

Homework 6

due 5/3-2012

Iterative Methods

A common method for the computation of time-dependent incompressible flows is the so-called *projection method*, which will be discussed in detail during the course. In this method, the main effort at each time step is to solve a Poisson equation for the pressure. When discretising the Navier–Stokes equations on a staggered grid, one can show that homogeneous Neumann boundary conditions for the pressure can be imposed.

Let us consider a two-dimensional flow. Then we have

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f, \quad \frac{\partial p}{\partial n} = 0 \text{ on boundaries.} \quad (1)$$

Since the boundary conditions on all boundaries involve only derivatives of pressure, to find an unique solution, we need to specify the value of pressure at some point in the domain.

In this homework you are asked to apply an iterative method to solve the equation above for a domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The forcing function f is explicitly given as

$$f(x, y) = \cos(\pi x) \cos(\pi y) . \quad (2)$$

Equation (1) should be discretised with second-order central differences. Use a grid with $N \times N$ cells with spacing $h = 1/(N)$, see figure 1. Assemble the numerical solution \underline{p} and the inhomogenous part \underline{f} as one-dimensional vectors of the form

$$\underline{p} = (p_{1,1}, p_{2,1}, \dots, p_{N,1}, p_{1,2}, \dots, p_{N-1,N}, p_{N,N})^T , \quad (3)$$

$$\underline{f} = (f_{1,1}, f_{2,1}, \dots, f_{N,1}, f_{2,1}, \dots, f_{N-1,N}, f_{N,N})^T , \quad (4)$$

where $p_{i,j}$ and $f_{i,j}$ are the values of p and f on the cell centres with the coordinates $x_i = (i-1/2)h$ and $y_j = (j-1/2)h$ (the black dots in figure 1).

Then equation (1) is rewritten as a linear system of N^2 equations of the form $\underline{A} \underline{p} = \underline{f}$, in which the boundary conditions yet have to be implemented. The homogeneous Neumann condition means that the pressure in cells close to boundary should be equal to those at dummy nodes (gray dots in figure 1), i.e.,

$$p_{1,j} = p_{0,j}, \quad p_{N,j} = p_{N+1,j}, \quad p_{i,1} = p_{i,0}, \quad p_{i,N} = p_{i,N+1} . \quad (5)$$

This corresponds to a discretised homogeneous Neumann-condition of order one.

Two iterative methods will be used in this problem, the Gauss-Seidel and the Successive Over-Relaxation (SOR) method. Both of these methods when applied to the discretized Poisson equation using central differences, can be written explicitly as

$$p_{i,j}^{(m+1)} = (1 - \omega)p_{i,j}^{(m)} + \frac{\omega}{4} \left[p_{i+1,j}^{(m)} + p_{i-1,j}^{(m+1)} + p_{i,j+1}^{(m)} + p_{i,j-1}^{(m+1)} - h^2 f_{i,j} \right] , \quad (6)$$

where the Gauss-Seidel method is recovered for $\omega = 1$. $p^{(m)}$ denotes the iterative solution after m number of iterative steps.

Applying SOR to discrete Laplace operator, including boundary conditions (5), along $i = 1$ yields

$$p_{1,1}^{(m+1)} = (1 - \omega)p_{1,1}^{(m)} + \frac{\omega}{2} [p_{2,1}^{(m)} + p_{1,2}^{(m)} - h^2 f_{1,1}] , \quad (7)$$

$$p_{1,j}^{(m+1)} = (1 - \omega)p_{1,j}^{(m)} + \frac{\omega}{3} [p_{2,j}^{(m)} + p_{1,j+1}^{(m)} + p_{1,j-1}^{(m+1)} - h^2 f_{1,j}] \quad (j = 2, N - 1) , \quad (8)$$

$$p_{1,N}^{(m+1)} = (1 - \omega)p_{1,N}^{(m)} + \frac{\omega}{2} [p_{2,N}^{(m)} + p_{1,N-1}^{(m+1)} - h^2 f_{1,N}] . \quad (9)$$

Note that SOR is applied after inclusion of the boundary conditions. The equations along $i = N$, $j = 1$ and $j = N$ should be modified in a similar way. As mentioned before, since the boundary conditions are only on the derivatives of p , the value of the pressure will be only determined up to an additive constant. To fix this constant replace the equation at node (N, N) with $p_{N,N} = 0$.

Your task is to

- Investigate if the problem is well posed. Consider in particular the boundary conditions.
- Choose $N = 30$. First find the numerical solution $\underline{p}_{\text{direct}}$ of the problem by computing the direct solution of the 900×900 matrix. Present the solution in a suitable plot (contour or 3D).
- Implement the Gauss-Seidel and the SOR methods using the relaxation factor $\omega = 1.5$. Choose the initial condition for inner points $\underline{p}^{(0)} = \underline{0}$. The iterative solution after m iterations is denoted $\underline{p}^{(m)}$. Show the convergence by plotting the residual

$$R^{(m)} = \|\underline{A} \underline{p}^{(m)} - \underline{f}\|_2 = \sqrt{(\underline{A} \underline{p}^{(m)} - \underline{f})^T \cdot (\underline{A} \underline{p}^{(m)} - \underline{f})} \quad (10)$$

versus the number of iterations m in a semilogarithmic plot.

- Plot the L_2 norm of the difference between the Gauss-Seidel and SOR methods from part c) and the direct solution from part b),

$$\varepsilon^{(m)} = \|\underline{p}^{(m)} - \underline{p}_{\text{direct}}\|_2 , \quad (11)$$

with respect to m in a semilogarithmic plot. Compare and discuss the various iterative methods, and compare with the residuals obtained in part c).

- Repeat part d) with $N = 20$. Compare the results for $\varepsilon^{(m)}$ versus m for $N = 20$ and $N = 30$. How does the convergence rate change with respect to N ? Estimate the convergence rate from the slope of the residual plot.

Hint: A template of the program can be found on the homepage.

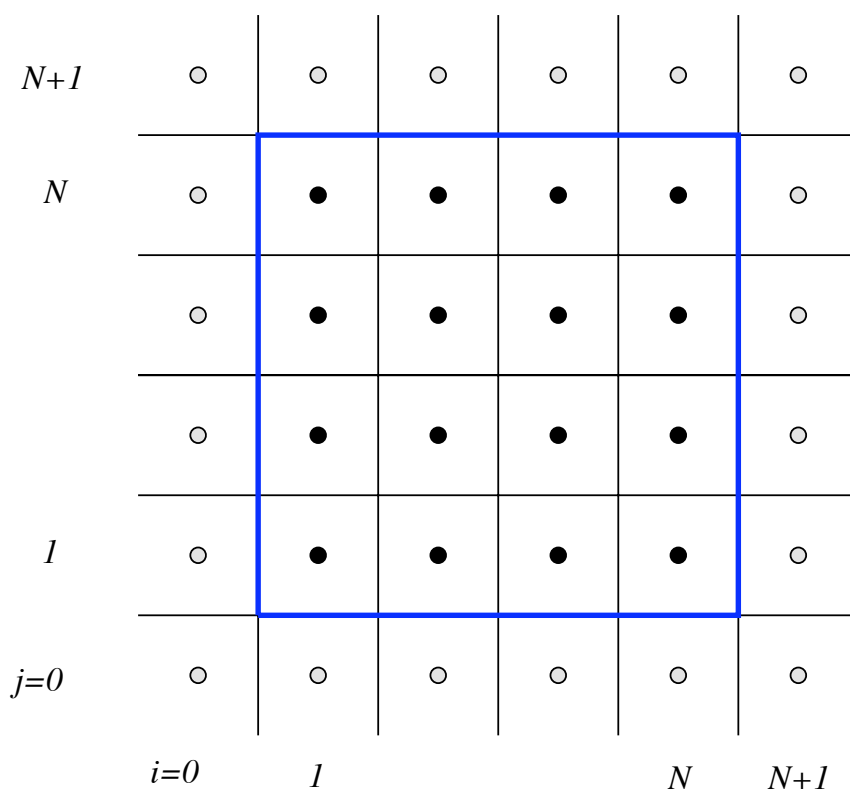


Figure 1: Sketch of the grid. Black dots denote the $(N \times N)$ positions in which the pressure is evaluated, whereas gray dots indicate dummy cells situated outside the computational domain.