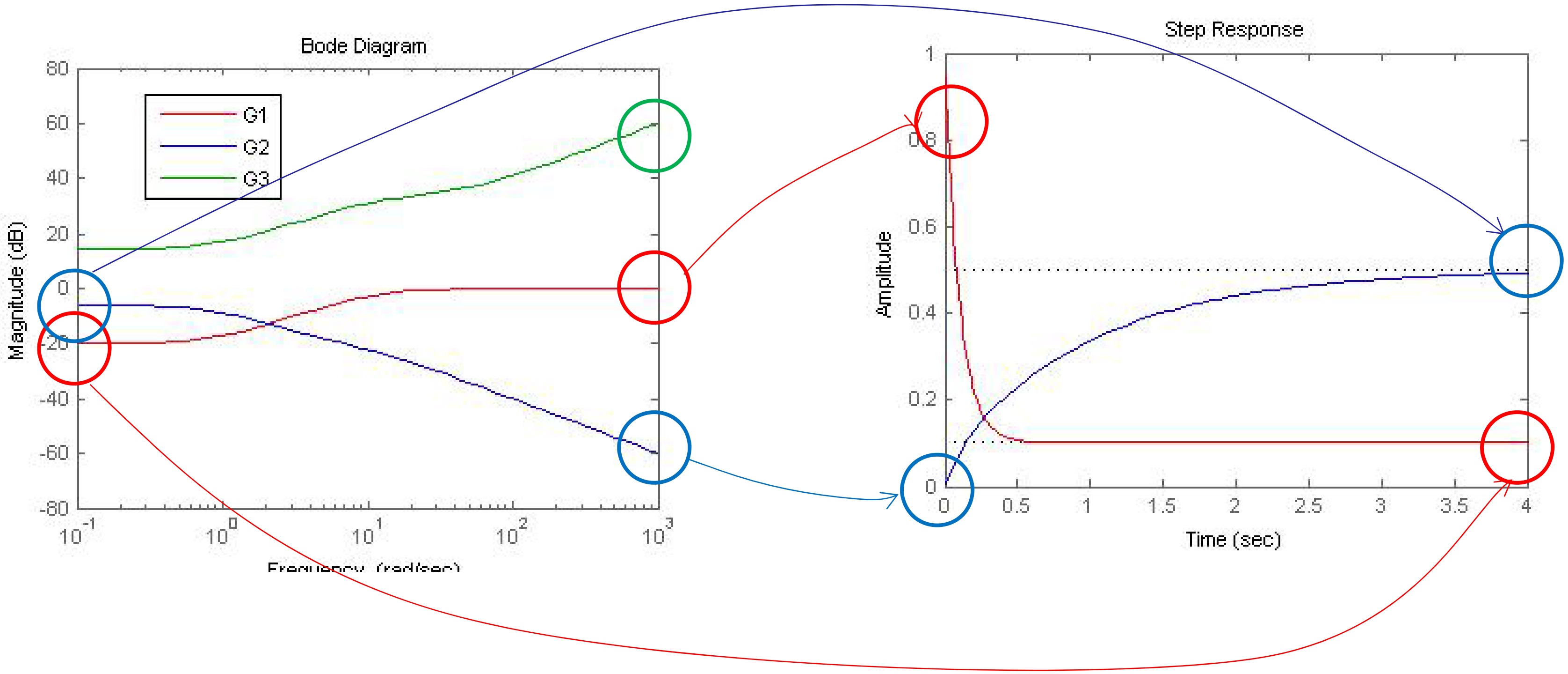
$$G_1(s) = \frac{s+1}{s+10}, \quad n_A = 0$$

$$G_2(s) = \frac{s+5}{(s+10)(s+1)}, \quad n_A = 1$$

$$G_3(s) = \frac{(s+1)(s+50)}{(s+10)}, \quad n_A = -1$$

Relationship between initial value and Dc-gain in frequency and time domain for models with different pole access
OBS! No step response for $G_3(s)$

$$\mathrm{dB} = 20\log 10(\mathrm{mag})$$



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Modelling from physical properties

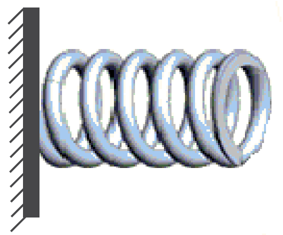
Mechatronic system design *Janscheck*

- section 2.3-2.3.4 (except the parts with Lagrange and Hamilton)
- section 2.3.8

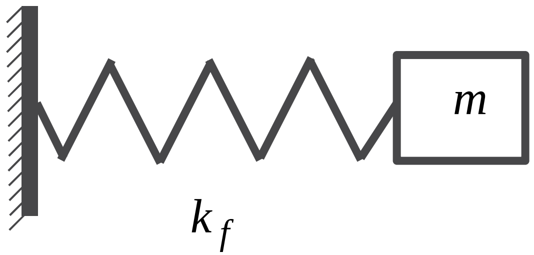
- **Lumped models**
- **Descriptions of basic elements**
- **Energy storage and dissipative energy**
- **Mechanical, translational and rotation**
  - mass, inertia, damping, friction, stiffness
- **Electric**
  - Resistors, inductors, capacitors

Distributed vs. lumped parameters models

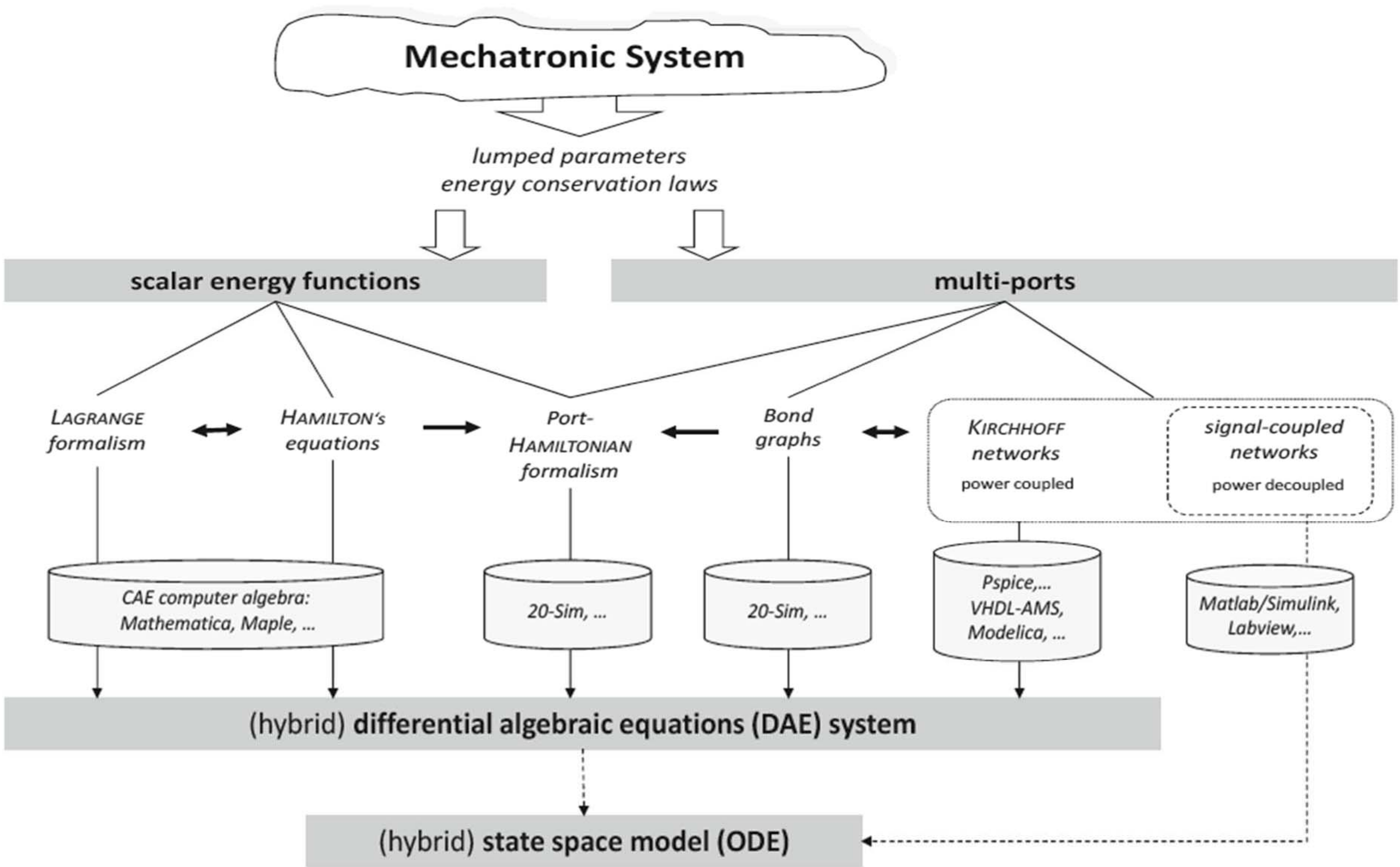
- A spring has a distributed mass, it gives a force when compressed or extended
    - The model is a partial differential equation with mass distribution
    - If the spring is first compressed and then released it starts to oscillate with zero speed at the fixed end.



- Modeling the spring as a massless spring and a point mass gives a lumped model with two elements.
- The spring can now be modeled using ordinary differential equations with an equivalent mass *m*, and spring stiffness k_f.

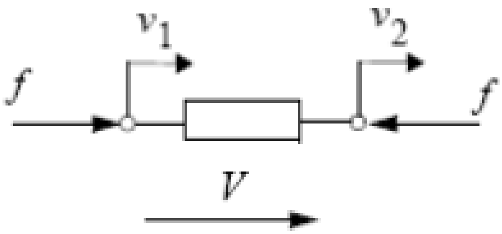


Different concepts of modelling



Trough and across variables

Mechanical components

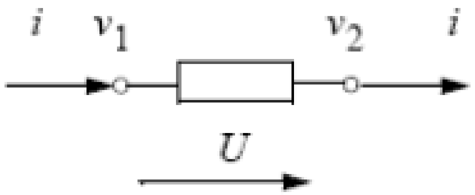


v_i velocity at node i

f force **trough** the component

$V = v_1 - v_2$ velocity **across** the component

Electrical components



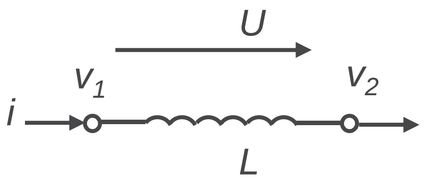
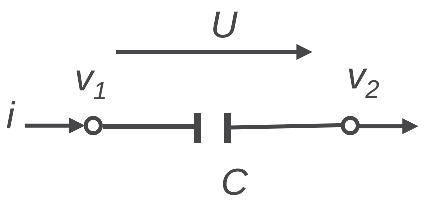
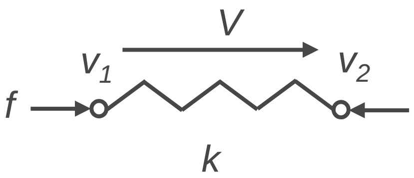
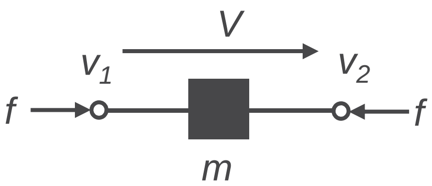
v_i voltage at node i

i current **trough** the component

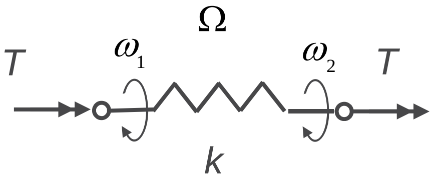
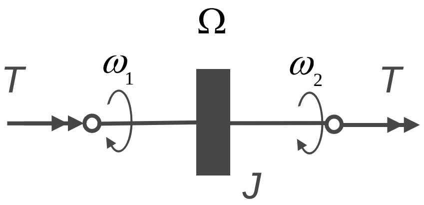
$U = v_1 - v_2$ voltage **across** the component

How do you measure the variables?

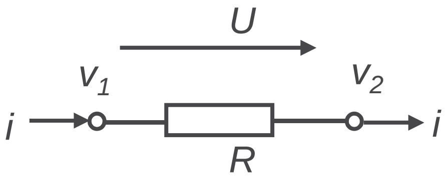
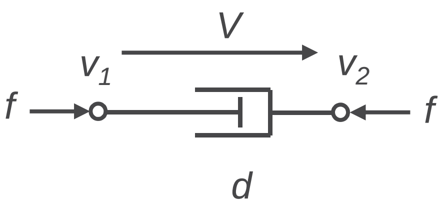
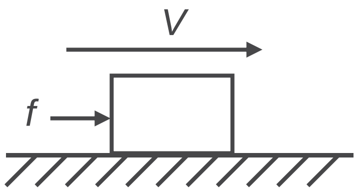
Elements that can store enegy

Symbol	*Physical element*	*Constitutive relation*	*Stored energy*
	Inductance	$L\frac{di}{dt} = U = v_1 - v_2$	$E = \frac{1}{2}Li^2$
	Capacitance	$C\frac{dU}{dt} = i, U = v_1 - v_2$	$E = \frac{1}{2}Cu^2$
	Translational Spring	$f = ky, y = \int V dt$ $\frac{1}{k}\frac{df}{dt} = V$	$E = \frac{1}{2}ky^2$
	Translational Mass	$m\frac{dV}{dt} = f, V = v_1 = v_2$	$E = \frac{1}{2}mV^2$

Rotational mechanical elements

Symbol	*Physical element*	*Constitutive relation*	*Stored energy*
	Rotational Spring	$T = k\varphi, \varphi = \int \Omega dt$ $\dot{T} = k\dot{\varphi}$	$E = \frac{1}{2} k\varphi^2$
	Rotational mass	$J\frac{d\Omega}{dt} = T, \Omega = \omega_1 = \omega_2$	$E = \frac{1}{2} m\Omega^2$

Elements that only dissipates energy

Symbol	*Physical element*	*Constitutive equation*	*Stored energy*
	Resistance	$U = Ri$	—
	Damping (friction)	$f = dV$	—
	Dry friction	$f = \mu mg$	—

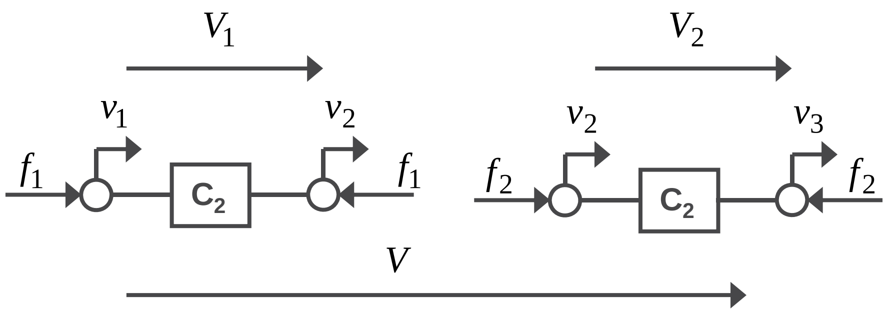
Connecting basic elements

- Mechanical properties -> Newtons laws
- Electrical properties -> Kirchhofs laws
- Parallel and series equations
  - Node and loop equations
- The principles of impedance and mobility
- The order of the differential equations equals the number of energy storage elements

Connection of elements

<table>
<tr><th></th><th>The velocity/voltage across the component</th><th>The force/current through the component</th></tr>
<tr><td>

Series connection



</td><td>

$V_1 = v_1 - v_2$

$V_2 = v_2 - v_3$

$\sum V = 0$

$V = V_1 + V_2$

$V = (v_1 - v_2) + (v_2 - v_3)$

$V = v_1 - v_3$

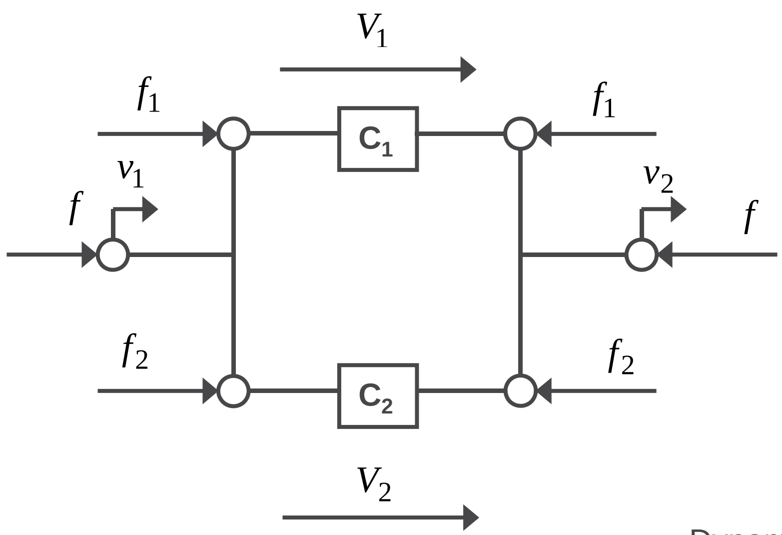
</td><td>

The same:

$f_1 = f_2$

</td></tr>
<tr><td>

Parallel connection



</td><td>

The same:

$V_1 = V_2$

$= v_1 - v_2$

</td><td>

Trough C_1

f_1

Trough C_2

f_2

$\sum f = 0$

$f = f_1 + f_2$

</td></tr>
</table>

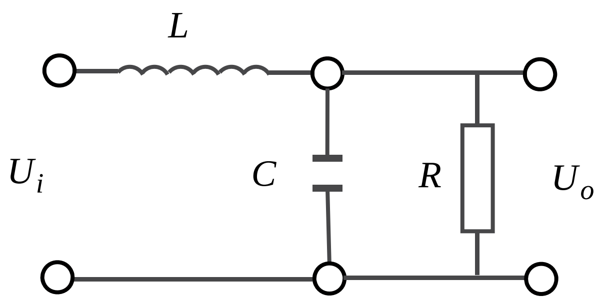
Connection of components

	Parallel Node equation	Series Loop equation
Mechanical:	$m\frac{dv}{dt} = \sum f$ force balance equation or Newtons 2 : nd law	$\frac{df}{dt} = k_i(v_{1_1} - v_{1_2}) - \cdots - k_n(v_{n_1} - v_{n_2})$ compatibility equation
Electrical:	$C\frac{dU}{dt} = \sum i$ Kirchofs current law	$L\frac{di}{dt} = R_1(u_{1_1} - u_{1_2}) - \cdots - R_n(u_{n_1} - u_{n_2})$ Kirchofs voltage law

State space modelling steps

- Make a lumped sketch of the elements (for mechanical modeling)
- Make a free-body figure (mechanical) or circuit diagram (electrical)
- Give notation to parameters, node and loop variables
- Write the constitutive equations
  - Gives the states of the model
- Write the node and loop equations
- Eliminate unwanted variables
- Write the equations in matrix form

Example : Electric circuit



Constitutive equations

$$L\frac{di_L}{dt} = U_L$$

$$C\frac{dU_C}{dt} = i_C$$

$$U_o = Ri_R$$

Eliminate $U_L \quad i_C \quad i_R \quad U_R \quad U_o$

Loop equations

$$U_C = U_R = U_o$$

$$U_L + U_o = U_i$$

Model

$$\dot{x}_1 = \frac{1}{L}(U_i - x_2)$$

$$\dot{x}_2 = \frac{1}{C}(x_1 - \frac{1}{R}x_2)$$

Node equation

$$i_L = i_C + i_R$$

State and output

$$x_1 = i_L \quad x_2 = U_C$$

$$y = \begin{bmatrix} U_o \\ i_L \end{bmatrix}$$

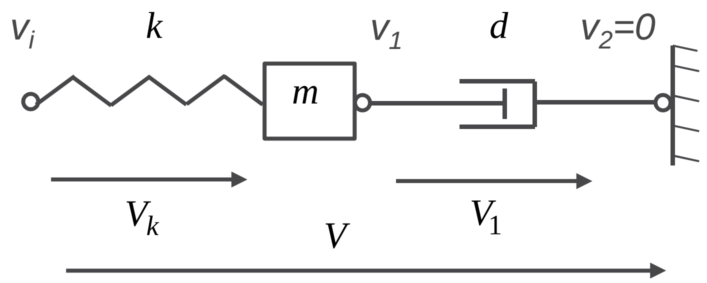
Matrix form

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U_i$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

input U_i

Example : Mechanical

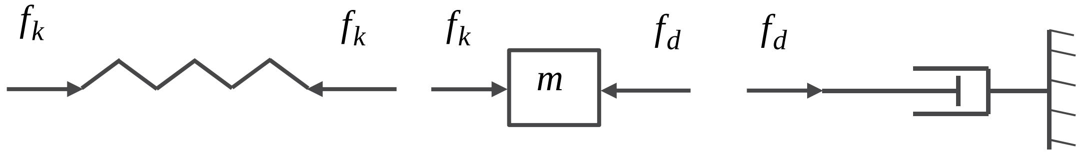


Constitutive equations

$$\frac{1}{k}\frac{df_k}{dt} = V_k$$

$$m\frac{dV_1}{dt} = f_m$$

$$f_d = dV_1$$



Loop equations

$$V = V_k + V_1$$

$$V_k = v_i - v_1$$

$$V_1 = v_1$$

Eliminate $V_k \quad f_m \quad V_1 \quad V \quad f_d$

Node equation

$$f_m = f_k - f_d$$

Model

$$\dot{x}_1 = k(v_i - x_2)$$

$$\dot{x}_2 = \frac{1}{m}(x_1 - dx_2)$$

State and outputs

$$x_1 = f_k, \quad x_2 = v_1$$

$$y = \begin{bmatrix} v_1 \\ f_m \end{bmatrix}$$

Matrix form

$$\dot{x} = \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} k \\ 0 \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix} x$$

Input

v_i

Compare the mechanical and electric systems

$$\dot{x} = \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} k \\ 0 \end{bmatrix} v_i$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U_i$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

Mechanical: $\frac{1}{m}$ k d

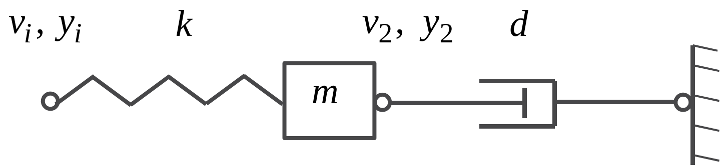
Electrical: $\frac{1}{C}$ $\frac{1}{L}$ $\frac{1}{R}$

The mechanical system:

the mass and damper are in parallell !

Alternative selection of states in mechanical systems

Sometimes the position is needed as state or output of the model



y is the position that corresponds to the velocity.
Select the states as position and velocity

$$x_1 = y_1, \quad x_2 = v_1$$

$$m\dot{v}_2 = f_k - f_d$$
$$f_k = k(y_i - y_1)$$
$$f_d = dv_1$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{k}{m}(y_i - x_1) - \frac{d}{m}x_2$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix} y_i$$
$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Using impedance and mobility as modelling tools

For electric circuits $U = Zi$ where Z is the impedance

For mechanical systems $V = Mf$ where M is the mobillity

Equivalente impeadance and mobillity for series conections $Z_e = Z_1 + Z_2 + \cdots + Z_n$

Equivalente impeadance and mobillity for parallellconections $\frac{1}{Z_e} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$

Two elements in parallell $Z_e = \frac{Z_1 Z_2}{Z_1 + Z_2}$

Electrical

$$Z_C = \frac{1}{Cs}$$

$$Z_L = Ls$$

$$Z_R = R$$

Mechanical

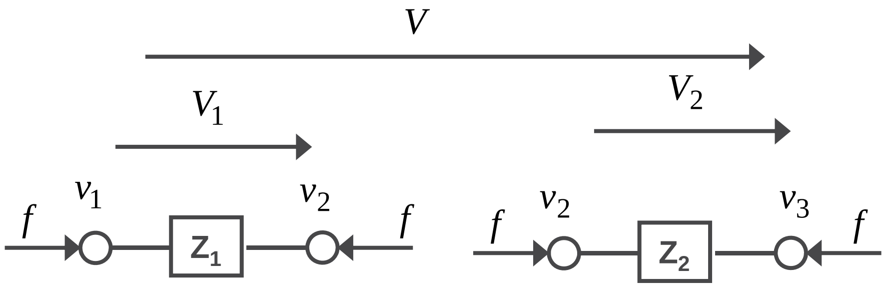
$$M_m = \frac{1}{ms}$$

$$M_k = \frac{1}{k}s$$

$$M_d = \frac{1}{d}$$

Divisions, getting other outputs

Voltage and velocity divisions

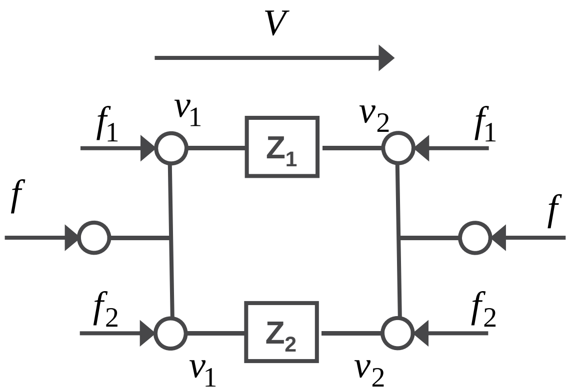


$$f = \frac{1}{Z_1 + Z_2} V$$
$$V_1 = Z_1 f$$
$$V_2 = Z_2 f$$

$\Rightarrow$

$$V_1 = \frac{Z_1}{Z_1 + Z_2} V$$
$$V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

Current and force divisions



$$V = \frac{Z_1 Z_2}{Z_1 + Z_2} f$$
$$f_1 = \frac{1}{Z_1} V$$
$$f_2 = \frac{1}{Z_2} V$$

$\Rightarrow$

$$f_1 = \frac{Z_2}{Z_1 + Z_2} f$$
$$f_2 = \frac{Z_1}{Z_1 + Z_2} f$$

Same example as in slide xx

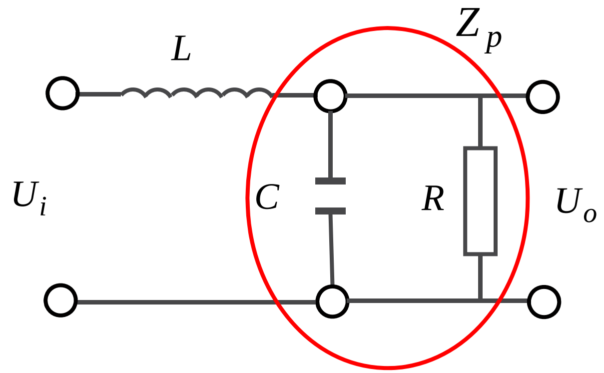
The equivalent impedance
from i to U_i.

$$U_i = Z_e i$$

$$C \text{ and } R \text{ in parallel, } Z_p = \frac{R/Cs}{1/Cs + R} = \frac{R}{RCs+1}$$

$$\text{L in serie } Z_e = Ls + Z_p = \frac{LRCs^2 + Ls + R}{RCs+1}$$

What is the order of the model ?



The output impedance
from U_i to U_o .

$$U_o = Z_o U_i$$

$$Z_o = \frac{Z_p}{Z_p + Z_L}$$

$$Z_o = \frac{R}{RCLs^2 + Ls + R}$$

What is the dc-gain ?

State space model

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} U_i$$

$$y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

Same example as in slide xx

The equivalent mobility from *f* to *V*.

$$V = v_i = M_e f$$

$$M_e = M_k + \frac{M_m M_d}{M_m + M_d}$$

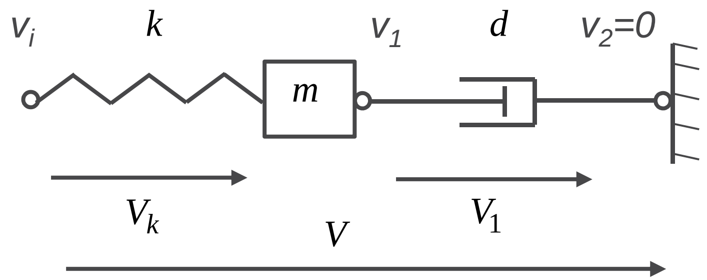
$$M_e = \frac{s}{k} + \frac{1/(dms)}{1/(ms) + 1/d}$$

$$M_e = \frac{ms^2 + ds + k}{(ms + d)k}$$

The velocity at node 1, v_1 using velocity division

$$v_1 = \frac{\frac{M_m M_d}{M_m + M_d}}{M_k + \frac{M_m M_d}{M_m + M_d}} v_i$$

$$v_1 = \frac{k}{ms^2 + ds + k} v_i$$



$$\dot{x} = \begin{bmatrix} 0 & -k \\ \frac{1}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} k \\ 0 \end{bmatrix} v_i$$

$$v_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

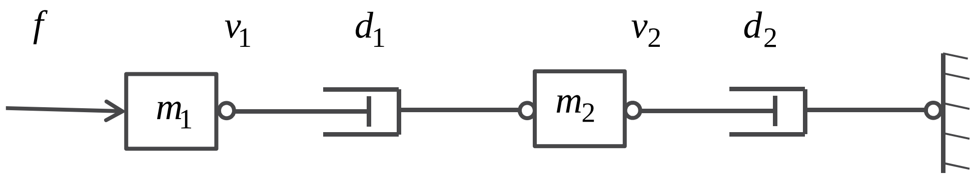
Transferfunction from v_i to v_1 by:

$$v_1(s) = C(sI - A)^{-1} B\, v_i(s)$$

Calculated in Maple we get:

$$v_1 = \frac{k}{ms^2 + ds + k} v_i$$

Example where state space technique is simpler



$$m_1\dot{v}_1 = f - f_{d1}$$
$$m_2\dot{v}_2 = f_{d1} - f_{d2}$$
$$f_{d1} = d_1(v_1 - v_2)$$
$$f_{d2} = d_2 v_2$$

Using node and loop equations

$\Rightarrow$

$$m_1\dot{v}_1 = f - d_1(v_1 - v_2)$$
$$m_2\dot{v}_2 = d_1 v_1 - (d_1 + d_2)v_2$$

Differential eq.

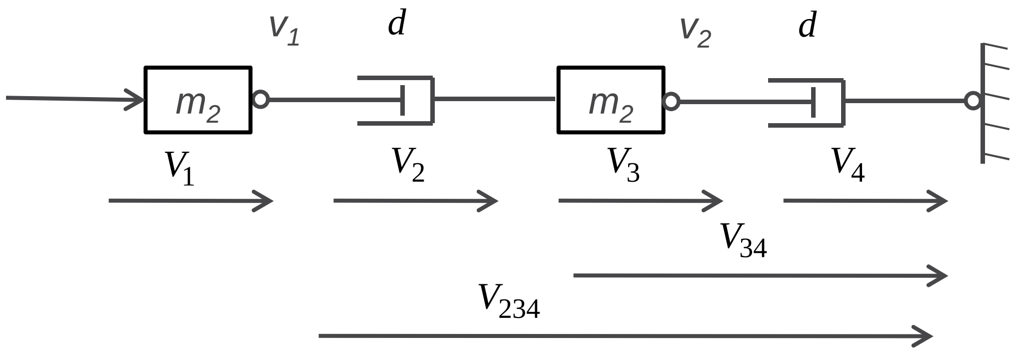
$$x_1 = v_1, \qquad x_2 = v_2$$

states

$$\dot{x} = \begin{bmatrix} -\frac{d_1}{m_1} & \frac{d_1}{m_1} \\ \frac{d_1}{m_2} & -\frac{d_1 + d_2}{m_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{m_1} \\ 0 \end{bmatrix} f$$

model

Same example with mobility technique



$V_3 = V_4 = V_{34} = v_2$ Parallel -> $M_{34} = \frac{M_{d1} M_{m1}}{M_{d1} + M_{m1}}$

$V_2 = v_1 - v_2 \neq V_{34}$ Series -> $M_{234} = M_2 + M_{34}$

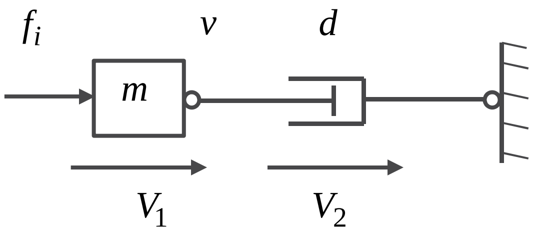
$V_{234} = V_2 + V_{34} = (v_1 - v_2) + v_2 = v_1$

$V_{234} = V_2 + V_{34} = (v_1 - v_2) + v_2 = v_1$

$V_1 = v_1 = V_{234}$ Parallel -> $M_{1234} = \frac{M_1 M_{234}}{M_1 + M_{234}}$

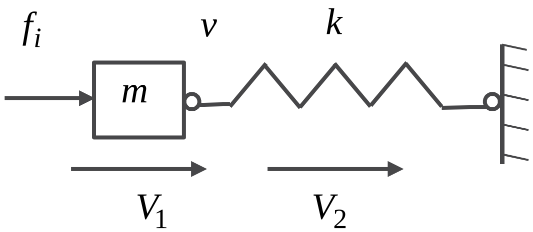
Model order with position as output of a model

Two systems with parallel connections, $V_1=V_2$



First order model to velocity $$v = \frac{1}{ms + d} f_i$$

Second order model to position $$y = \frac{1}{ms^2 + ds} f_i$$



Second order model to velocity $$v = \frac{s}{ms^2 + k} f_i$$

Second order model to position $$y = \frac{1}{ms^2 + k} f_i$$

Draw the step response for each model

Lecture outline

1. Introduction
2. Mathematical descriptions of models
3. Dynamic analysis
4. Basic modeling
5. **Linearization**
6. Models of typical components and phenomena in mechatronic systems.
7. Example: Hydraulic actuator
8. Example: Brushless DC-Motor

Linearization

- A nonlinear model $\dot{x} = f(x, u)$ $\quad y = g(x, u)$ can be linearized around some operating point $\{x_Q, u_Q\}$ by considering a neighbourhood around the operating point and approximating the nonlinear model with a truncated Taylor series.
  Set $x = x_Q + \Delta x$, $u = u_Q + \Delta u$ and $y = y_Q + \Delta y$, then

$$\dot{x} \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_Q \\ u = u_Q}} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x_Q \\ u = u_Q}} \Delta u$$

$$y \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x = x_Q \\ u = u_Q}} \Delta x + \left.\frac{\partial g}{\partial u}\right|_{\substack{x = x_Q \\ u = u_Q}} \Delta u$$

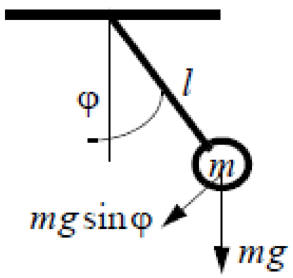
→

$$\Delta\dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x = x_Q \\ u = u_Q}} \quad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x_Q \\ u = u_Q}}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x = x_Q \\ u = u_Q}} \quad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x = x_Q \\ u = u_Q}}$$

Example: pendelum



Differential equation

$$J\ddot{\varphi} = \sum M_y = -mgl\sin\varphi$$

$$J = ml^2$$

Nonlinear state space model

$$x_1 = \varphi, x_2 = \dot{\varphi}$$

$$\dot{x} = f(x_1, x_2)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

Equilibrium point and Linearization

$$(\dot{x}_2 = 0) \Rightarrow x_{1Q} = 0$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x = x_Q} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos 0 & 0 \end{bmatrix}$$

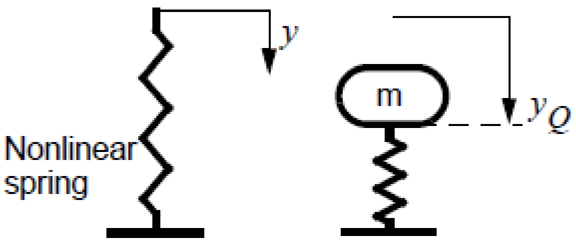
Linear model

$$x = x_Q + \Delta x = \Delta x$$

$$\Delta\dot{x}_1 = \Delta x_2$$

$$\Delta\dot{x}_2 = -\frac{g}{l}\Delta x_1$$

Example: nonlinear spring, *f=ky²*



Differential eq.

$$m\ddot{y} = mg - ky^2$$

Nonlinear model

$$x_1 = y,\; x_2 = \dot{y}$$

$$\dot{x} = f(x)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(mg - kx_1^2\right)$$

Equilibrium point

$$\dot{x} = 0$$

$$\Rightarrow$$

$$x_{1Q} = \sqrt{\frac{mg}{k}}$$

$$x_{2Q} = 0$$

Linearization

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_Q} = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\frac{kg}{m}} & 0 \end{bmatrix}$$

Linearized model

$$\Delta\dot{x} = A\Delta x$$

$$y = \Delta x_1 + x_{1Q}$$

See Simulink model

Lecture outline

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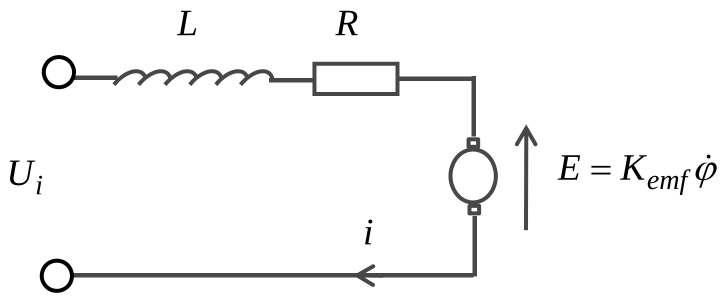
DC motor with permanent magnets in the stator

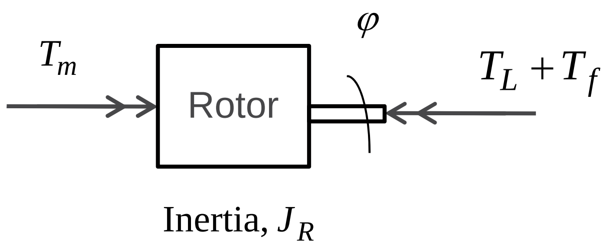
Electric part:
The rotor winding has an inductance, a resistance and a back-emf voltage proportional to rotor velocity



$$U_i = U_L + U_r + E$$

$$U_i = L\frac{di}{dt} + Ri + K_{emf}\dot{\varphi}$$

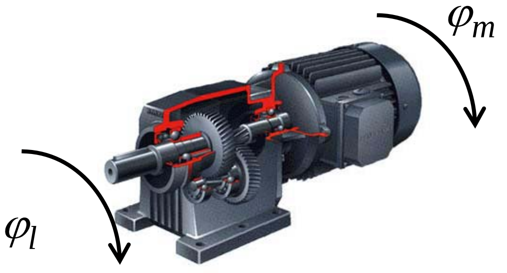
Mechanical part:
A torque T_m between rotor and stator is proportional to rotor current.
The rotor inertia, J_R
A load T_L on the outgoing shaft.



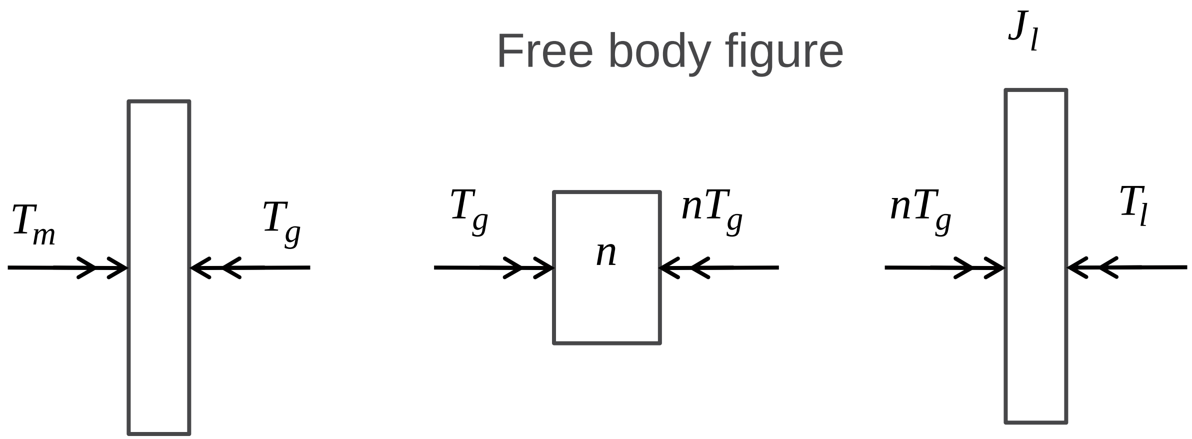
$$J_r\ddot{\varphi} = T_m - T_L - T_f$$

$$T_m = k_T i$$

Gearbox model



$$\varphi_l = n\varphi_m$$
$$n > 1$$



J_r is the motors rotor inertia, J_g the gearbox inertia calculated on the motor side, J_l the inertia of a load connected to the gearbox output and *n* the gear ratio.

$$(J_r + J_g)\ddot{\varphi}_r = T_m - T_g$$

$$J_l\ddot{\varphi}_l = nT_g - T_l$$

solve for T_g in one eq, and put in the other.

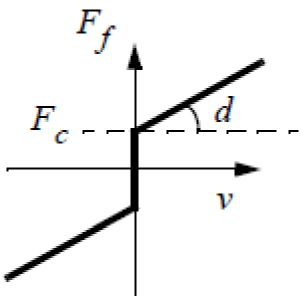
$$T_g = \frac{J_l\ddot{\varphi}_l}{n} + \frac{T_l}{n} = \frac{J_l\ddot{\varphi}_r}{n^2} + \frac{T_l}{n}$$

$$\left(J_r + J_g + \frac{J_l}{n^2}\right)\ddot{\varphi}_r = T_m - \frac{T_l}{n}$$

Compare with an electrical transformer

Nonlinear friction model

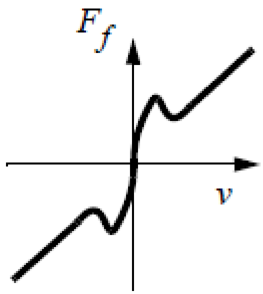
Standard Coulomb model



$$F_f(v) = F_c \operatorname{sgn}(v) + dv$$

$F_f(v)$ is discontinuous and not unique, numerical problems. Friction force can be higher then applied force.

Dynamic zero velocity model



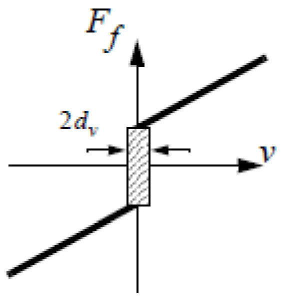
$$\frac{dF}{dt} = \sigma_0\left(1 - \frac{F}{F_c}\operatorname{sgn}(v)\right)v$$

High stiffness,

$\sigma_0 = [10^3 \ldots 10^5]$

numerical problems.

Karnop stick-slip model



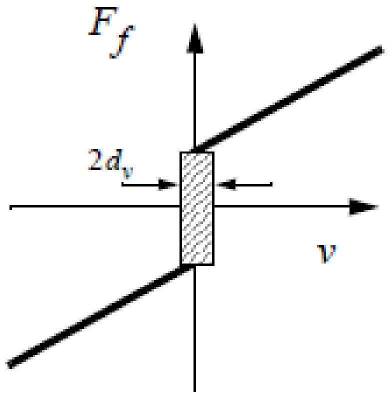
$$-d_v < v < d_v \quad \begin{cases} v = 0 \\ F_f = F_{applied} \end{cases}$$

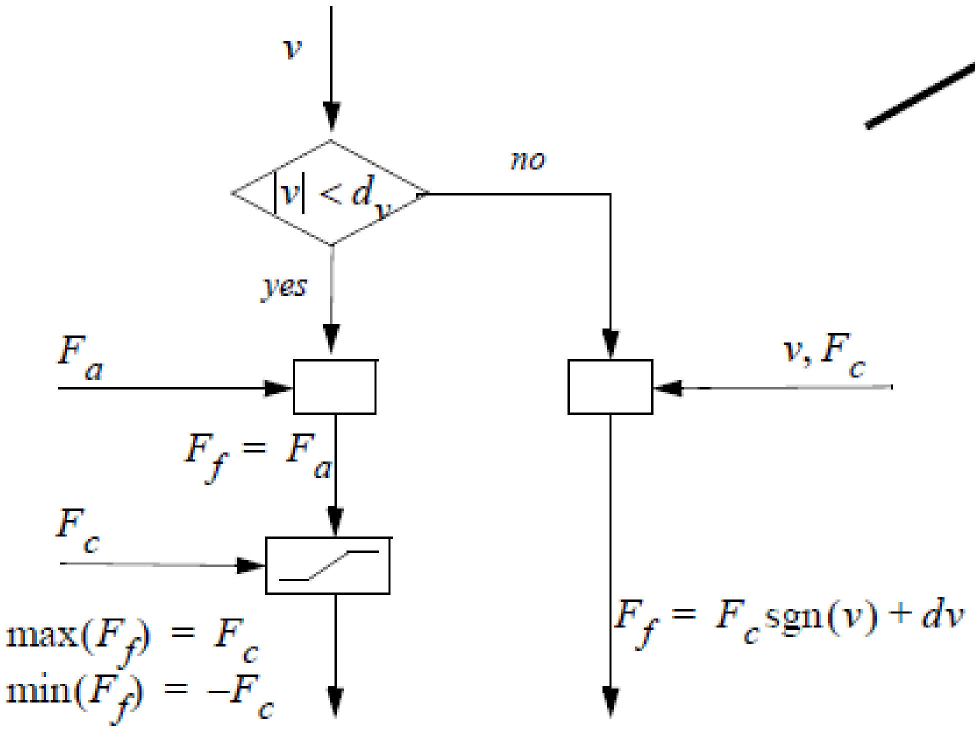
$$-d_v > v > d_v \quad \{ \; F_f = f(v)$$

Small velocity error around zero velocity.

Implementation of Karnop’s friction model

v = velocity
F_a = applied torque
F_f = friction torque
F_c = Coulomb friction level
d = velocity proportional friction
d_v = velocity deadband





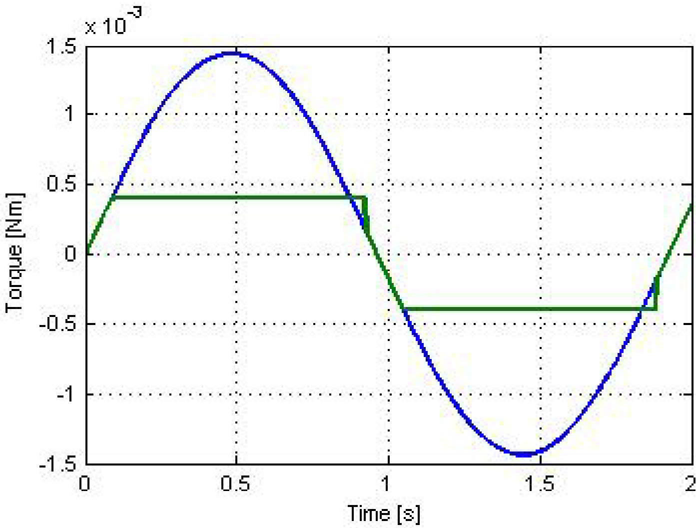
Coulumb friction

Dc-motor simulation with torque input and Coulumb friction

$$T_{applied} = 1.5e^{-3}\sin(0.5t) \qquad T_c = 0.4e^{-3}$$

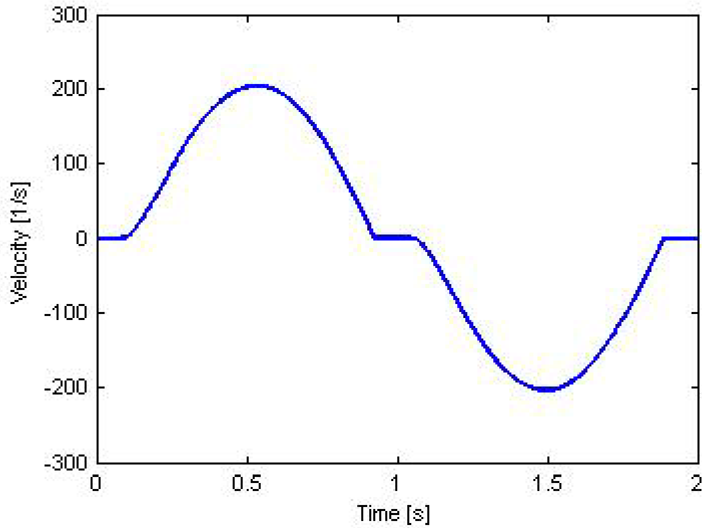
Blue line is applied torque

Green line is friction torque



$$J\ddot{\varphi} = T_{applied} - T_{friction}$$

Typical velocity with Coulomb friction

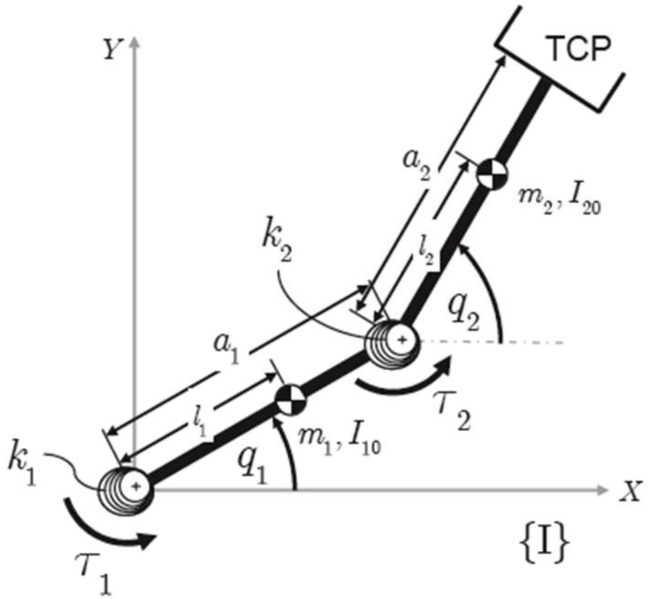


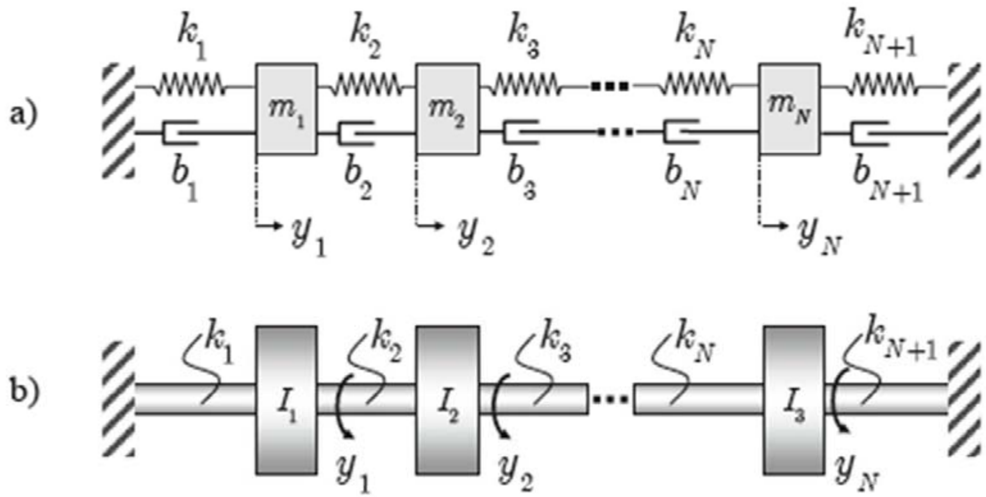
Higher order dynamics in moving machine parts

- All material has finite stiffness
- Lumped models with mass, spring and damper
  - Multi Body Systems, MBS
- Resonance and anti resonance frequencies,
- Gives phase lag which can make feedback systems instable
- For a general theory on MBS see any textbook in Robotics or for an introduction, Jansheck chapter 4.
- Reading material Jansheck section 4.4 – 4.7.5
- Which frequencies can affect a feedback system in a negative way

General MBS

Two basic types of MBS systems





Machines where parts can move with relative motion in different coordinate systems

Machines where the relative motion is because of flexible (not stiff) parts. Same coordinate system.

General nonlinear model of MBS systems

Based on Newton Euler can a general matrix based equations of motion be written as the nonlinear model

$$M(q,t)\ddot{q} + g(q,\dot{q},t) = f(q,\dot{q},t)$$

Where:

$q \in \Re^{N_{DOF}}$ are N_{DOF} the minimal number of generalized coordinates

$M \in \Re^{N_{DOF} \times N_{DOF}}$ is the mass matrix

$g \in \Re^{N_{DOF}}$ generalized spring, damping, Coriolis forces

$f \in \Re^{N_{DOF}}$ generalized external forces

Linearized model of MBS

Linearizing around a stable position q_{*0} gives that $q(t) = q_{*0} + y(t)$ and the equations of motion as

$$M\ddot{y} + (B+G)\dot{y} + (K+N)y = f(t)$$

Where all matrices are $N_{DOF} \times N_{DOF}$

$M = M^T$, $M\ddot{y}$ are the inertial forces

$B = B^T$, $B\dot{y}$ are the damping forces

$G = -G^T$, Ny are the gyroscopic forces

$K = K^T$, Ky are the spring forces

$N = -N^T$, Ny are the non - conservative forces

N is always zero for our models

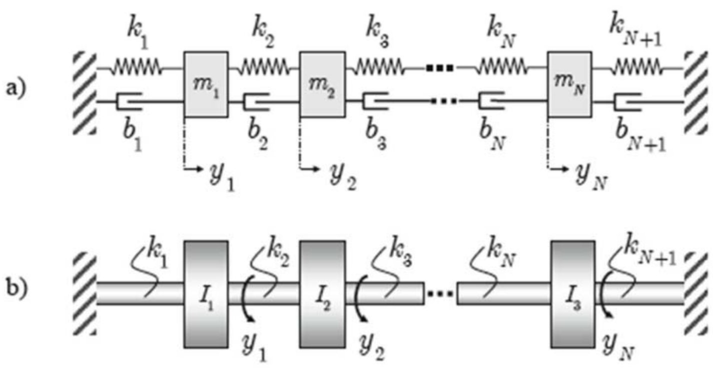
Structured modeling of MBS with flexible linkage

$$M = diag(m_1 \quad m_2 \quad m_3 \quad m_4 \quad \cdots \quad m_N)$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & -k_3 & k_3+k_4 & -k_4 & & \\ & & & \ddots & & -k_N \\ & & & & -k_N & k_N+k_{N+1} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1+b_2 & -b_2 & & & & \\ -b_2 & b_2+b_3 & -b_3 & & & \\ & -b_3 & b_3+b_4 & -b_4 & & \\ & & & \ddots & & -b_N \\ & & & & -b_N & b_N+b_{N+1} \end{bmatrix}$$

$$y = (y_1 \quad y_2 \quad y_3 \quad y_4 \quad \cdots \quad y_N)^T$$



equation of motion

$$M\ddot{y} + B\dot{y} + Ky = f(t)$$

Define a state vector

$$x_i = (y_i \quad \dot{y}_i)^T$$

Gives the state space model

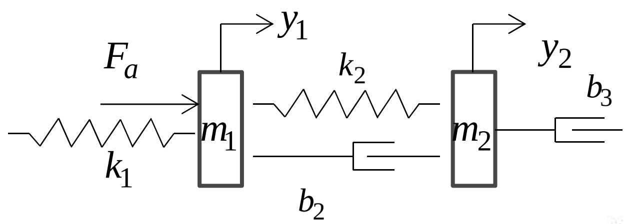
$$\dot{x} = Ax + Bf = \begin{bmatrix} 0 & E \\ -M^{-1}K & -M^{-1}B \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} f$$

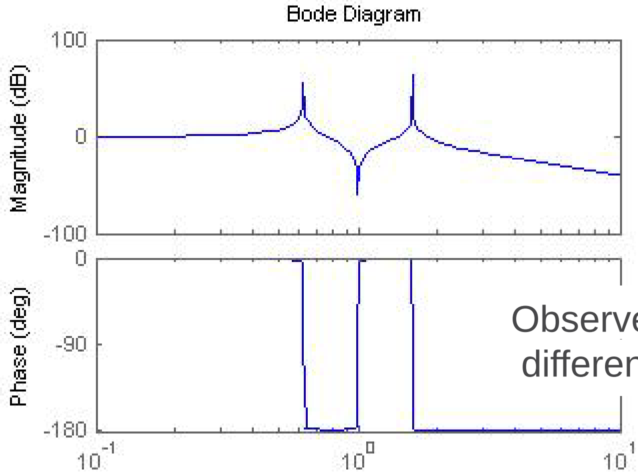
Dynamic of the MBS

Poles are simply calculated as the eigenvalues of the *A* matrix

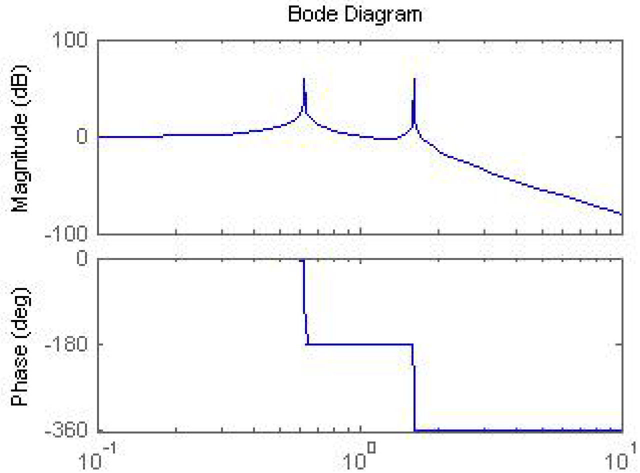
The **zeros** and therefore also the frequency response depends on which mass is actuated and which mass is measured. That is, on which row in the B matrix and which column in the C matrix.

Example: 2 mass





Observe phase difference



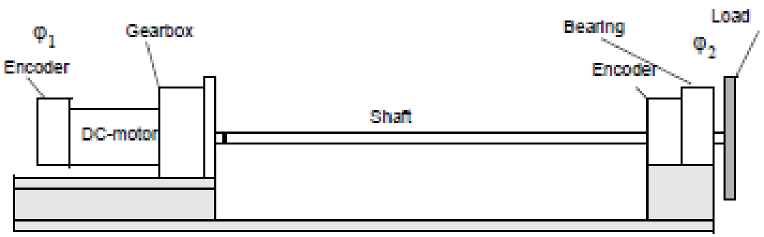
From F_a to y_1

Four poles and two zeros

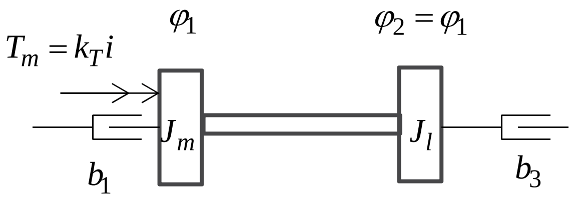
From F_a to y_2

Four poles and no zeros

Dc motor with load and week shaft



Stiff shaft model
(with gear ratio 1.0)



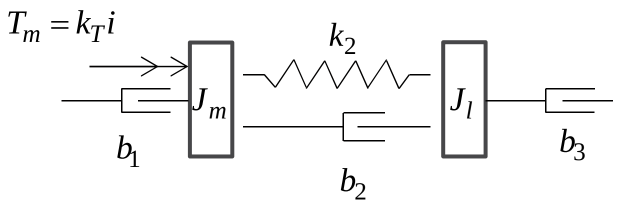
$$\varphi_1 = \frac{b_s}{s(s+a_s)} i$$

Where:

$$b_s = \frac{k_T}{J_m + J_l}$$

$$a_s = \frac{b_1 + b_3}{J_m + J_l}$$

Shaft with torisional spring and damper model



$$M = \mathrm{diag}(J_m \quad J_l)$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \quad k_1 = k_3 = 0$$

$$B = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix}$$

Gives:

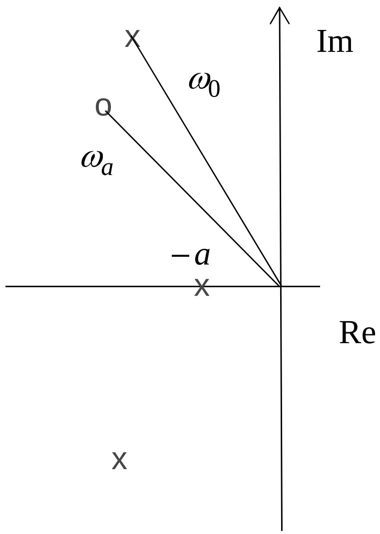
$$\varphi_1 = \frac{b_w}{s(s+a_w)} \frac{s^2/\omega_a^2 + 2\zeta_a s/\omega_a + 1}{s^2/\omega_0^2 + 2\zeta_0 s/\omega_0 + 1} i$$

Dc motor with week shaft

$$G_w(s) = \frac{b_w}{s(s+a_w)} \frac{s^2/\omega_a^2 + 2\zeta_a s/\omega_a + 1}{s^2/\omega_0^2 + 2\zeta_0 s/\omega_0 + 1}$$

$$G_s(s) = \frac{b_s}{s(s+a_s)}$$

If $|a_s|$ is sufficiently smaller than ω_0



Then:

$$G_w(s) \approx \frac{b_s}{s(s+a_s)} \frac{s^2/\omega_a^2 + 2\zeta_a s/\omega_a + 1}{s^2/\omega_0^2 + 2\zeta_0 s/\omega_0 + 1}$$

Example: Identify MBS model

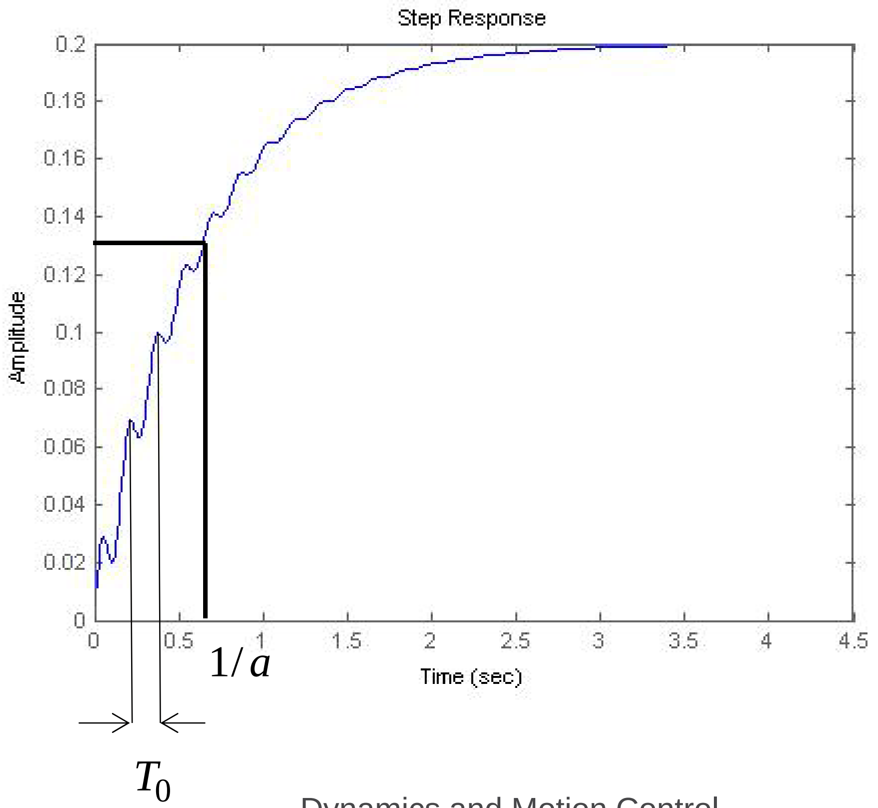
Simplest approach is to make a step response to velocity and measure time constant, *1/a* and resonance frequency ω_0

Model parameters:

$m_1 = 1, m_2 = 2$

$k_2 = 1000$

$b_1 = 4, b_2 = 1, b_3 = 1$



$$a \approx 1/0.6 = 1.67$$

$$\omega_0 \approx \frac{2\pi}{T_0} = 42$$

Example continued

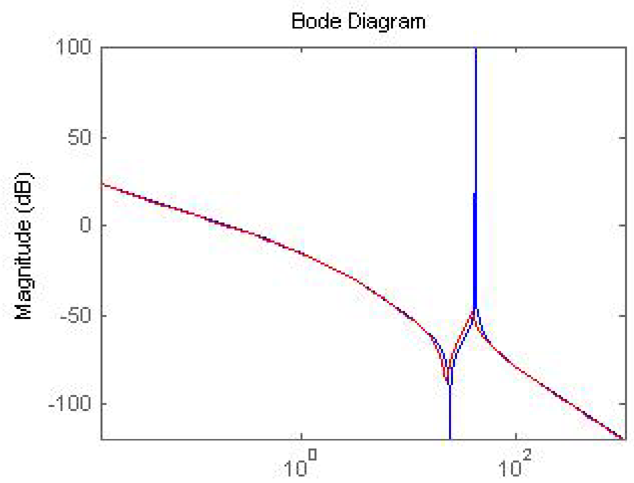
Antiresonance frequency

$$\omega_a = \omega_0 \sqrt{\frac{m_1}{m_1 + m_2}} = 24.2$$

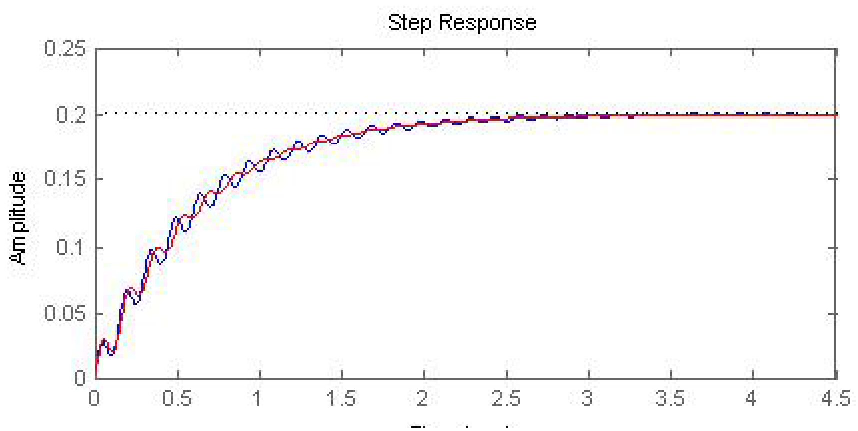
Gives the parametric model

$$G = \frac{0.2 \cdot 1.67}{s + 1.67} \frac{(s/24.2)^2 + 1}{(s/42)^2 + 1}$$

Compare frequency response



Compare step response

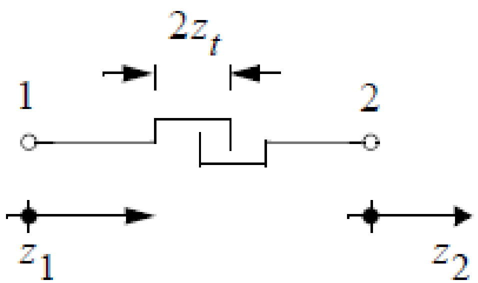


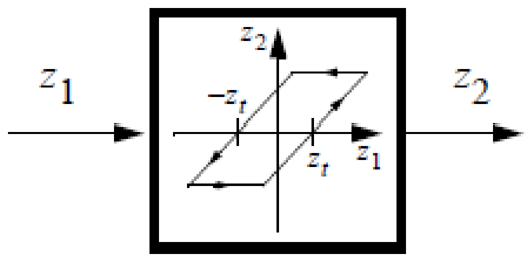
Red lines original model
Blue lines identified model

Model of: Backlash or play (glapp på svenska)

Simple model

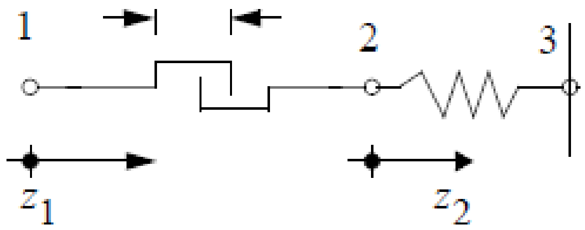
$$z_2 = z_1 - z_t \quad \text{for} \quad \dot{z}_1 > 0$$
$$z_2 = z_1 + z_t \quad \text{for} \quad \dot{z}_1 < 0$$

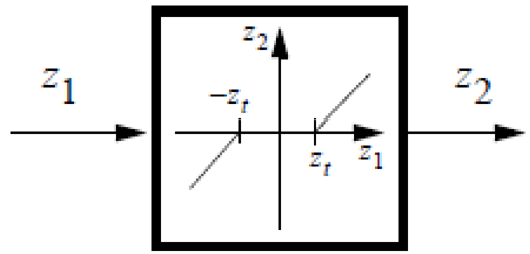




Spring loaded model

$$z_2 = z_1 - z_t \quad \text{for} \quad z_1 \geq 0$$
$$z_2 = z_1 + z_t \quad \text{for} \quad z_1 \leq 0$$
$$z_2 = 0 \quad \text{for} \quad -z_t < z_1 < z_t$$





1. Introduction
2. Mathematical descriptions of models
3. Dynamic analysis
4. Basic modeling
5. Linearization
6. Models of typical components and phenomena in mechatronic systems.
7. **Example: Hydraulic actuator**
8. Example: Brushless DC-Motor

Hydraulic systems

- Pressure difference is the *across* variable
- Volume flow is the *through* variable
- Node and loop equations
- Fluid capacitance and fluid resistance
- Volume and pressure sources -> Pumps
- Flow and pressure control valves -> servo valves
- Fluid to mechanical transformers -> cylinders and motors
- Modeling example : flow controlled hydraulic cylinder

Hydraulic components

Across type

$$C_f \frac{dP}{dt} = Q$$

C_f = fluid capacitance

Through type

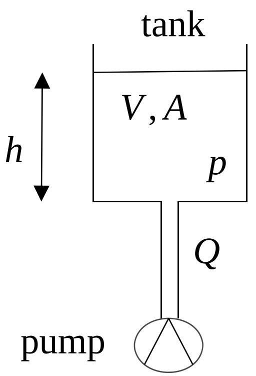
$$I_f \frac{dQ}{dt} = P$$

I_f = fluid inertance

Disipative type

$$R_f Q = P$$

R_f = fluid resistance



$$p = \rho g h = \frac{\rho g}{A} V$$

$$C_f = \frac{A}{\rho g}$$

$$C_f \frac{dp}{dt} = \frac{dV}{dt} = Q$$

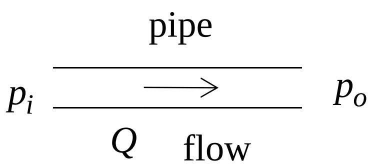
V = liquid volume $[m^3]$

A = cross sectional area $[m^2]$

Q = volume flow $\left[\frac{m^3}{s}\right]$

p = pressure $[Pa, N/m^2]$

Not so important !



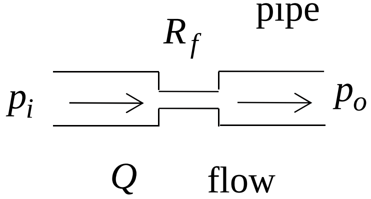
for a circular pipe

$$I_f = \frac{\rho l}{A}$$

ρ = density $\left[\frac{kg}{m^2}\right]$

l = length of pipe $[m]$

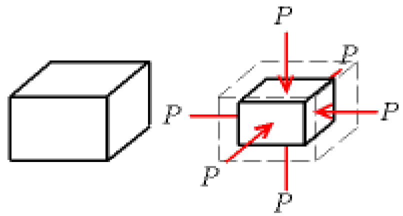
A = cross sectional area of pipe $[m^2]$



$$Q = R_f \sqrt{(p_i - p_o)}$$

Compressibility of hydraulic oil

Density increase, (volume decrease) of hydraulic oil is more than 100 times larger then that of steel. So it can not be neglected.



Bulk modulus, $$\beta = -V\left(\frac{\partial p}{\partial V}\right) = \rho\left(\frac{\partial p}{\partial \rho}\right) \approx 2\cdot 10^9 \left[\frac{N}{m^2}\right]$$

density, $$\rho = \frac{m}{V}\left[\frac{kg}{m^3}\right]$$

Mass flow into a constant volume, $$\dot{m} = Q\rho = \frac{d}{dt}(V_0\rho) = V_0\frac{d\rho}{dt}$$

From definition, $$d\rho = \frac{\rho}{\beta}dp$$

Hence, $$Q = \frac{V_0}{\beta}\frac{dp}{dt}$$

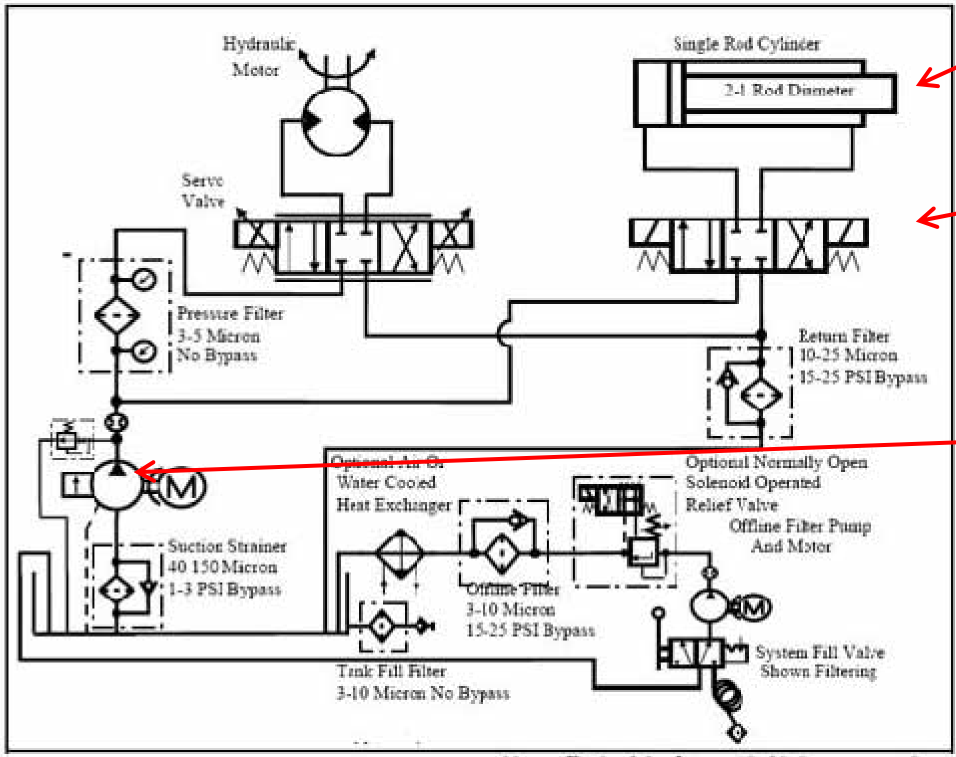
With, $$C_f = \frac{V_0}{\beta}$$

$$C_f\frac{dp}{dt} = Q$$

Constitutive equation

Hydraulic circuits

There are a lot of hydraulic details in a system but we will concentrate on a few components that are important for the dynamics.

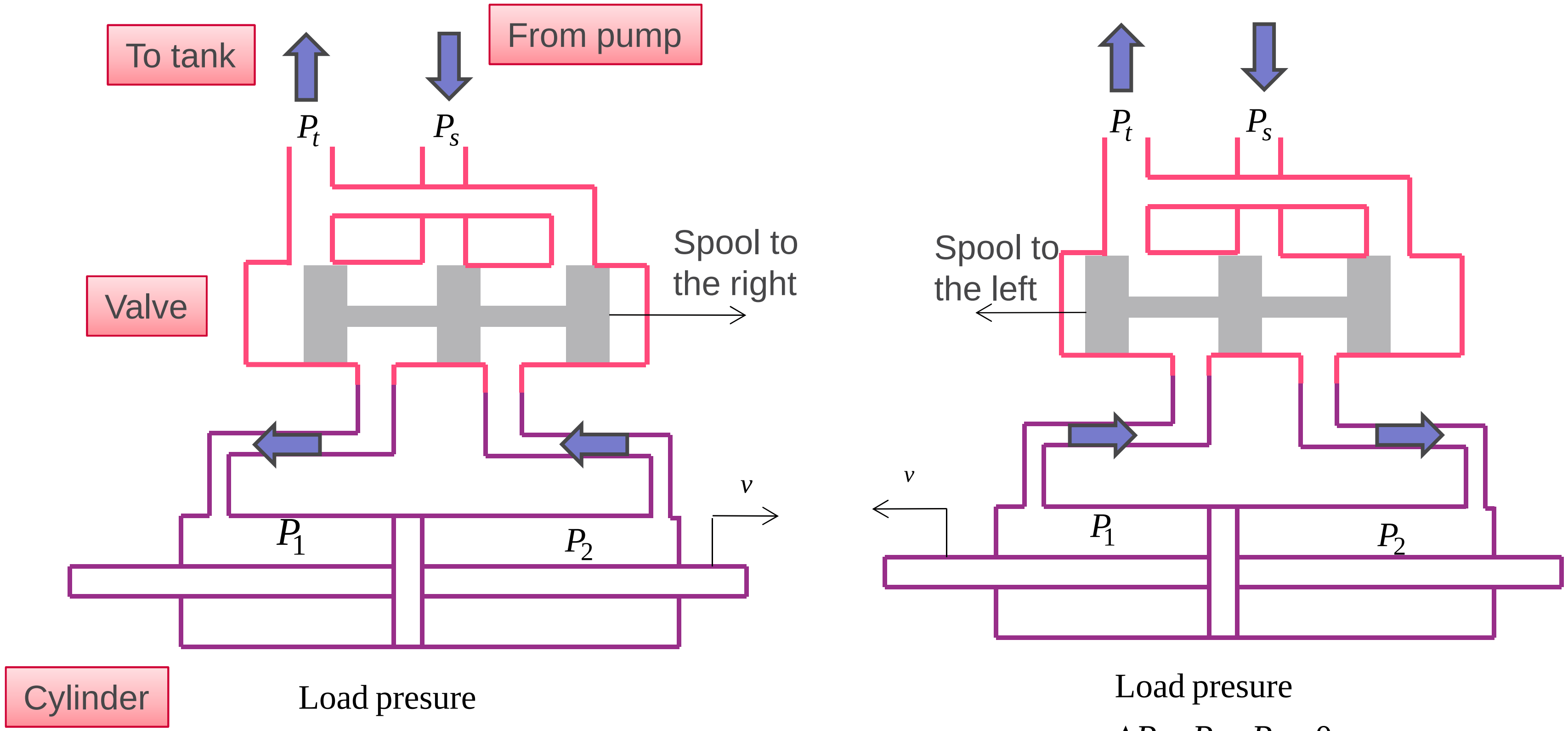


Cylinder

Flow control valve

Variable displacement pump

4-way 3 position directional valve (closed center)

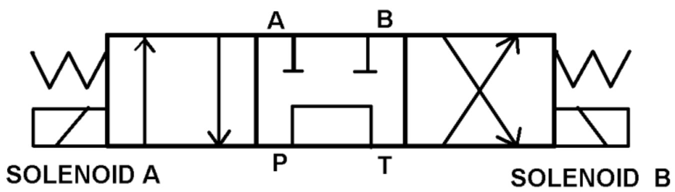


Load presure

$\Delta P = P_1 - P_2 > 0$

Load presure

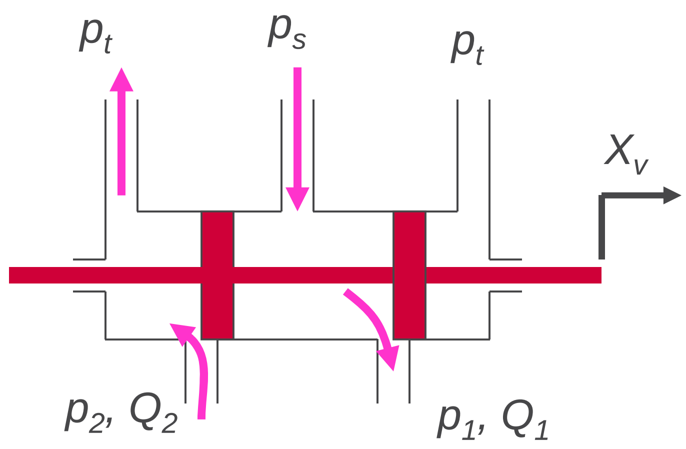
$\Delta P = P_1 - P_2 < 0$



Standard hydraulic valve component sketch

Spool valve model

- Spool assumptions
  - No leakage, equal cylinder actuator areas
  - Sharp edged, steady flow
  - Opening area proportional to x_v
  - Return pressure is zero
  - Symmetrical



Orifice model for sharp edged orifice:

$$Q = C_d A_o \sqrt{\frac{2}{\rho}\Delta p} \quad \left[\frac{m^3}{s}\right]$$

Q, flow
C_d , Discharge constant
A_o , effective opening area
ρ, density
Δp, pressure drop over orifice
R_v , a constant given by valve data sheet

set:

$$Q_1 = R_v \sqrt{p_s - p_1}\, x_v$$
$$Q_1 = R_v \sqrt{p_2 - p_t}\, x_v \qquad x_v > 0$$

$$Q_1 = R_v \sqrt{p_1 - p_t}\, x_v$$
$$Q_1 = R_v \sqrt{p_s - p_2}\, x_v \qquad x_v < 0$$

The complete model

Constitutive equations:

$$m\dot{v} = f_m$$
$$C_f \dot{p}_1 = Q_1$$
$$C_f \dot{p}_2 = Q_2$$
$$Q_{1v} = R_v \sqrt{p_s - p_1}\, x_v$$
$$Q_{2v} = R_v \sqrt{p_2}\, x_v$$

Node eq.

Loop eq.

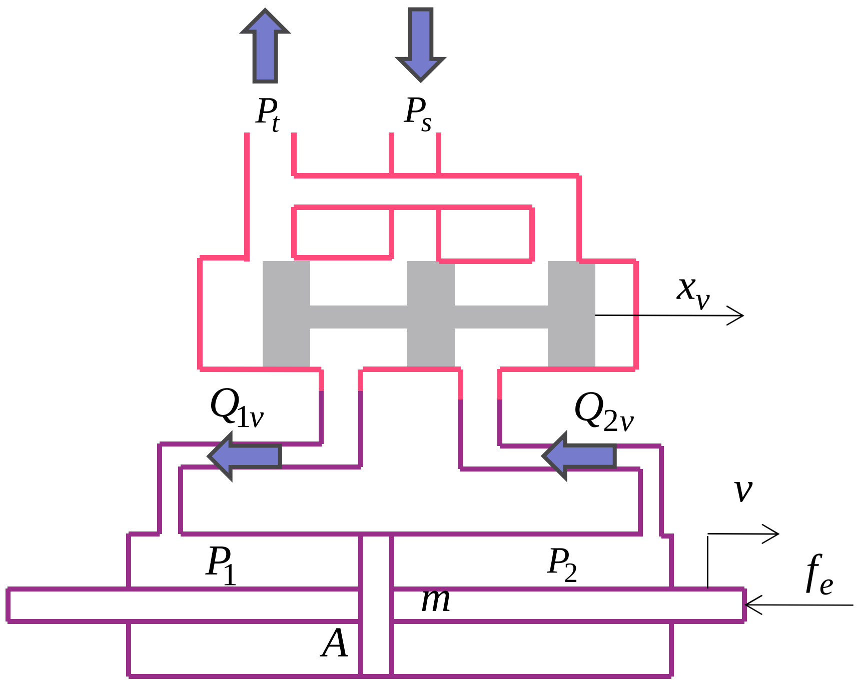
$$f_m = p_1 A - p_2 A - f_f - f_e$$
$$Q_1 = Q_{1v} - Q_c$$
$$Q_2 = -Q_{2v} + Q_c$$

The valve dynamics, spool mass and solenoid must be modeled. Physical model is difficult, flow forces on spool.

A second order model from valve input signal to spool position is usually sufficient.
Parameters from valve data sheets.

$$x_v = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} u$$

m, mass of piston, piston rod and load
A, effective piston area
f_f ,friction force, f_e , external force
$x_v > 0$



Volume flow due to piston velocity. $Q_c = Av$

Linearizing the model

Linearize around an operating point p_{1Q}, p_{2Q} and x_{vQ}, assume $p_t = f_f = 0$

$$\left.\begin{aligned}
&0 = p_{1Q}A - p_{2Q}A - f_e \Rightarrow p_{1Q} = p_{2Q} - \frac{f_e}{A} \\
&0 = R_v\sqrt{p_s - p_{1Q}}\,x_v - Av \Rightarrow \left(\frac{Av}{C_v x_v}\right)^2 = p_s - p_{1Q} \\
&0 = -R_v\sqrt{p_{2Q}}\,x_v + Av \Rightarrow \left(\frac{Av}{C_v x_v}\right)^2 = p_{2Q}
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
&p_{1Q} = \frac{p_s}{2} + \frac{f_e}{2A} \\
&p_{2Q} = \frac{p_s}{2} - \frac{f_e}{2A}
\end{aligned}$$

x_{vQ}, must be manualy selected

Define, R_i, K_i $\Longrightarrow$

$$Q_1 = K_1\Delta x_v + R_1\Delta p_l$$
$$Q_2 = K_2\Delta x_v + R_2\Delta p_2$$

where:

$$x_v = x_{vQ} + \Delta x_v$$
$$p_1 = p_{1Q} + \Delta p_1$$
$$p_2 = p_{2Q} + \Delta p_2$$

$$R_1 = \left|\frac{\partial Q_1}{\partial p_1}\right|_{\substack{p_1 = p_{1Q} \\ x_v = x_{vQ}}} = -\frac{R_v x_{vQ}}{2\sqrt{p_s - p_{1Q}}} = -\frac{1}{\sqrt{2}}\frac{R_v x_{vQ}}{\sqrt{p_s - \frac{f_e}{A}}}$$

$$R_2 = \left|\frac{\partial Q_2}{\partial p_2}\right|_{\substack{p_2 = p_{2Q} \\ x_v = x_{vQ}}} = -\frac{R_v x_{vQ}}{2\sqrt{p_{2Q}}} = -\frac{1}{\sqrt{2}}\frac{R_v x_{vQ}}{\sqrt{p_s - \frac{f_e}{A}}}$$

$$K_1 = \left|\frac{\partial Q_1}{\partial x_v}\right|_{\substack{p_1 = p_{1Q} \\ x_v = x_{vQ}}} = R_v\sqrt{p_s - p_{1Q}} = R_v\sqrt{\frac{p_s}{2} - \frac{f_e}{2A}}$$

$$K_2 = \left|\frac{\partial Q_2}{\partial x_v}\right|_{\substack{p_2 = p_{2Q} \\ x_v = x_{vQ}}} = -R_v\sqrt{p_{2Q}} = -R_v\sqrt{\frac{p_s}{2} - \frac{f_e}{2A}}$$

Linear model

Select states:

$$x_1 = x_v, x_2 = v_v, x_3 = v, x_4 = p_1, x_5 = p_2$$

d, linear friction coeficient

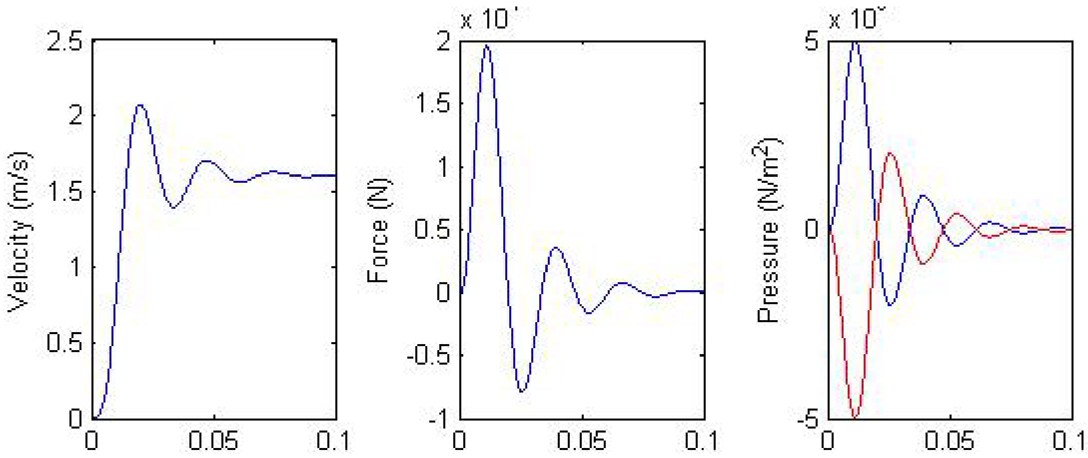
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\omega_v^2 & -2\zeta\omega_v & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{m} & \frac{A}{m} & -\frac{A}{m} \\ \frac{K_1}{C_f} & 0 & -\frac{A}{C_f} & \frac{R_1}{C_f} & 0 \\ \frac{K_2}{C_f} & 0 & \frac{A}{C_f} & 0 & \frac{R_2}{C_f} \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega_v^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

Step response

$$p_s = 20\,[MPa]$$

$$m = 100\,[kg]$$

$$A_c = \pi 0.025^2\,[m^2]$$



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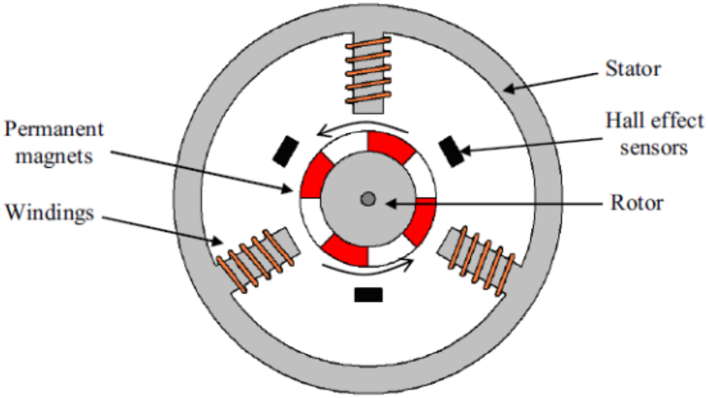
3-phase electric motors

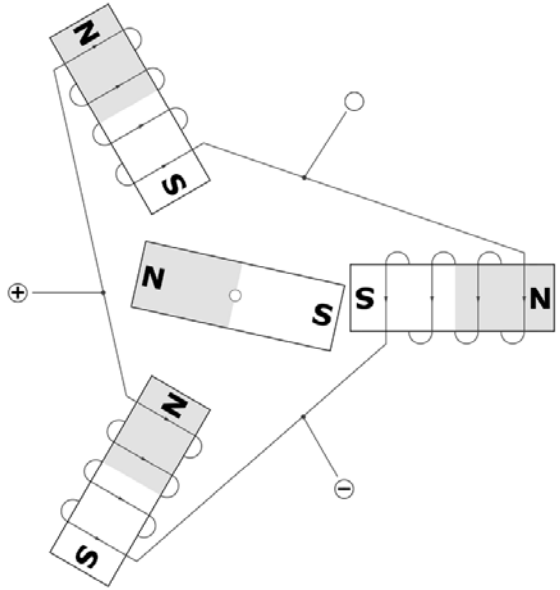
- Asynchronous machines have windings in both stator and rotor
- Permanent magnet 3-phase motors have only winding in stator
  - Also called Synchronous motors (rpm synchronous to electric field rotation)
- Two types
  - Brushless DC motor BLDC or Trapezoidal motor
  - Permanent Magnet Synchronous Machine PMSM or Sinusoidal motor
- Advantage over DC-motor
  - cooling -> higher currents and/or smaller size
- Disadvantage over DC-motor
  - More advanced control -> electronic commutation (software)

Electromechanical design

8-pole motor (4 magnets)



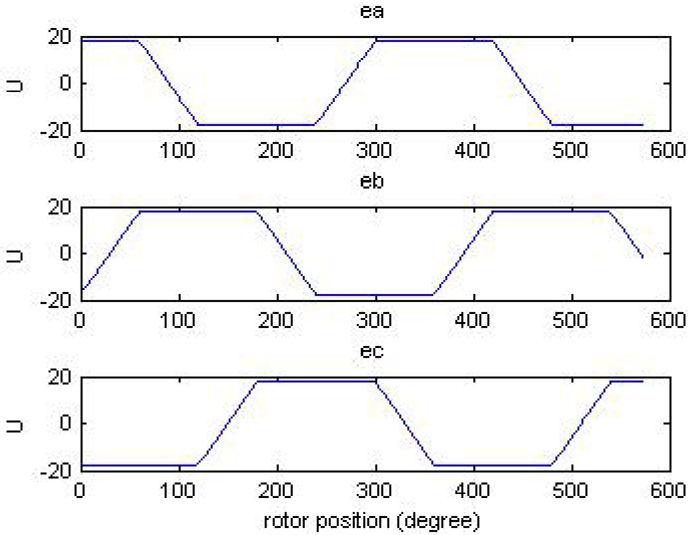
2-pole motor (1 magnet)



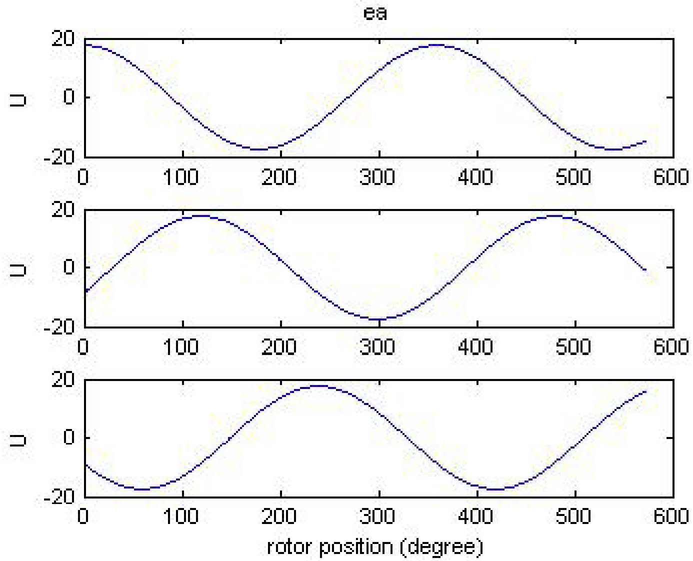
http://www.stefanv.com/rcstuff/qf200212.html

Back EMF depends on motor design

Trapezoidal EMF



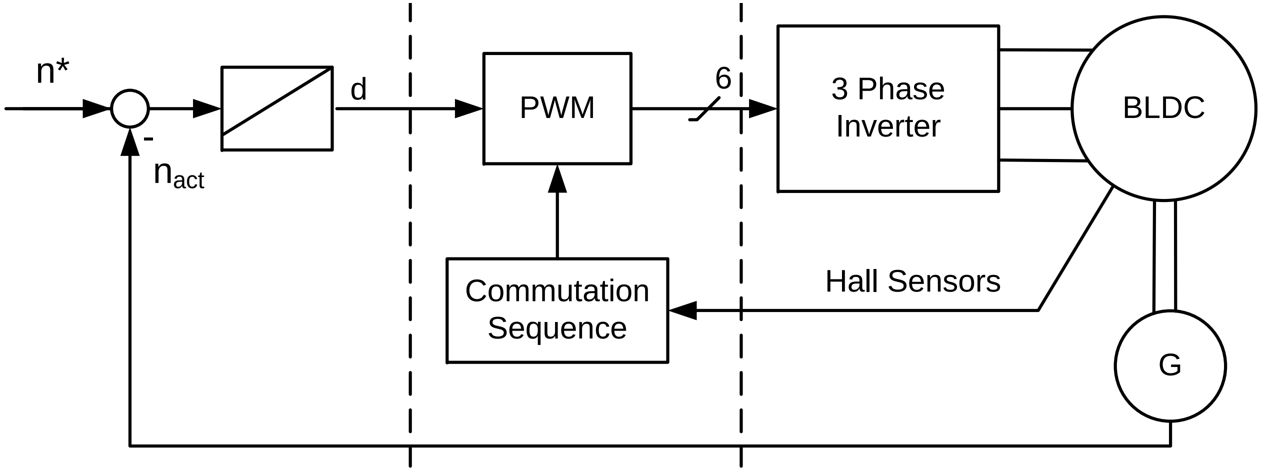
Sinewave EMF



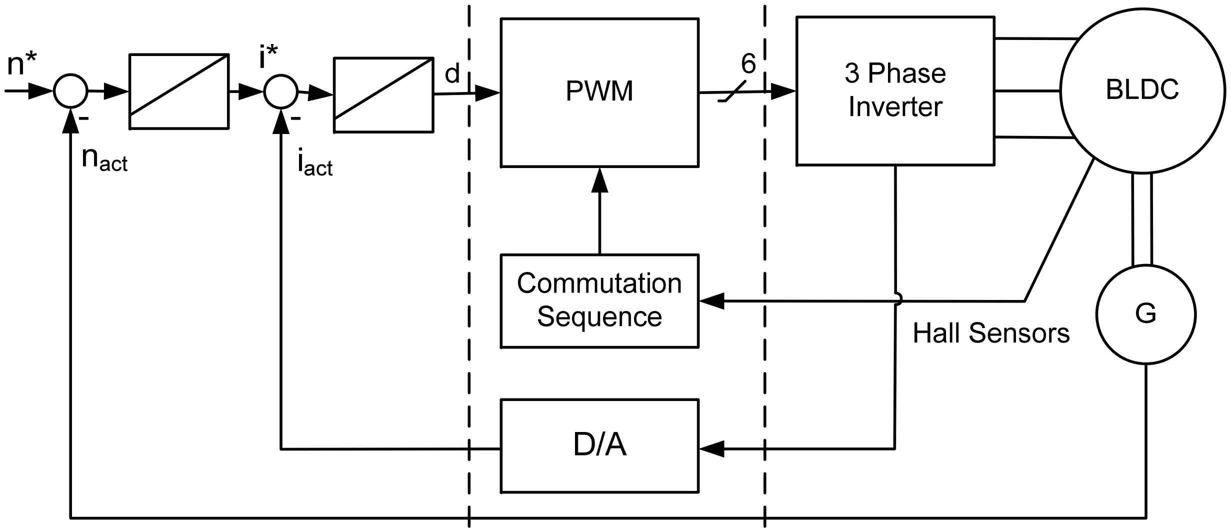
Different Modeling and control strategies are used for the two kinds

BLDC control structure (trapezoidal)

BLDC speed control

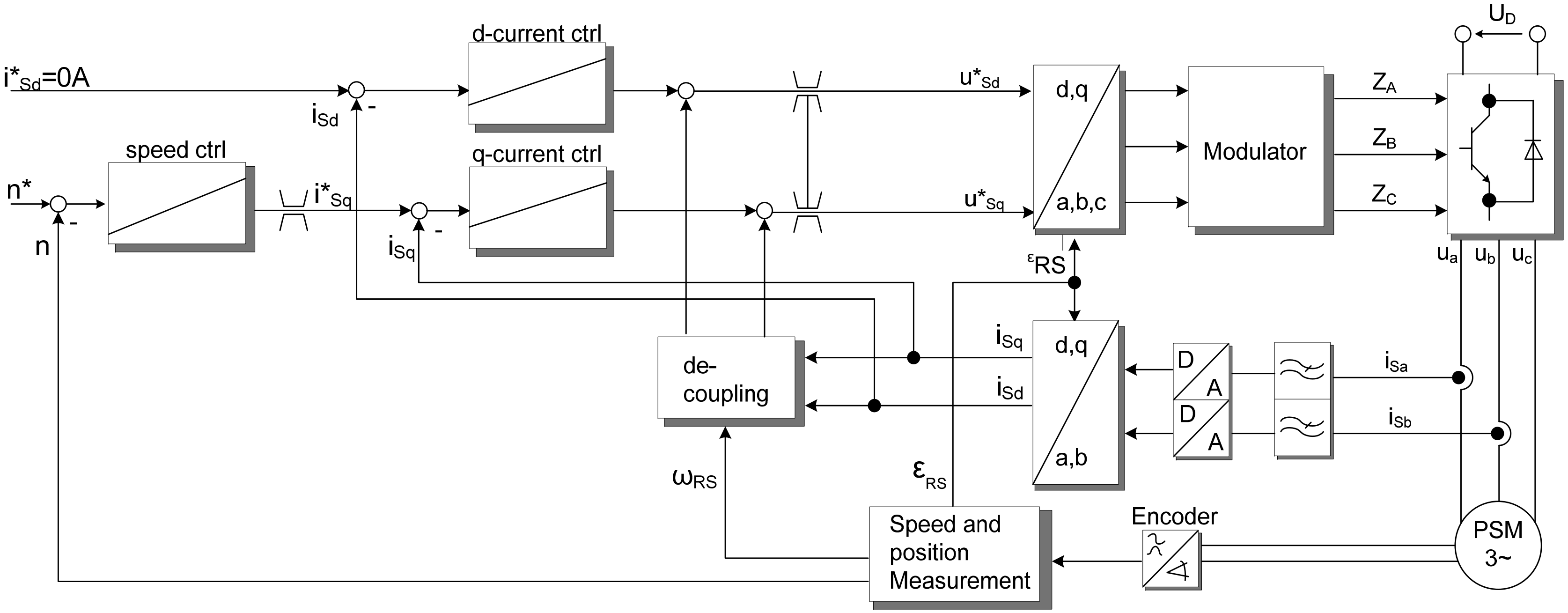


BLDC speed and current cascaded control



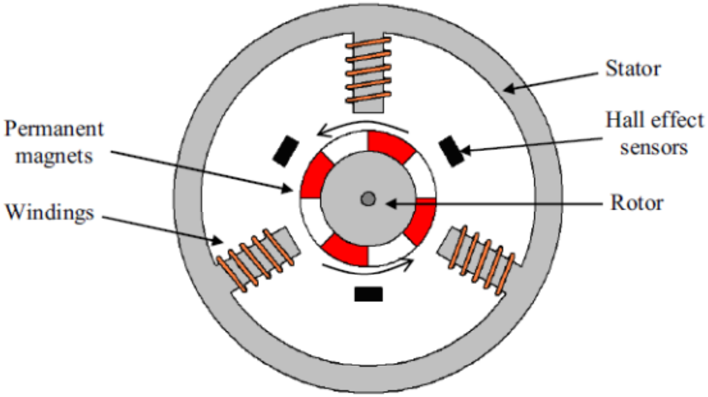
PMSM control structure

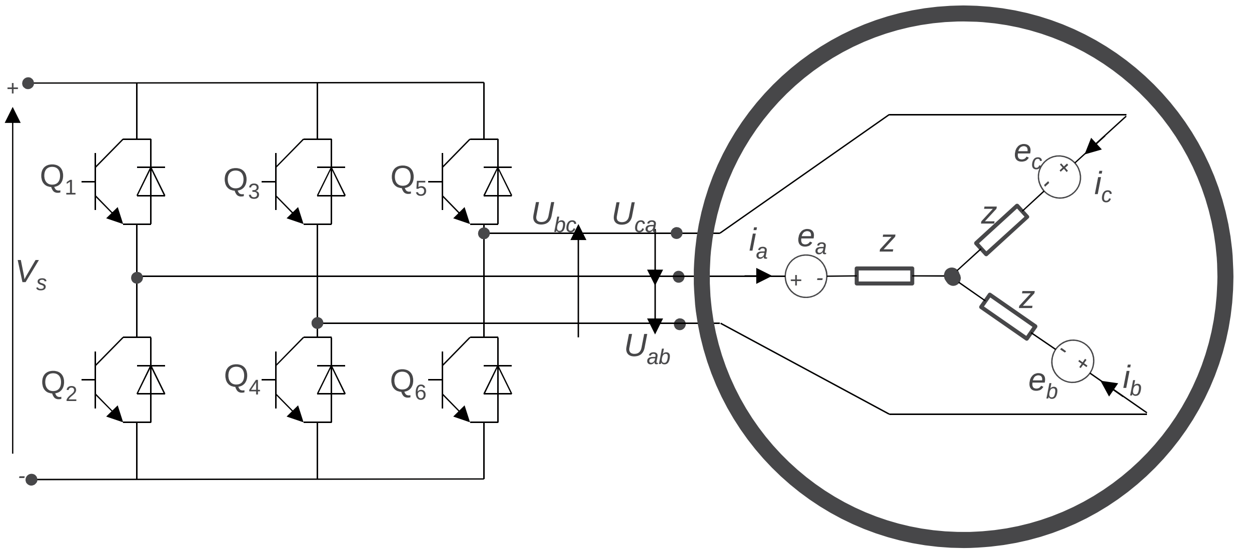
Field Oriented Control

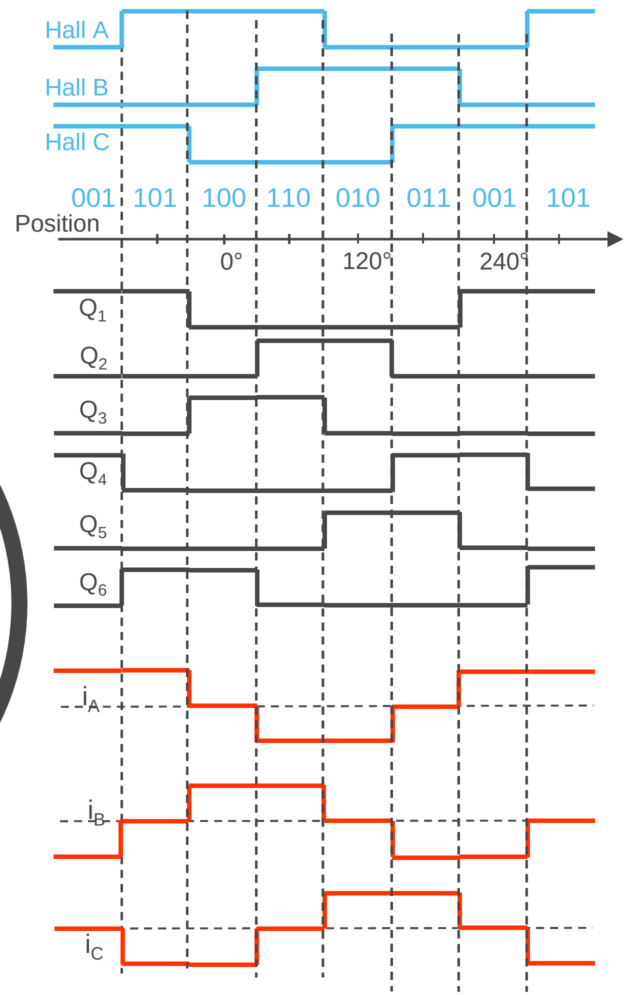


Phase currents are sampled synchronously to PWM signals

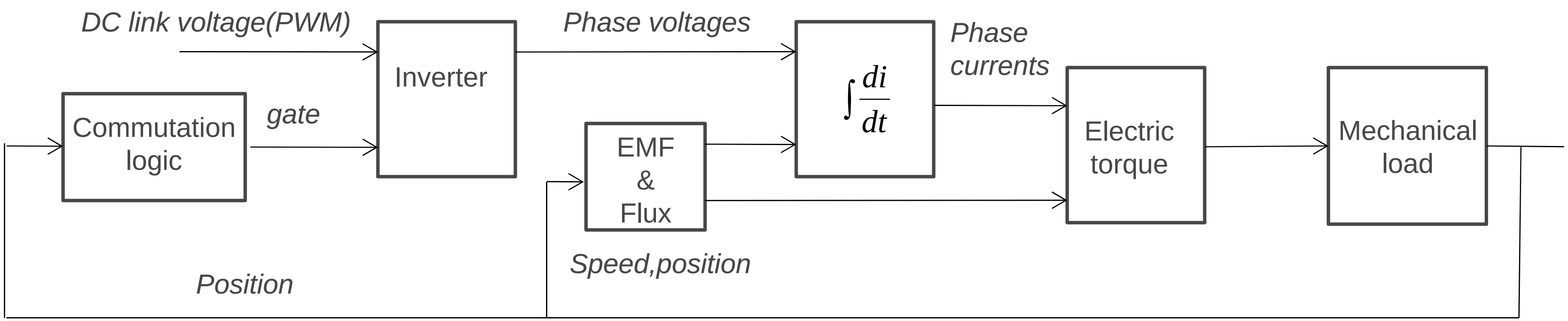
Commutation of trapezoidal motor (BLDC)







BLDC model structure



Modeling steps:

1. Set up the differential equations for the phase currents
2. Model the shape of the EMF and flux
3. Calculate the electric torque
4. Model the commutation logic based on hall sensors or position
5. Model the inverter

4 and 5 can be modeled in one state machine (state flow)

1. Differential equations for phase currents

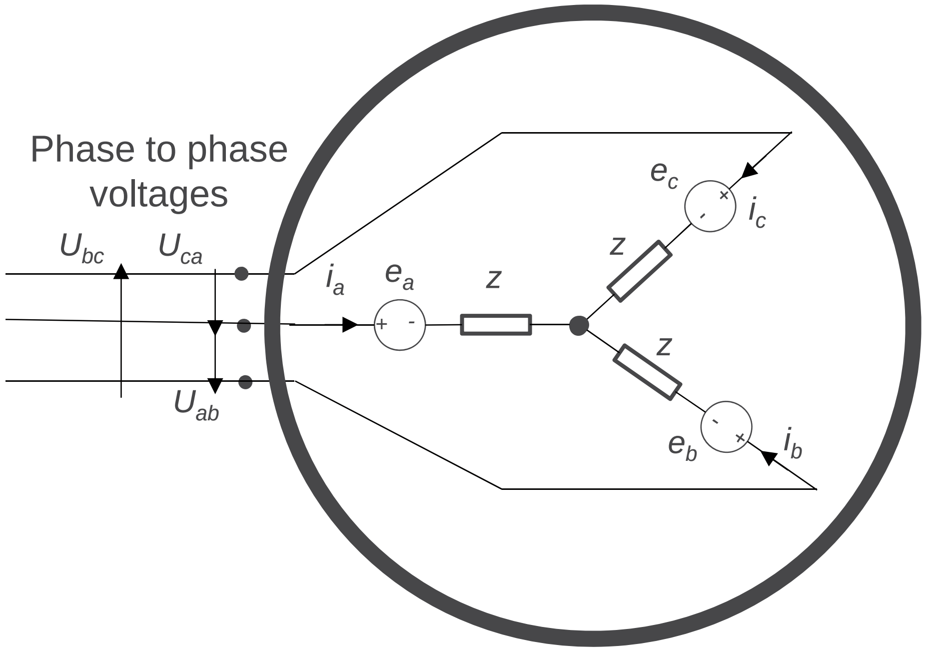
$$\begin{bmatrix} U_{ab} \\ U_{bc} \\ U_{ca} \end{bmatrix} = \left(R + L\frac{d}{dt} \right) \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix}$$

$$z = Ri + L\frac{d}{dt}i$$

Same R and L in each phase

$$i_c = -i_a - i_b$$

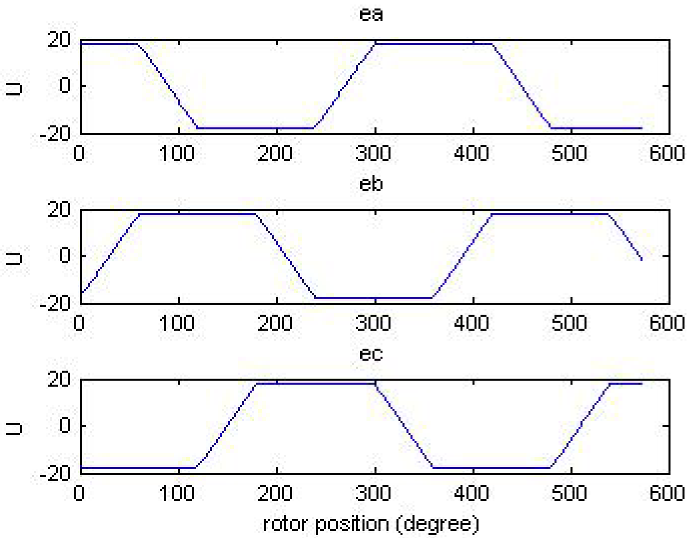
$$\begin{bmatrix} U_{ab} \\ U_{bc} \end{bmatrix} = \left(R + L\frac{d}{dt} \right) \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix}$$



$$\frac{d}{dt}i_a = \frac{1}{L}\left(-Ri_a + \frac{2}{3}(U_{ab} - E_{ab}) + \frac{1}{3}(U_{bc} - E_{bc}) \right)$$

$$\frac{d}{dt}i_b = \frac{1}{L}\left(-Ri_b + \frac{1}{3}(U_{ab} - E_{ab}) + \frac{1}{3}(U_{bc} - E_{bc}) \right)$$

2. Back EMF model



K_e Electromotoric constant [V/rad]

ω_e Electric velocity [rad/s]

If magnetic poles is 2, $\omega_e = \omega_\mathrm{m}$

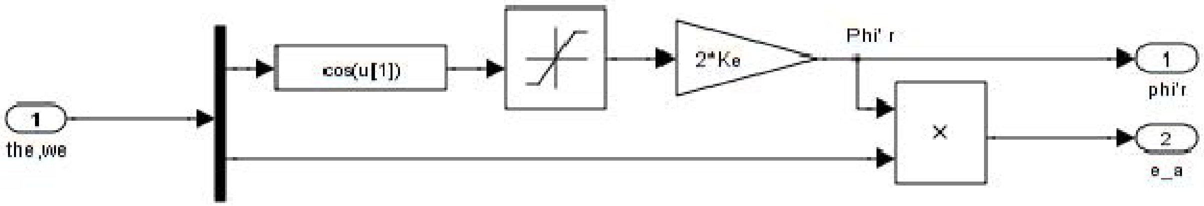
$$e_a = f(\theta) K_e \omega_e$$

$$e_b = f\left(\theta - \frac{2\pi}{3}\right) K_e \omega_e$$

$$e_c = f\left(\theta - \frac{4\pi}{3}\right) K_e \omega_e$$

$$f = \begin{cases} 1, 0 \le \theta < \frac{2\pi}{3} \\ 1 - \frac{6}{\pi}\left(\theta - \frac{2\pi}{3}\right), \frac{2\pi}{3} \le \theta < \pi \\ -1, \pi \le \theta < \frac{5\pi}{3} \\ \frac{6}{\pi}\left(\theta - \frac{5\pi}{3}\right) - 1, \frac{5\pi}{3} \le 2\pi \end{cases}$$

A simple way to simulate is to, take cos(*f*()) and saturate it between -0.5…0.5 and then multiply it with 2. Which is how the plot above has been done.



3. Electric torque

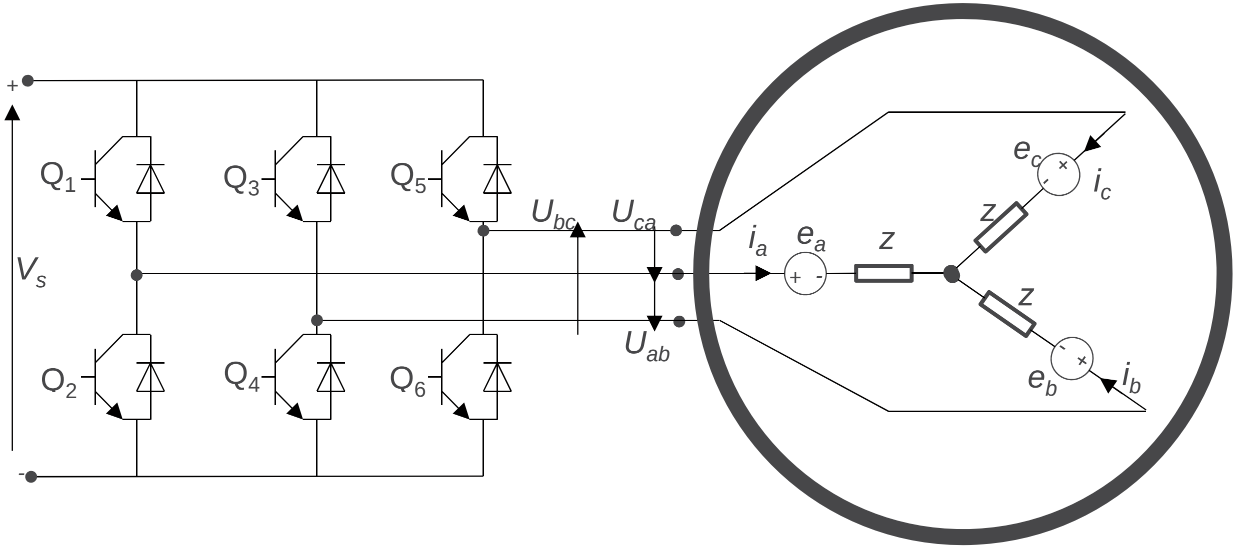
Same shape as the EMF

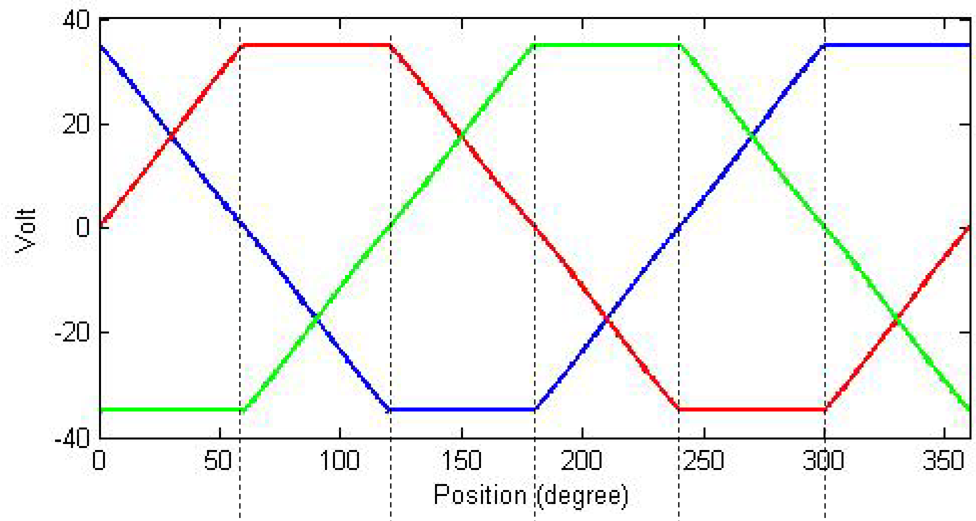
$$T_e = K_t \left(f(\theta) i_a + f\left(\theta - \frac{2\pi}{3}\right) i_b + f\left(\theta - \frac{4\pi}{3}\right) i_c \right)$$

4. Commutation logic

One way to find the correct commutation sequence is to calculate the phase to phase EMF. $E_{xy} = e_x - e_y$
Maximum magnetic torque is achieved when the phase currents are flow in the same direction, for example for E_{ab} should $i_a > 0$ and $i_b < 0$. Which is achieved with
$U_{ab} = V_s$.

See xxx for proof.





Blue E_{ab}

Red E_{bc}

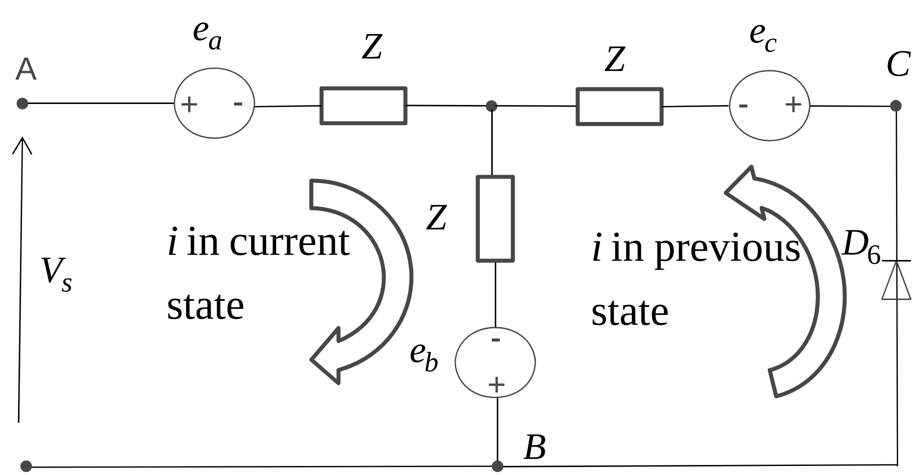
Green E_{ca}

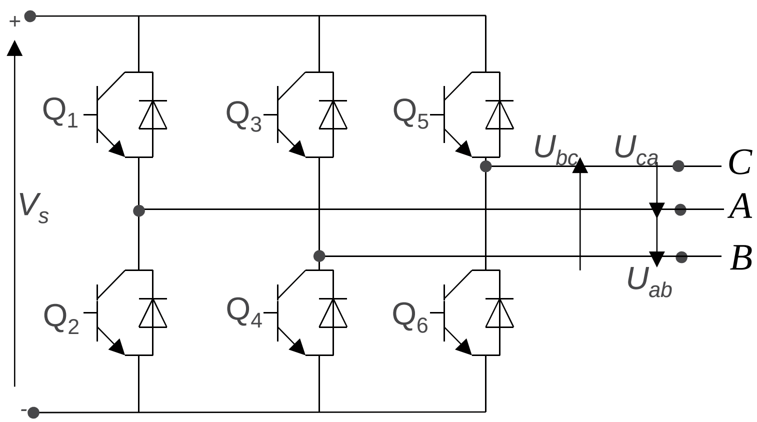
Closed transistors ->	Q_1Q_6	Q_3Q_6	Q_3Q_2	Q_5Q_2	Q_5Q_4	Q_1Q_4
Energized phases ->	$U_{ca}=-V_s$	$U_{bc}=V_s$	$U_{ab}=-V_s$	$U_{ca}=V_s$	$U_{bc}=-V_s$	$U_{ab}=V_s$

5. Model the inverter

Each energized state must be modeled separately
Let's start with state Q_1Q_4 when $U_{ab} = V_s$
What is then U_{bc}?

Redraw the motor inverter system for easier analysis





Is phase C connected to plus or ground?

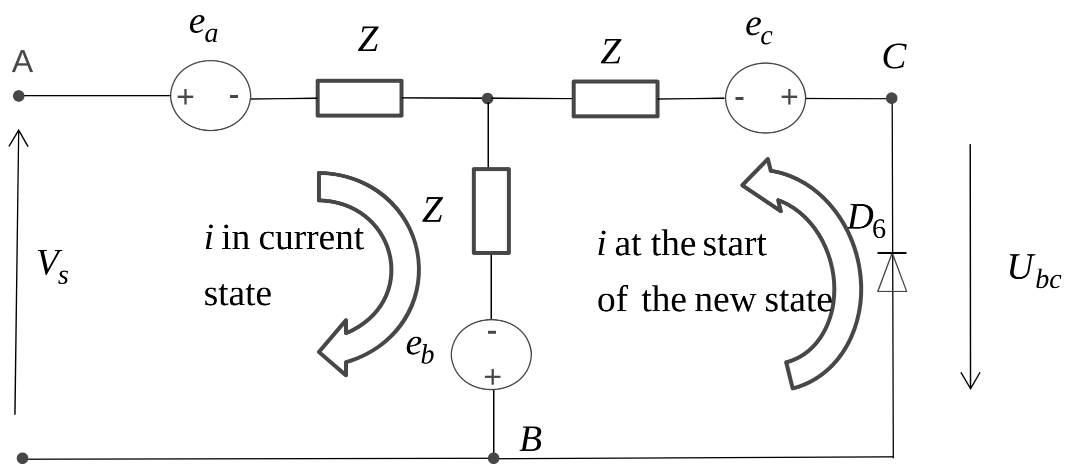
It depends on the direction of the current in C from previous state

For positive direction (rotation) was previous state U_{bc} = - V_s , Q_5 Q_4 closed

Gives $i_c > 0$

5. Model the inverter

Now can we calculate U_{bc}

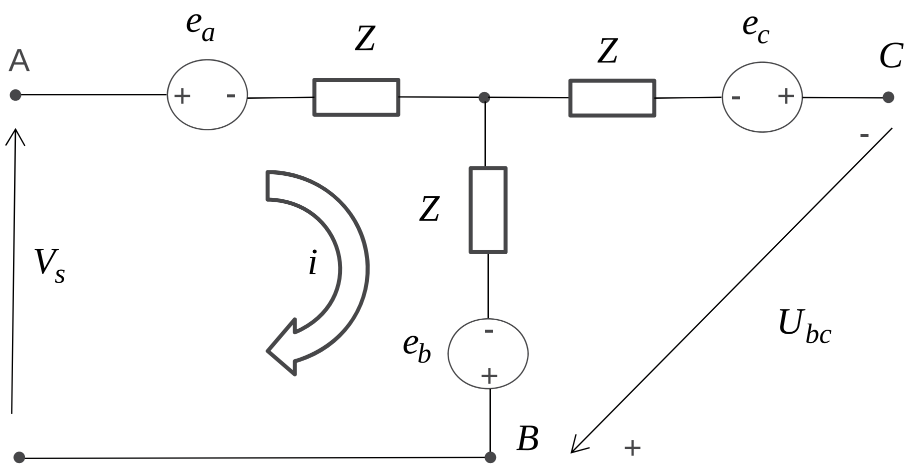


$U_{bc} = 0$ short circuit between B and C

After some time the current in C will become zero, what happens then?

5. Model the inverter

Equivalent circuit when $i_c = 0$



There are two loops, one directly from B to C and one via A

$$\text{Loop BC}, -e_b - Zi_b + e_c + U_{bc} = 0$$

$$\text{Loop BAC}, \; V_s - e_a - Zi_a + e_c + U_{bc} = 0$$

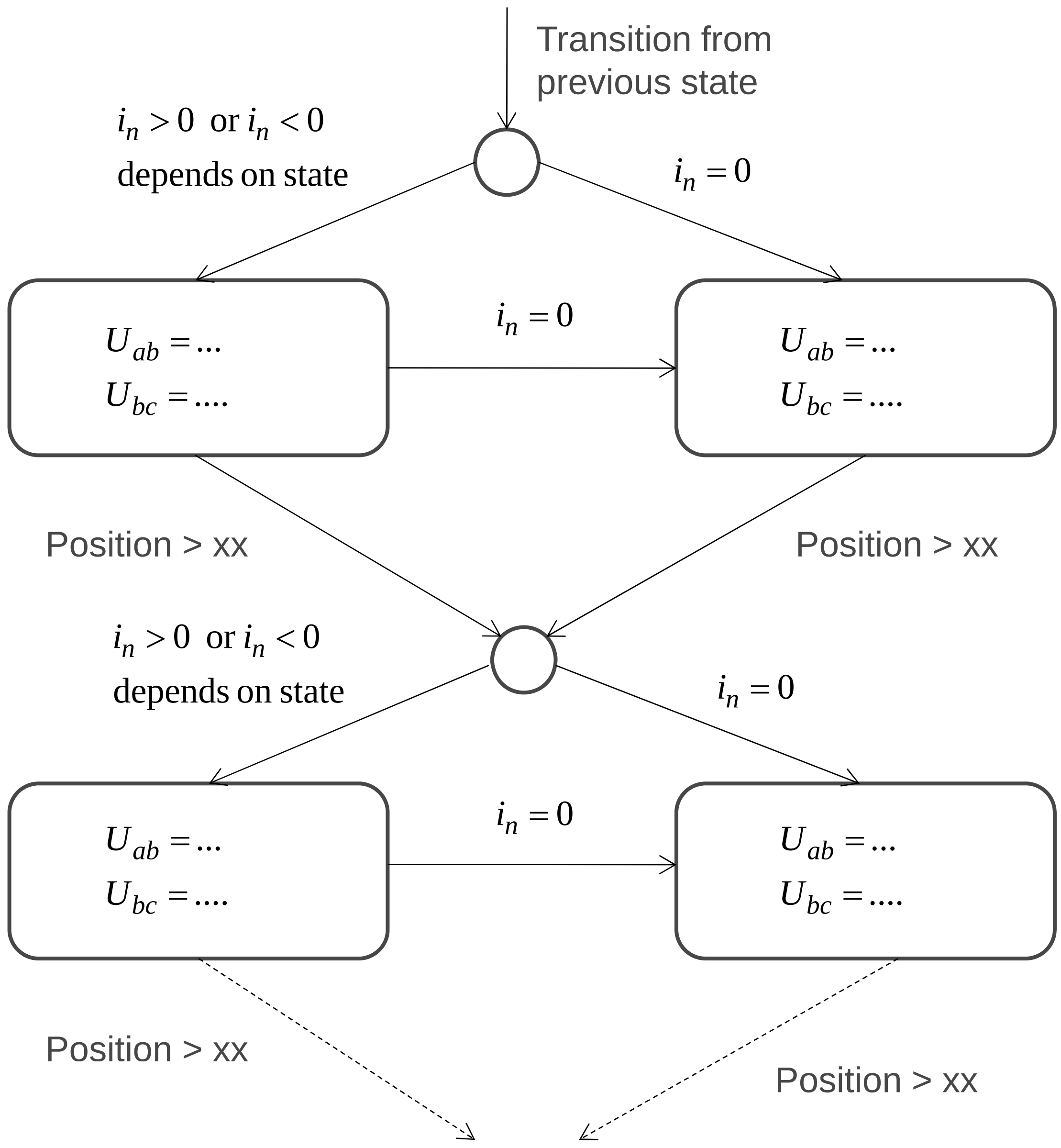
$$i_c = 0, \; \text{gives } i_b = -i_a$$

hence:

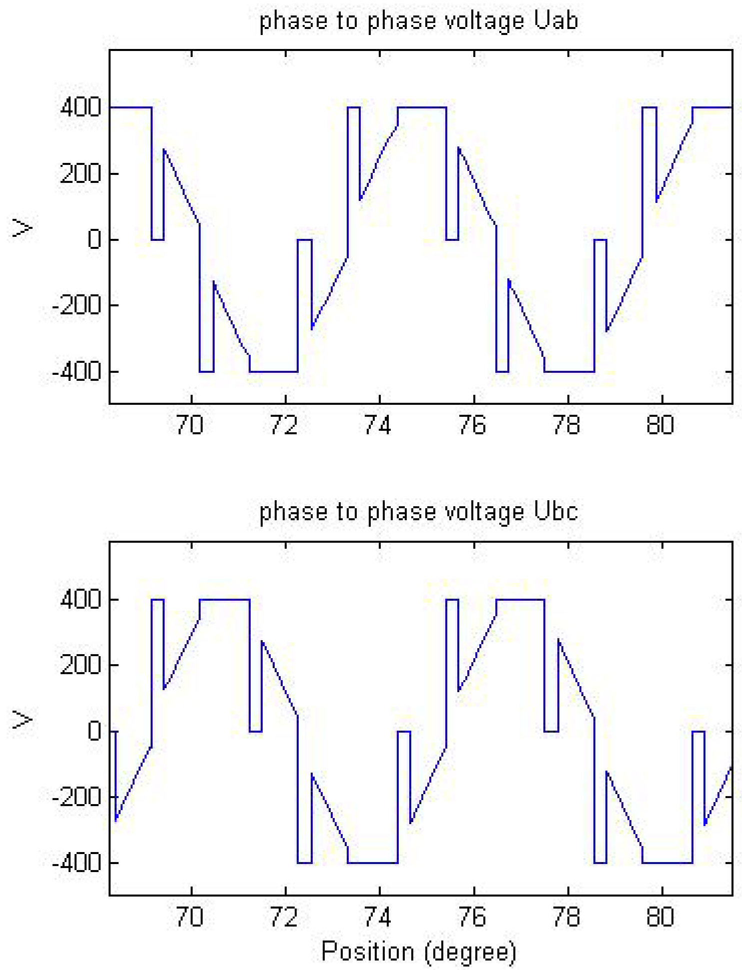
$$U_{bc} = \frac{1}{2}\left(-V_s + e_a + e_b - 2e_c\right) = \frac{1}{2}\left(-V_s + E_{ac} + E_{bc}\right)$$

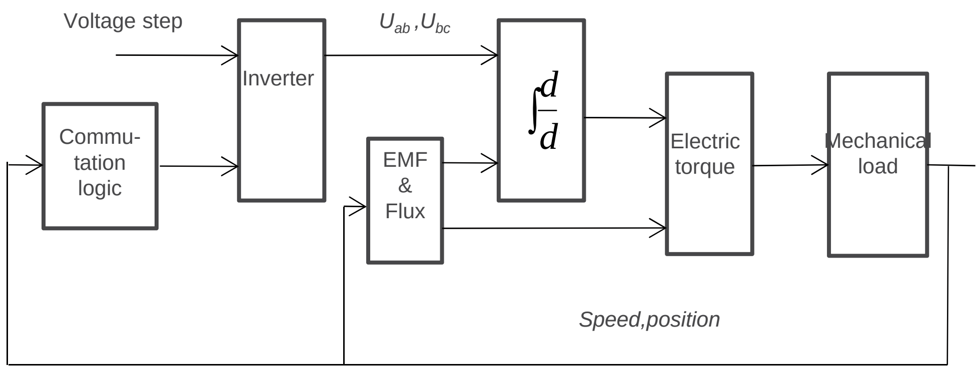
This must be done for all six states in both directions of rotation but, only for U_{ab} and U_{bc} since U_{ca} is not needed in the differential equations.

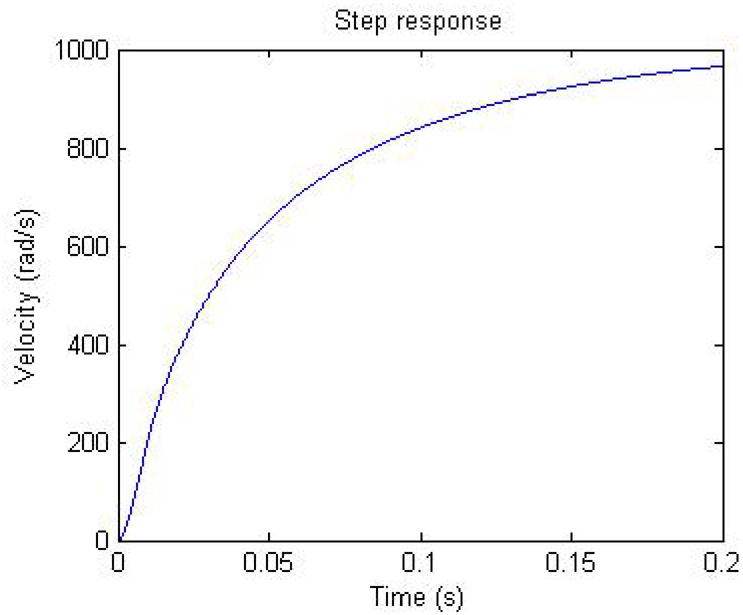
5. Model the inverter



Voltage step input

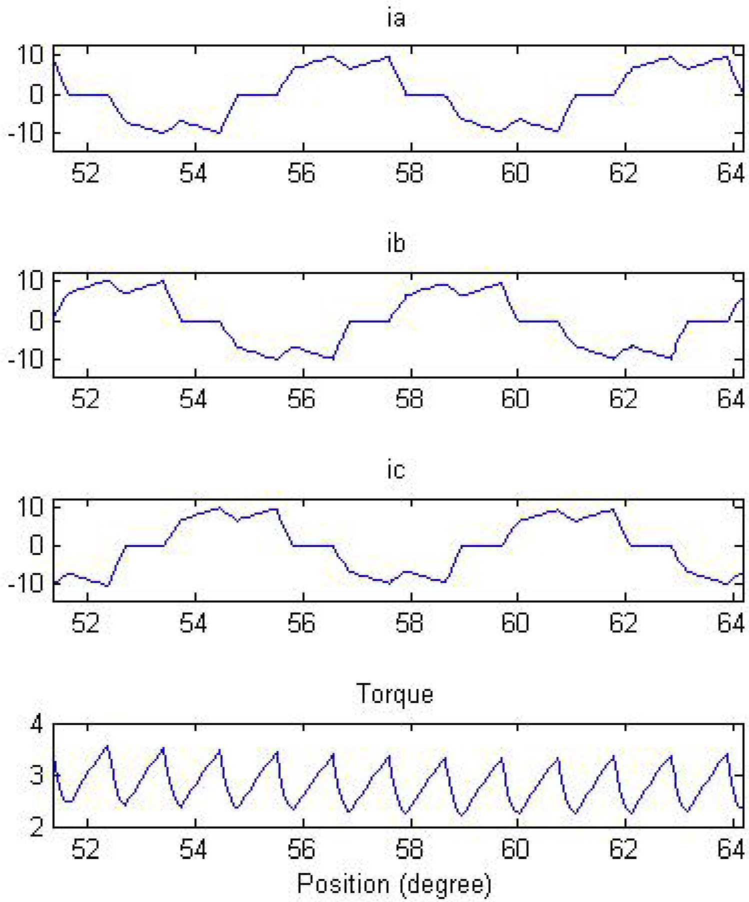






Step response cont.

Currents



Torque ripple because of the commutation