

## Homework 3

## Finite Differences and Absolut Stability

due 6/2-2012

**Task 1: Finite Difference Scheme**

Find the highest order approximation possible of the first derivative based on the grid values  $u_{i-1}$ ,  $u_i$ ,  $u_{i+1}$  and  $u_{i+2}$ . Assume equidistant grid spacing  $\Delta x$ .

$$\left. \frac{\partial u}{\partial x} \right|_{x=x_i} \approx f(u_{i-1}, u_i, u_{i+1}, u_{i+2}) \quad (1)$$

- Give the approximation for the derivative
- What is the leading error term? Of what order is the scheme?
- Implement this derivative in a similar way as in Task 2 of Homework 2 and assess numerically the order of its accuracy.

**Task 2 : Stability criteria**

The range of absolute stability of the Runge-Kutta 4<sup>th</sup>-order method is studied. This method for an initial value problem of the form  $\frac{du}{dt} = f(u, t)$ ,  $u(t_0) = u_0$  is

$$u^{n+1} = u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \quad (2)$$

$$t^n = n\Delta t \quad (3)$$

where

$$f^n = f(u^n, t^n) \quad (4)$$

$$k_1 = f(u_1, t^{n+\frac{1}{2}}), u_1 = u^n + \frac{\Delta t}{2}f^n, t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2} \quad (5)$$

$$k_2 = f(u_2, t^{n+\frac{1}{2}}), u_2 = u^n + \frac{\Delta t}{2}k_1 \quad (6)$$

$$k_3 = f(u_3, t^{n+1}), u_3 = u^n + \Delta tk_2 \quad (7)$$

Consider a simple linear test equation (*Dahlquist equation*):

$$\frac{du}{dt} = \lambda u. \quad (8)$$

Show that  $u^{n+1}$  can be written as a function of  $u^n$  and  $z = \Delta t\lambda$  as follows

$$u^{n+1} = u^n \left( 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \right). \quad (9)$$

The absolute stability criterion is

$$|G(z)| = \left| \frac{u^{n+1}}{u^n} \right| \leq 1. \quad (10)$$

Draw the region that corresponds to equation (10) on the complex  $z$ -plane (Hint: The curve  $|G(z)| = 1$  cuts the imaginary axis at  $\pm 2.83$ ).

### Task 3: The modified wavenumber

On an equidistant grid, the finite-difference derivative of a Fourier mode  $e^{ikx}$  can be found by multiplying the function value on each node with the so called modified wavenumber  $\tilde{k}(k)$ .

To better understand that concept, consider a periodic function

$$f(x) : \mathbb{R} \rightarrow \mathbb{C}, f(x + 2\pi) = f(x) \quad \forall x. \quad (11)$$

Let  $\underline{f}$  be the discrete representation of function  $f(x)$  on an equidistant grid where  $x_j = j\Delta x$ ,  $\Delta x = 2\pi/N$ ,  $j = 0, 1, \dots, N-1$  with  $N = 20$ ,

$$\underline{f} = [f_0, f_1, \dots, f_{N-1}] \quad \text{where } f_j = f(x_j). \quad (12)$$

- a) Write a Matlab script that computes the matrix  $\underline{\underline{D}}$  corresponding to left-sided finite differences of first order. The matrix  $\underline{\underline{D}}$  is defined as

$$\underline{f}'_{num} = \underline{\underline{D}} \underline{f} \quad (13)$$

with the vector  $\underline{f}'_{num}$

$$\underline{f}'_{num} = [\delta f_0, \delta f_1, \dots, \delta f_{N-1}]^T, \quad (14)$$

and the operator  $\delta f_j$

$$\delta f_j = \frac{f_j - f_{j-1}}{\Delta x}. \quad (15)$$

Remember that  $f(x)$  is periodic when computing the derivative at the point  $x = x_0$ , i.e.  $f_{-1} = f_{N-1}$ .

- b) Assume  $f(x) = e^{ikx}$  and derive the expression for the modified wavenumber  $\tilde{k}$  for these left-sided finite-difference scheme. Non-dimensionalise the wavenumber with the grid spacing, i.e. derive  $\Delta x \tilde{k}(k\Delta x)$ .
- c) From now on assume that  $k = 5$  (i.e. a specific wave). Compute the derivative in a discrete ( $\delta f_j$ ) and analytical ( $f'_{x=x_j}$ ) manner in every grid point. Use the previously defined  $\underline{\underline{D}}$  for the discrete derivative. Plot the real part for both the numerical and analytical derivative as a function of  $x$ .
- d) Plot the real and imaginary part of the complex quantity

$$\mu_j = \frac{\delta f_j}{f_j} \quad (16)$$

as a function of  $x$ . What can you say about the significance of  $\mu_j$  (see part f) below)?

- e) The result of part c) indicates that the vector corresponding to the discrete Fourier harmonic  $\underline{F}_k = [e^{ikx_0} \quad e^{ikx_1} \quad \dots \quad e^{ikx_{N-1}}]^T$  is in special relation with the matrix  $\underline{\underline{D}}$ . Can you say what that relation between  $\underline{F}_k$  and  $\underline{\underline{D}}$  is in terms of linear algebra?
- f) Compare the values of  $\mu_j$  with the complex number  $i\tilde{k}$ ,  $k = 5$  where  $\tilde{k}$  is the modified wavenumber for the left-sided finite differences as derived in b).