

1 Mathematics

1.1 Determine the one-sided Laplace transform of the following signals.

a) $u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$, where A is a constant.

b) $u(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$, where A is a constant.

c) $u(t) = e^{-2t}$ for $t \geq 0$.

d) $u(t) = \cos 5t$ for $t \geq 0$.

Express the following in $U(s)$, the Laplace transform of $u(t)$.

e) $\dot{u}(t)$

f) $\dot{u}(t)$, when $u(t) = 0$ for $t \leq 0$.

g) $\ddot{u}(t)$

h) $\ddot{u}(t)$, when $u(t) = \dot{u}(t) = 0$ for $t \leq 0$.

i) $u(t - T)$

1.2 Consider the differential equation

$$\dot{y}(t) + 2y(t) = u(t)$$

a) If $u(t)$ is constant then $\dot{y}(t) \approx 0$ when time goes to infinity. What value will $y(t)$ approach as $t \rightarrow \infty$ if $u(t) = 5$?

b) Determine the transfer function relating $U(s)$ and $Y(s)$ for the differential equation above.

1.3 Determine the general solution of the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = 3\sin(2t)$$

1.4 Below, differential equations that describe dynamic systems are given together with system inputs and initial conditions. Use the Laplace transform to determine the system outputs.

a)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sigma(t)$$

$$\sigma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\frac{dy}{dt}(0) = y(0) = 0$$

b)

$$\dot{y}(t) + y(t) = u(t)$$

$$u(t) = 1 + \sin t$$

$$y(0) = 0$$

1.5 Write the following complex numbers in polar form, that is, determine their absolute value and argument.

a) $1 + i$

b) $\frac{1+i}{5i(1+\sqrt{3}i)}$

Write the following complex numbers on rectangular form:

c) $2e^{i\frac{\pi}{3}}$

d) $5e^{-i\pi}$

1.6 A system has amplification 100. What is the amplification expressed in decibel (dB_{20})? What is the amplification corresponding to 20 dB_{20} , -3 dB_{20} , 0 dB_{20} , -10 dB_{20} , and 10 dB_{20} respectively?

1.7 Verify that the following rule for inversion of 2×2 matrices holds.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

1.8 Determine eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -6 & 2 & 1 \end{pmatrix}$$

1.9 Determine a transformation matrix T , such that $T^{-1}AT$ is a diagonal matrix, where

$$A = \frac{1}{3} \begin{pmatrix} 6 & 0 & -3 \\ -1 & 5 & -1 \\ -2 & -2 & 7 \end{pmatrix}$$

1.10 Characterize the range space and null space of the matrix A by specifying bases for them. Find the rank of the matrix.

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

1.11 What are the time functions corresponding to the Laplace transforms below? What values will the time functions approach as time goes to infinity?

a)

$$F(s) = \frac{1}{s^2 + s}$$

b)

$$F(s) = \frac{1}{s^2 - 1}$$

c)

$$F(s) = \frac{1}{(s + 1)^2}$$

1.12 The water level, y , in a tank is modelled by the differential equation

$$\dot{y}(t) + y(t) = z(t)$$

where z denotes the inflow. The inflow is a function of a valve position, which in turn is controlled by the electric control signal u . The relation between control signal and flow is given by the differential equation

$$\ddot{z}(t) + \dot{z}(t) + z(t) = u(t)$$

What differential equation relates the water level y to the control signal u ?

2 Dynamic Systems

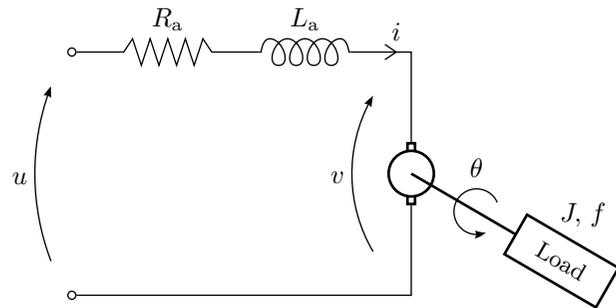


Figure 2.1a

2.1 A common component in a control system is a DC-motor. A schematic picture of the motor is shown in Figure 2.1a. The motor is characterized by a number of physical relationships as will now be explained. The rotating axis is described by

$$J\ddot{\theta} = -f\dot{\theta} + M,$$

where θ is the angle of rotation, M is the torque, J is the moment of inertia of the load and f is the frictional coefficient.

The interplay between rotor and stator is given by

$$M = k_a i \quad \text{and} \quad v = k_v \dot{\theta}$$

where i is the anchor current, k_a a proportional constant characteristic for the motor, v is voltage induced by the rotating axis and k_v is a proportional constant. The input voltage u is the control signal and θ is the output.

- Use the equations above and Kirchhoff's voltage law to write a differential equation that relates u and θ . The inductance L_a can be neglected.
- Determine the transfer function of the system from u to θ .
- Study the behavior of the system by calculating θ when u is a step.

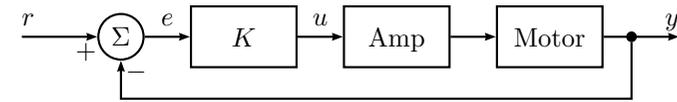


Figure 2.2a

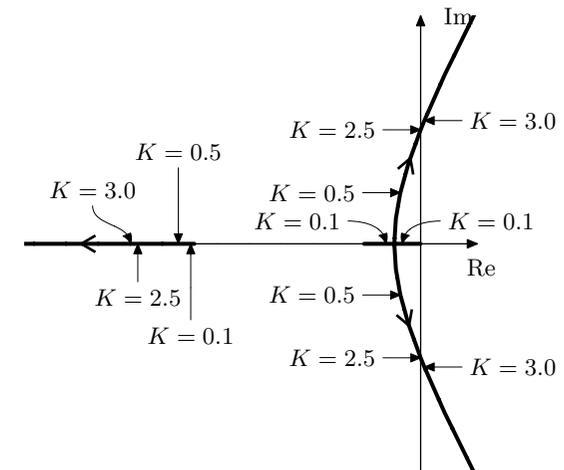


Figure 2.2b

2.2 A servo system for positioning of a tool in a tooling machine is depicted in Figure 2.2a. In Figure 2.2b, the poles of the closed loop system are plotted for different values of the gain K . Find (without calculations), for each of the step responses in Figure 2.2c, the corresponding value of K used when generating the step response.

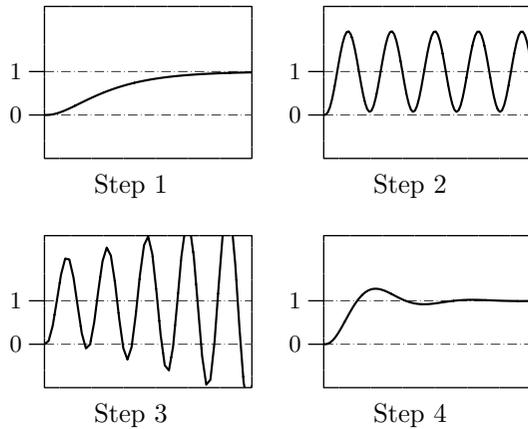


Figure 2.2c. All comparable axes have equal scaling.

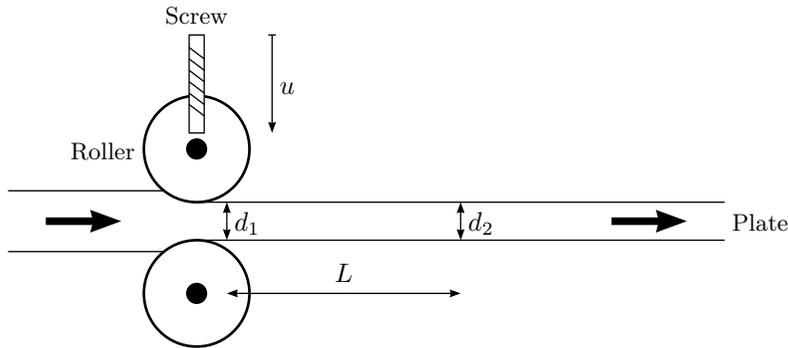


Figure 2.3a

2.3 Consider the simple model of the roller depicted in Figure 2.3a. To obtain a simple model we describe the relationship between the position of the screw and the thickness of the sheet d_1 directly after the rollers as a first order transfer function

$$G(s) = \frac{\beta}{1 + sT}$$

To determine the constants β and T we register the effect of a sudden change in the position of the screw. The units used in the model are chosen such that a unit step will make an appropriately sized input for identification purposes,

and that is the input used in the experiment for which the resulting thickness profile $d_1(t)$ is shown in Figure 2.3b. In production the thickness cannot be measured directly behind the rollers for practical reasons, and instead the thickness $d_2(t)$ is measured L length units after the rollers. Find the transfer function from the position of the screws to the thickness d_2 . The sheet moves with speed V .

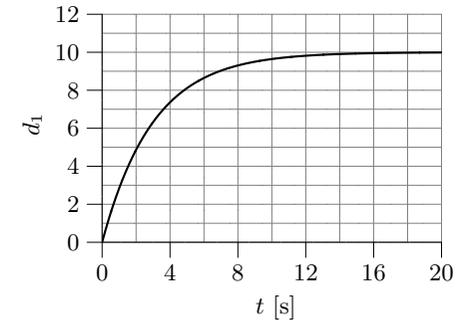


Figure 2.3b

2.4 The step response of the following system

$$G(s) = \frac{1}{s^2 + s + 1}$$

is shown as the dashed line in each part of Figure 2.4a.

a) The step response of the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

is shown as the solid line in the left of Figure 2.4a. Determine if $a > 1$ or $a < 1$.

b) The step response of the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

is shown as the solid line in the right of Figure 2.4a. Find b .

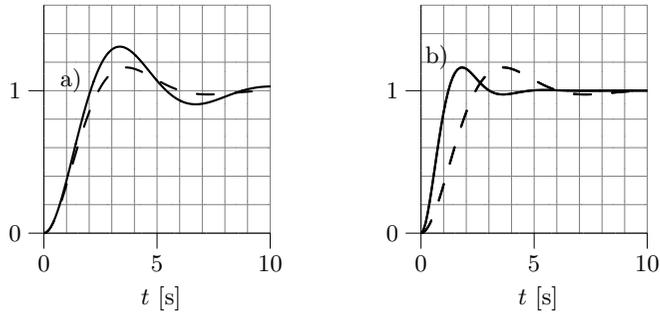


Figure 2.4a. Dashed: original system. Solid left: part a). Solid right: part b).

2.5 Pair the step responses and pole-zero diagrams in Figure 2.5a.

2.6 Consider the systems

$$G_A(s) = \frac{1}{s^2 + 2s + 1} \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1} \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

- Use MATLAB to plot the step responses of the systems. Find T_r (rise time), T_s (settling time) and M (overshoot) for the five step responses.
- Compute the poles of the systems $G_A(s)$, $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ respectively.
- How is the location of the poles related to the properties of the step responses?

2.7 Consider a system with the transfer function

$$G(s) = \frac{\alpha s + 1}{s^2 + 2s + 1}$$

Compute and plot the step response of the system for some different values of α in the range $-10 < \alpha < 10$. How are the properties of the step response affected by the location of the zero of the system?

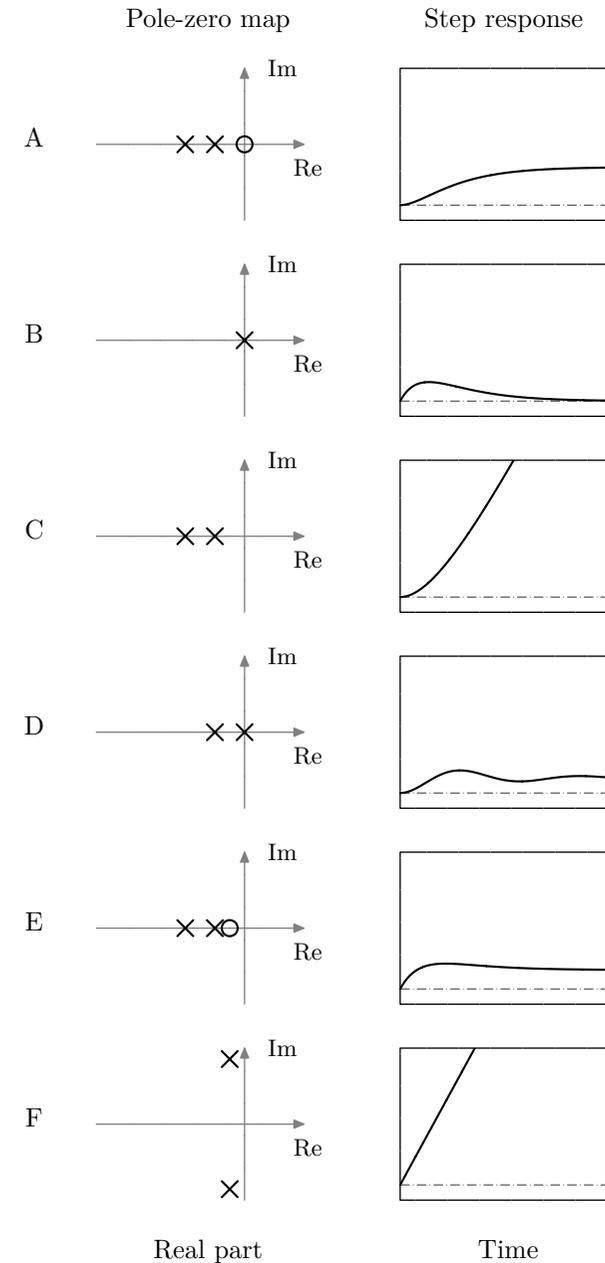


Figure 2.5a. All comparable diagrams have equal scaling. In the pole-zero maps, imaginary and real parts have equal scaling, \times marks poles, and \circ marks zeros.

2.8 Consider a system described by the model $Y(s) = G(s)U(s)$ as shown in Figure 2.8a. Given $G(s)$, how is the step response computed? How can the step response be determined using experiments?

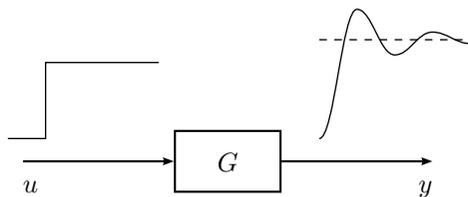


Figure 2.8a

2.9 Figure 2.9a shows the step response of a system $Y(s) = G(s)U(s)$. The input step has amplitude 1. Use the figure and determine

- Steady state value.
- Overshoot M in % of the final value.
- Rise time T_r .
- Settling time T_s .

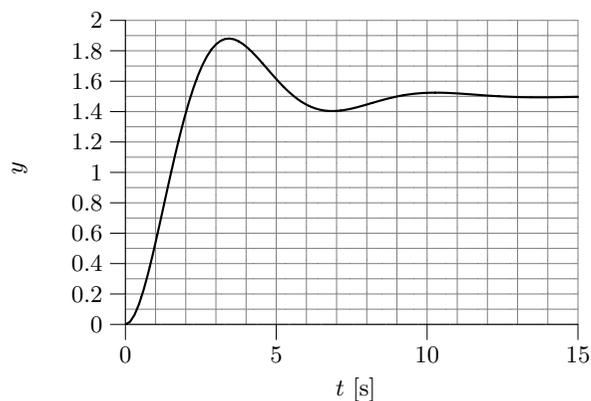


Figure 2.9a

2.10 Figure 2.10a shows the step responses of four different systems. Combine each step response with a transfer function from the alternatives below.

Transfer function	Poles	Zeros	$ G(0) $
$G_1(s) = \frac{100}{s^2+2s+100}$	$-1 \pm 10i$		1
$G_2(s) = \frac{1}{s+2}$	-2		1/2
$G_3(s) = \frac{10s^2+200s+2000}{(s+10)(s^2+10s+100)}$	$-10, -5 \pm 8.7i$	$-10 \pm 10i$	2
$G_4(s) = \frac{200}{(s^2+10s+100)(s+2)}$	$-2, -5 \pm 8.7i$		1
$G_5(s) = \frac{600}{(s^2+10s+100)(s+3)}$	$-3, -5 \pm 8.7i$		2
$G_6(s) = \frac{400}{(s^2-10s+100)(s+2)}$	$-2, 5 \pm 8.7i$		2

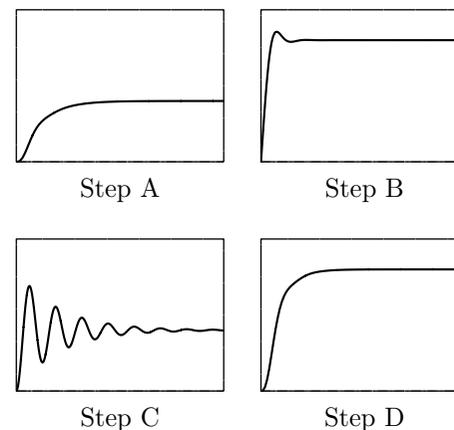


Figure 2.10a. All comparable axes have equal scaling.

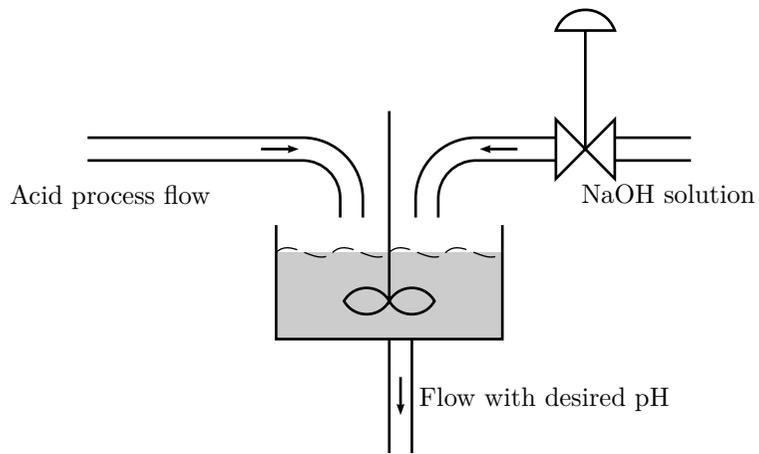


Figure 2.11a

2.11 In the continuously stirred tank, see Figure 2.11a, an acid process flow is neutralized by adding a concentrated NaOH solution. The acid process flow has a tendency to vary its pH with time, which gives undesired variation of the pH in the outflow. In an effort to reduce these variations one has decided to use control.

- Classify the different signals as input, output, and disturbance signal.
- Draw a block diagram of the system with a control strategy.

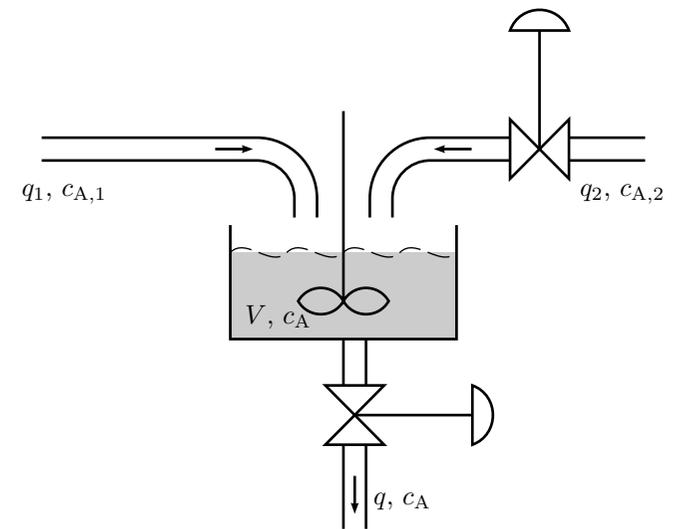


Figure 2.12a

2.12 Two flows with different concentrations of a chemical component A are mixed in a continuously stirred tank, see Figure 2.12a. The volume can be assumed constant, $V = 1 \text{ m}^3$.

- During a stationary period the values of the concentrations and flows are, $c_{A,1}^* = 1.0 \text{ kmol/m}^3$, $q_1^* = 1.0 \text{ m}^3/\text{min}$, $c_{A,2}^* = 4.0 \text{ kmol/m}^3$ and $q_2^* = 0.5 \text{ m}^3/\text{min}$. What are $c_A(t)$ and $q(t)$ during this period?
- Write down the dynamical balance equation for component A. State all your assumptions. Is the dynamical model linear?
- Assume that $c_{A,1}$ changes value from 1.0 kmol/m^3 to 1.2 kmol/m^3 at $t = 0$. Show that the expression for $c_A(t)$ can be written as

$$c_A(t) = k_0 + k_1 (1 - e^{-t/\tau})$$

Determine the constants k_0 , k_1 and τ .

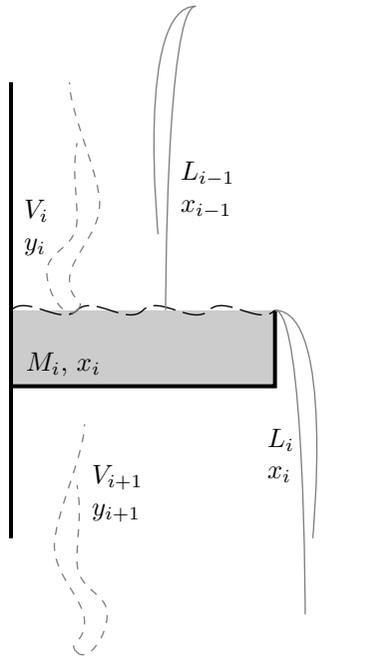


Figure 2.13a

2.13 The distillation plate, see Figure 2.13a, has the following variables:

L_i - Liquid flow from plate i (kmol/min)

V_i - Steam flow from plate i (kmol/min)

M_i - Amount of liquid on plate i (kmol)

x_i - Mole fraction of the most volatile component in the liquid on plate i .

y_i - Mole fraction of the most volatile component in the steam from plate i .

Assumptions:

- Perfect mixing of liquid on the plate.
- Binary separation
- y_i is in equilibrium with x_i .

- The equilibrium is described by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

- a) Write the two differential equations that describe $M_i(t)$ and $x_i(t)$.
- b) Linearize the model under the assumption that the change of the mass on the plate is zero.

3 Feedback Systems

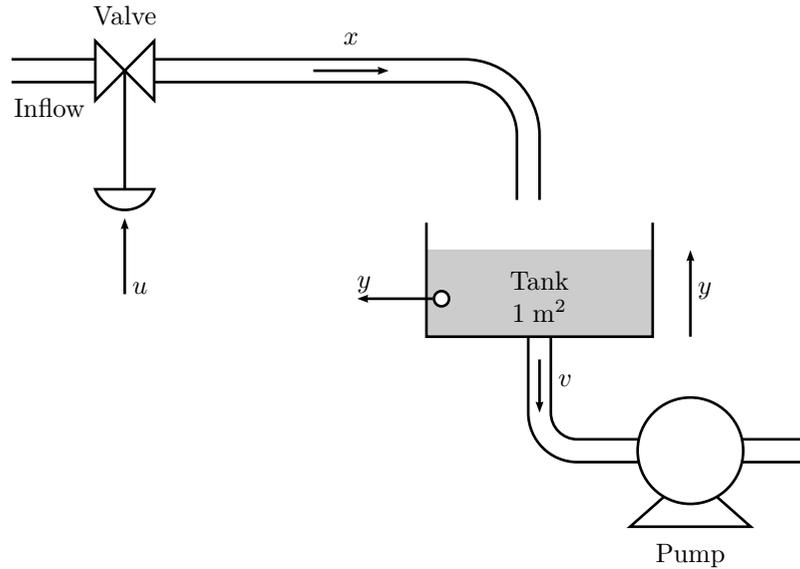


Figure 3.1a

3.1 Consider a part of a process industry, consisting of a valve and a tank. The system is described by Figure 3.1a and the following information:

- The valve is operated electrically, and the input voltage is denoted u . The resulting liquid flow is denoted x .
- The level in the tank is denoted y , and the flow out from the tank, v , is determined by a pump located further down the process. The cross section area of the tank is 1 m^2 .
- The pipes are always filled with incompressible liquid, so there are no transport delays in the system.

a) Use mass balance, that is, the fact that the change in tank volume per time unit is proportional to the difference between inflow and outflow, to determine a transfer function $G_t(s)$ for the tank.

b) The transfer function of the valve is assumed to be

$$G_v(s) = \frac{k_v}{1 + Ts}$$

To find k_v and T a unit step change in u has been applied. The step response, that is, the resulting liquid flow, is shown in Figure 3.1b. Determine the constants k_v and T .

c) Draw a block diagram of the system, in which the valve and the tank are represented by one block each.

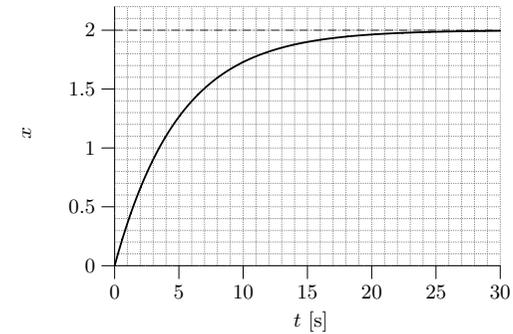


Figure 3.1b

3.2 Consider the system studied in Problem 3.1. The aim is now to control the tank level automatically, and therefore PID feedback control is introduced according to Figure 3.2a. This means that the valve input is determined according to the expression

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt} \quad (3.1)$$

where $e(t)$ denotes the control error, that is,

$$e(t) = r(t) - y(t)$$

and $r(t)$ denotes the reference level, that is, desired tank level.

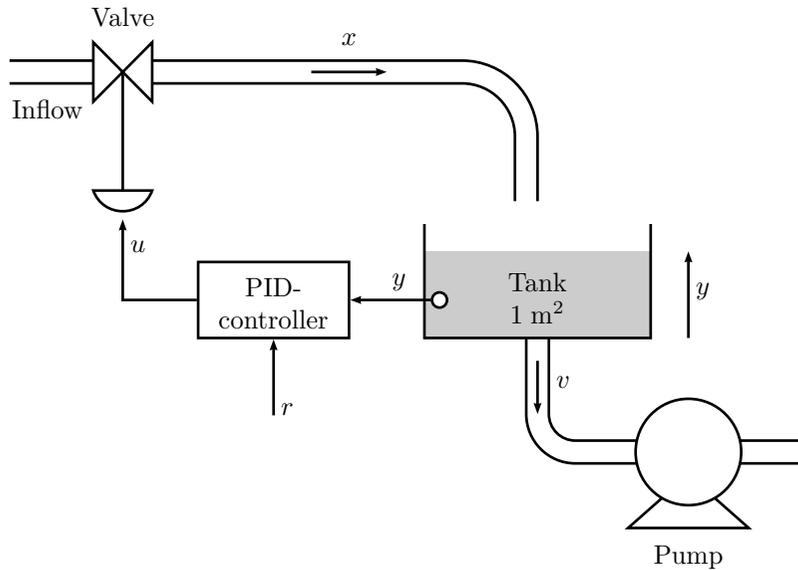


Figure 3.2a

- a) Verify, using Laplace transform, that the relationship in (3.1) can be expressed

$$U(s) = F(s)(R(s) - Y(s))$$

where

$$F(s) = K_P + K_I \frac{1}{s} + K_D s$$

- b) Let the PID controller be represented by a block and draw a block diagram of the entire feedback control system.
- c) Use the block diagram from above together with the transfer functions $G_v(s)$, $G_t(s)$, and $F(s)$, and derive the transfer functions of the closed loop system, that is, the transfer functions from the reference level and the outflow to the tank level.

3.3 Consider the level control system studied in Problems 3.1 and 3.2.

- a) Assume that the tank level is controlled using proportional control, that is, $F(s) = K_P$. Compute the poles of the closed loop system for $K_P =$

0.02 and $K_P = 1$ respectively. How does the choice of K_P affect the properties of the closed loop system?

- b) Assume that the reference level is constant, $r(t) = 5$, and that the outflow is constant, $v(t) = 2$. Determine the steady state tank level when the proportional feedback $F(s) = K_P$ is used. Is it possible for the output signal to reach the desired level? What happens with the steady state level and other properties of the closed loop system if K_P is chosen very large?
- c) Assume that a PI controller is used, that is,

$$u(t) = K_P e(t) + K_I \int e(\tau) d\tau$$

What can be said about the steady state tank level in this case? What is the possible benefit of introducing the integrating part in the feedback?

- d) Finally, assume that a PD controller is used, that is,

$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}$$

Assume also that $K_P = 1$ and calculate a value of K_D so the damping ratio of the closed loop poles will be greater than $1/\sqrt{2}$. This corresponds to placing the poles in the grey area in Figure 3.3a. What is the benefit of including the derivative part in the feedback?

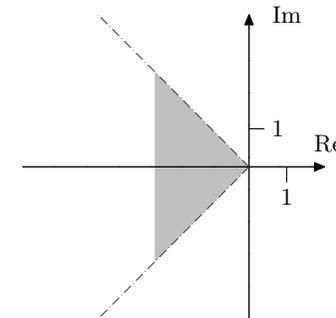


Figure 3.3a

3.4 Figure 3.4a shows the block diagram for a hydraulic servo system in an automatic lathe (Swedish: *svarv*). The output signal $y(t)$ represents the position of

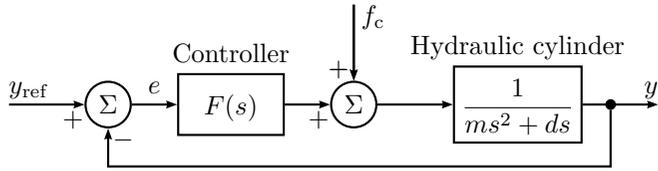


Figure 3.4a

the lathe tool (Swedish: *svarvstål*), $y_{\text{ref}}(t)$ is the desired position of the lathe tool, m is the mass of the tool slide (Swedish: *verktygsslid*) and the hydraulic piston (Swedish: *kolv*), d is the viscous damping of the tool slide, $F(s)$ is the transfer function for the controller, f_c is the cutting power.

- How large, in steady state, is the error $e(t)$ between the actual value of the lathe tool and its desired reference value, when there is a step disturbance in the cutting power $f_c(t)$? The controller is assumed to be an amplifier with a constant gain $F(s) = K$.
- How is this error changed if the amplifier is replaced by a PI controller with transfer function $F(s) = K_1 + K_2/s$?

3.5 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- Suppose $G(s)$ is controlled by a proportional controller with gain K_P , that is,

$$U(s) = K_P(R(s) - Y(s)).$$

Use MATLAB to compute the closed loop system, and to plot the step response of the closed loop system. Choose some values for K_P in the range 0.1 to 10. How are the properties of the step response affected by K_P ? What happens with the steady state error when K_P increases? Is it possible to obtain a well damped closed loop system and small steady state error using proportional control?

- Let us now introduce integration in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s})(R(s) - Y(s)).$$

Put $K_P = 1$ and try some values of K_I in the range $0 < K_I < 2$. How are the step response and the steady state error affected by the introduction of the integrating part and the value of K_I ?

- Finally we will introduce the differentiating part in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s} + \frac{K_D s}{sT + 1})(R(s) - Y(s)).$$

Since true differentiation is difficult to implement, the derivative part is approximated by $\frac{K_D s}{1 + sT}$. (This will low-pass filter the error signal before differentiation.) Put $K_P = 1$, $K_I = 1$ and $T = 0.1$ and try some values of K_D in the range $0 < K_D < 3$. How does the D-part affect the step response of the closed loop system?

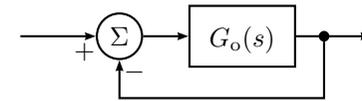


Figure 3.6a

3.6 Draw a root locus with respect to K for the system in Figure 3.6a, with $G_o(s)$ given below. For which values of K are the systems stable? What conclusions on the principal shape of the step response can be drawn from the root locus?

- A Ferris wheel (Swedish: *Pariserhjöl*):

$$G_o(s) = \frac{K(s + 2)}{s(s + 1)(s + 3)}$$

- A Mars rover:

$$G_o(s) = \frac{K}{s(s^2 + 2s + 2)}$$

- A magnetic floater:

$$G_o(s) = \frac{K(s + 1)}{s(s - 1)(s + 6)}$$

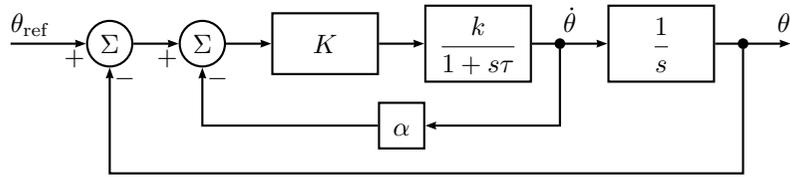


Figure 3.7a

3.7 Consider the servo system in Figure 3.7a with a DC-motor. Suppose that the angular velocity can be measured with a tachometer and let the control law be as in the block diagram. Let $\tau = 0.5$ and $k = 2$.

- Draw the root locus with respect to K for the system without the tachometer feedback (that is, $\alpha = 0$).
- Draw the root locus with respect to K for $\alpha = 1$.
- Draw the root locus with respect to K for $\alpha = 1/3$.
- Let $K = 1$ and draw the root locus with respect to α .

Discuss, using the results from a), b), c), and d), what is gained by using the tachometer.

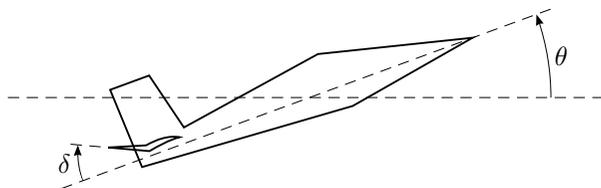


Figure 3.8a

3.8 Consider an aircraft where the pitch angle θ is controlled by the elevator deflection (Swedish: *höjdroderutslag*) δ , see Figure 3.8a. Let ω be the angular velocity,

$$\omega = \dot{\theta}.$$

If we consider small deviations from a reference value θ_0 , we get the transfer function from δ to ω for a specific aircraft as

$$G_2(s) = \frac{s + 1}{(s + 4)(s - 3)}$$

This model is valid when the aircraft is flown with a large θ_0 . The elevator (Swedish: *höjdroder*) is driven by a hydraulic servo amplifier with the transfer function

$$G_1(s) = \frac{10}{s + 10}$$

from elevator command δ_{ref} to δ .

- What happens with ω if one gives a constant elevator command δ_{ref} ? Motivate!
- The angular velocity ω is measured and a control law is used so that the input δ_{ref} to the servo amplifier is

$$K(\omega_{\text{ref}} - \omega).$$

Draw root locus with respect to K . For which values of K is the system stable?

- Is there any value of K such that the closed loop system is stable and all poles are real?

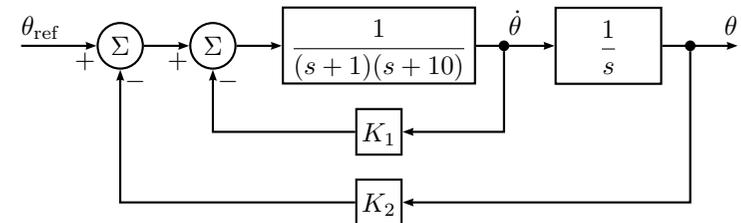


Figure 3.9a

3.9 The block diagram in Figure 3.9a shows a cascade controlled DC-motor where $K_1 > 0$ and $K_2 > 0$.

- Draw root locus with respect to K_2 for the characteristic equation of the closed loop system. For which $K_2 > 0$ are the closed loop system asymptotically stable?

- b) How is the stability requirement on K_2 affected by the size of the velocity feedback K_1 ?

3.10 We want to control the temperature of an unstable chemical reactor. The transfer function is

$$\frac{1}{(s+1)(s-1)(s+5)}$$

- a) Use a proportional controller and draw a root locus with respect to the amplification K . Calculate which K in the compensator that stabilizes the system.
 b) Use a PD controller. The control law is given by

$$u = K(e + T_D \frac{de}{dt})$$

where e is the error. Let $T_D = 0.5$ and draw a root locus with respect to K . For which values of K does the controller stabilize the system?

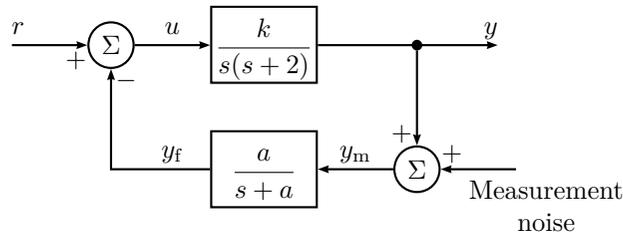


Figure 3.11a

3.11 Consider the system in Figure 3.11a. In a realistic situation what you really measure is not $y(t)$ but a signal $y_m(t)$ which is the sum of $y(t)$ and measurement noise. To avoid that the control is based on noisy measurements one uses $y_f(t)$ instead of $y_m(t)$. The signal $y_f(t)$ is $y_m(t)$ filtered through the low pass filter

$$\frac{a}{s+a}$$

- a) First we assume that the measurement noise is negligible. Choose $k = 6$. Draw a root locus for the closed loop system with respect to the time constant of the low pass filter $1/a$. Find for which $a > 0$ the system is stable.

- b) Use $k = 6$ and assume that the noise is a high frequency sinusoid. The amplitude of y_f when

$$y_m(t) = \sin(10t)$$

is used as a measurement of how effective the noise reduction is. What is the smallest value you can obtain (after transients) by choosing a suitable a . We also want y to tend to the steady state value 1 when $r(t)$ is a unit step.

3.12 Figure 3.12a shows the root locus for the characteristic equation of a P-controlled process G with respect to the gain K . In Figure 3.12b four step responses for the closed loop system with different values of K are shown. Match the plots in Figure 3.12b with the K -values below. Justify your answer.

$$K = 4 \quad K = 10 \quad K = 18 \quad K = 50$$

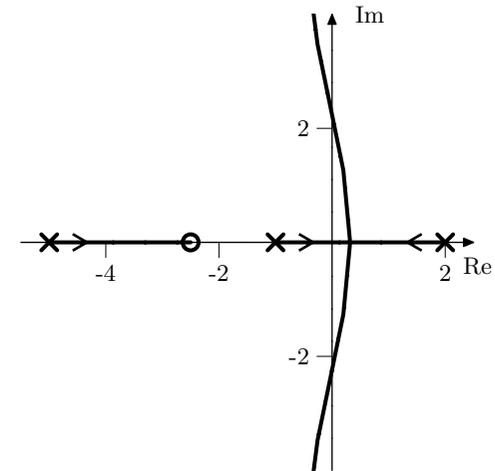


Figure 3.12a. Starting points are marked \times and end points \circ .

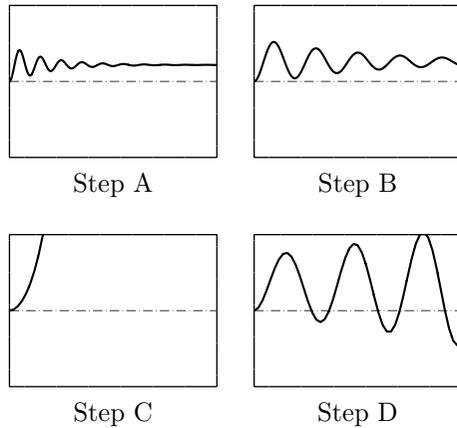


Figure 3.12b. All comparable axes have equal scaling.

3.13 Consider a system with the transfer function

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

that has all zeros strictly in the left half plane. Show that such a system always can be stabilized by

$$u(t) = -Ky(t)$$

if K is selected large enough.

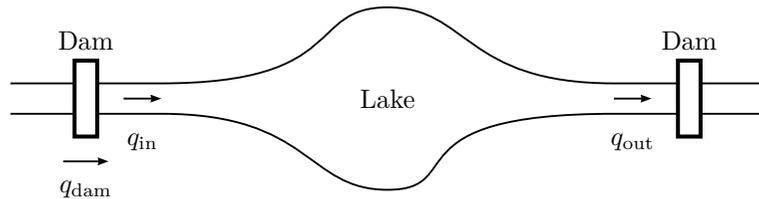


Figure 3.14a

3.14 We want to control the level in a lake by controlling the flow using flood gates, see Figure 3.14a. The relationship between changes in the level of the lake,

h_Δ , and changes in the flows, $q_{in,\Delta}$ and $q_{out,\Delta}$, is given by

$$\frac{d}{dt}(Ah_\Delta) = q_{in,\Delta} - q_{out,\Delta}$$

where A is the area of the lake. In order to try to keep the level of the lake constant the flows through the lake are controlled at the inflow such that

$$q_{dam,\Delta} = K(h_{ref,\Delta} - h_\Delta)$$

where $h_{ref,\Delta}$ is the reference value. The reservoir that controls the outflow is controlled so that q_{out} is constant, that is, $q_{out,\Delta} = 0$. Since it takes time before a change in q_{dam} gives a result in q_{in} we have

$$q_{in,\Delta} = q_{dam,\Delta}(t - T)$$

where $T = 0.5$ hours. How large can the quotient K/A be at the most before the system becomes unstable?

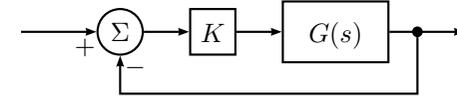


Figure 3.15a

3.15 A system $G(s)$ is controlled using feedback with a proportional controller according to Figure 3.15a.

- For $K = 1$, the open loop system $KG(s)$ has the Nyquist diagram according to (i), (ii), (iii), or (iv) in Figure 3.15b. Is the closed loop system stable in each case? $G(s)$ has no poles in the right half plane.
- If $K > 0$, for which values of K are the different closed loop systems stable?

- Draw the Nyquist curve for an integrator $G(s) = 1/s$.
- Draw the Nyquist curve for the double integrator $G(s) = 1/s^2$.

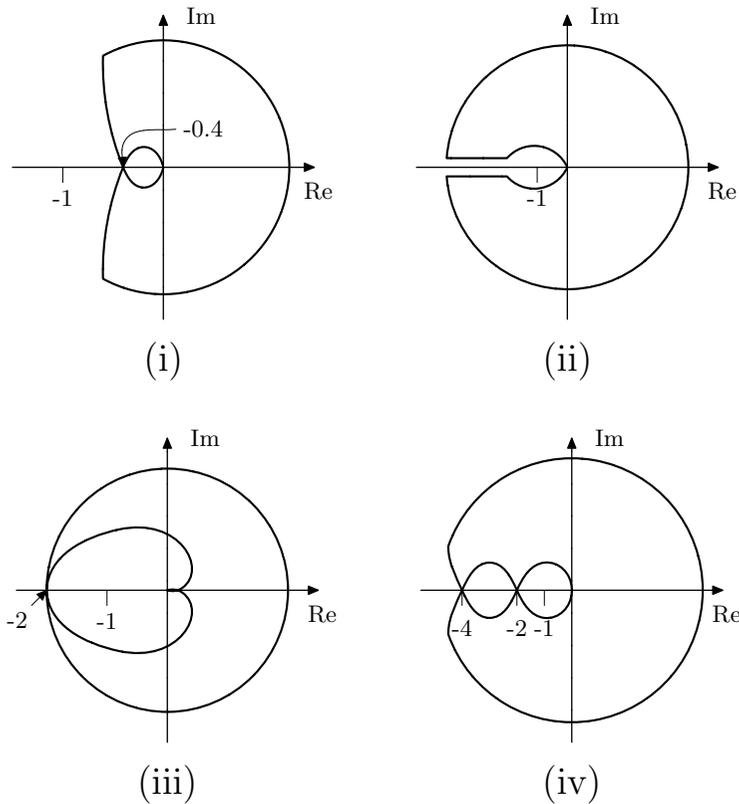


Figure 3.15b

3.17 The system $G(s)$ is asymptotically stable and has the Nyquist curve in Figure 3.17a. It is controlled using feedback according to Figure 3.17b.

- For what values of K ($K > 0$) is the closed loop system asymptotically stable?
- Determine the steady state error, e , as a function of K if y_{ref} is a unit step.
- Assume that G is controlled using an I controller according to Figure 3.17c. For what values of K is the closed loop system stable?

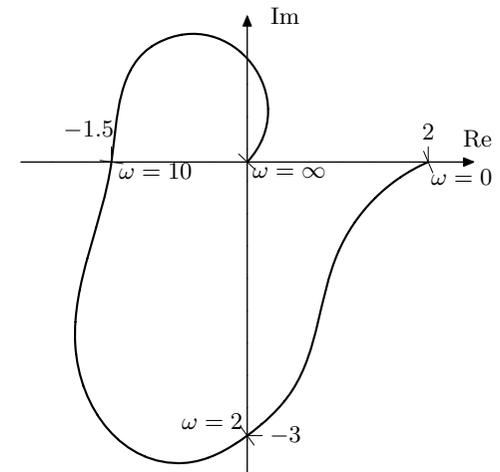


Figure 3.17a

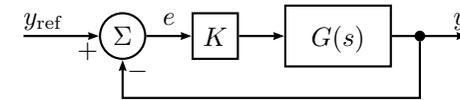


Figure 3.17b

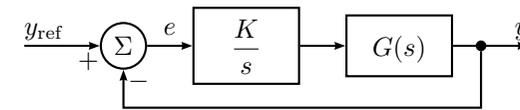


Figure 3.17c

3.18 Consider the DC-motor

$$\tau \ddot{y}(t) + \dot{y}(t) = u(t)$$

It is controlled by

$$u(t) = K(r(t - T) - y(t - T))$$

Here τ and T are positive constants. K is slowly increased until the system oscillates with the angular frequency $\omega = 1$. K is then set to 33% of this value. After a while the system starts to oscillate again, now with the angular frequency $\omega = 0.5$. This is due to the fact that the time delay T has changed

to T_1 . Can the parameters τ , T , and T_1 be decided from these data? If so determine τ , T and T_1 .

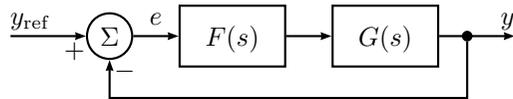


Figure 3.19a

3.19 A system $G(s)$ is to be controlled using the regulator

$$F(s) = \frac{K}{s+1}$$

according to the Figure 3.19a. The controller has positive gain K that, however, is not completely known. For what values of K is the closed loop system asymptotically stable? The Nyquist curve for $G(s)$ is given in Figure 3.19b.

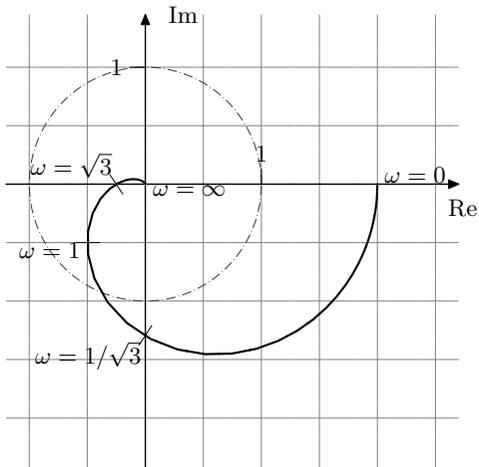


Figure 3.19b

3.20 The Nyquist curve for the system $G(s) = \frac{B(s)}{A(s)}$ can be seen in Figure 3.20a. Determine which one of the root loci in Figure 3.20b that matches

$$A(s) + KB(s) = 0$$

for this system.

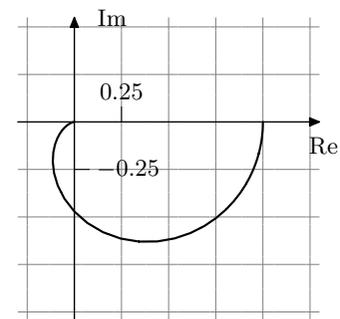


Figure 3.20a

3.21 The system $\frac{1}{s(s+1)}$ with input u and output y has the controller

$$U(s) = F(s)(R(s) - Y(s)) \quad F(s) = \frac{b_0 s^2 + b_1 s + b_2}{s}$$

How should the coefficients of F be chosen to achieve pure P, pure I and pure D control respectively?

3.22 The equations for the P, PI, and PID controllers to be used in this problem are given in Problem 3.5.

a) Let the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s)$$

be controlled by a proportional controller with gain K_P . Use MATLAB to plot the root locus with respect to K_P of the characteristic equation of the closed loop system. For which values of $K_P > 0$ is the closed loop system asymptotically stable?

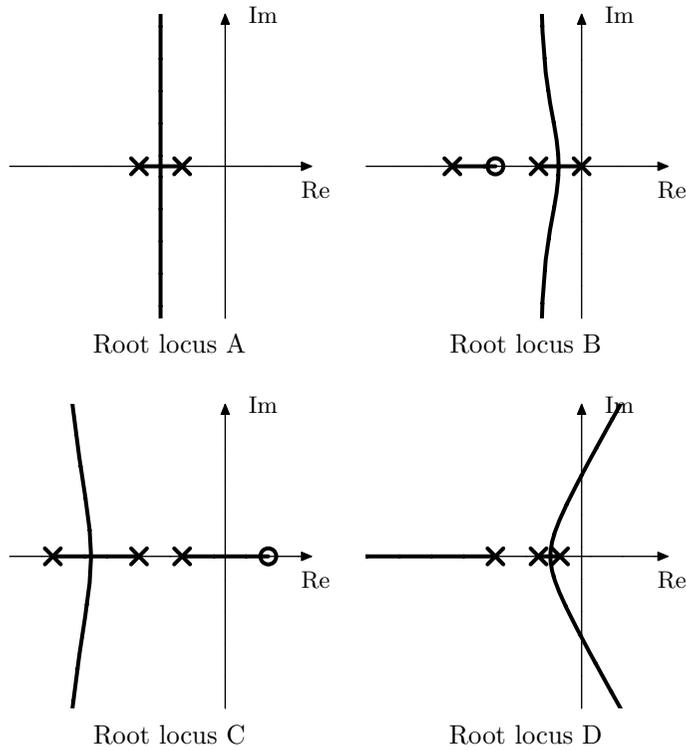


Figure 3.20b. Starting points ($K = 0$) are marked \times , and end points ($K \rightarrow \infty$) are marked \circ . All diagrams have equal scaling.

In Problem 3.5, we found that the step response was slow but well damped for small values of K_P , while it became faster but more oscillatory when K_P was increased. For large values of K_P the system became unstable. We also found that the steady state error was reduced when K_P was increased. Can these results be interpreted using the plot of the root locus?

- b) Now assume that the system is controlled by a PI controller where $K_P = 1$. Plot the root locus of the characteristic equation, with respect to K_I , and determine for which values of $K_I > 0$ the closed loop system is asymptotically stable.

Problem 3.5 showed that an integrator eliminates the steady state error.

A small value of K_I gives a large settling time, while a too large value gives an oscillatory, and perhaps unstable closed loop system. Give an interpretation of these results using the root locus.

- c) Finally, let the system be controlled by a PID controller where $K_P = 1$, $K_I = 1$ and $T = 0.1$. Plot a root locus of the closed loop characteristic equation, with respect to $K_D > 0$, and relate the behavior of the root locus to the simulation result in Problem 3.5, that is, that the derivative part increases the damping of the closed loop system, but a too large K_D will give an oscillation with a higher frequency, and finally an unstable closed loop system.

3.23 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- a) Use MATLAB to plot the Nyquist curve of the open loop system when $G(s)$ is controlled by a proportional regulator. Try some different values of K_P and find for which K_P the closed loop system is asymptotically stable. Compare your results with those from Problem 3.22a.
- b) Assume now that the system is controlled by a PI controller where $K_P = 1$. Investigate how K_I affects the Nyquist curve and determine for which values of K_I the closed loop system is asymptotically stable. Do you get the same results as in Problem 3.22b?
- c) Finally test a PID controller with $K_P = 1$, $K_I = 1$ and $T = 0.1$ (cf Problem 3.5). How is the Nyquist curve affected by the value of K_D ?

3.24 a) Assume that the system

$$Y(s) = G(s)U(s) = \frac{0.4}{(s^2 + s + 1)(s + 0.2)}U(s)$$

is controlled by a proportional controller where $K_P = 1$. Use MATLAB to make a Bode plot of the open loop system and determine ω_c (gain crossover frequency), ω_p (phase crossover frequency), φ_m (phase margin) and A_m (gain margin) respectively. Compute the closed loop system and plot the step response.

- b) Now let $K_P = 2.5$. How does the change of K_P affect ω_c , ω_p , φ_m , and A_m ? Simulate the step response of the closed loop system and plot the result. How have the properties of the step response changed?

- c) How much can K_P be increased before the closed loop system becomes unstable? How does this value relate to the value of A_m that was obtained for $K_P = 1$? Compute and plot the step response of the closed loop system for this value of K_P . How does the closed loop system behave in this case?

3.25 A system is controlled by a PID controller,

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)E(s)$$

In Figure 3.25a four step responses from a unit step for the parameter triples

- i) $K_P = 1 \quad K_I = 0 \quad K_D = 0$
- ii) $K_P = 1 \quad K_I = 1 \quad K_D = 0$
- iii) $K_P = 1 \quad K_I = 0 \quad K_D = 1$
- iv) $K_P = 1 \quad K_I = 1 \quad K_D = 1$

are shown. Match each one of the parameter triples to one of the step responses. Justify your answer!

3.26 Assume that a DC-motor of the type described in Problem 2.1 is controlled by a proportional controller, that is, $u(t) = K_P(\theta_{\text{ref}} - \theta)$.

- a) Write down a block diagram for the control system. Compute the closed loop transfer function and determine how the poles of the closed loop system depend on the control gain K_P . Discuss what this means for the behavior of the system for different values of K_P .
- b) Determine the transfer function from the reference signal to the error. Let the reference signal be a step and a ramp respectively and determine what the control error will be in steady state in these two cases.
- c) Let the controller be a PI controller. What will the steady state error be in this case if the reference signal is a ramp?

3.27 Determine the transfer function for the loop gain and the closed loop system for the control system given by the block diagram in Figure 3.27a.

3.28 Figure 3.28a shows a block diagram of a control system. Determine the transfer function

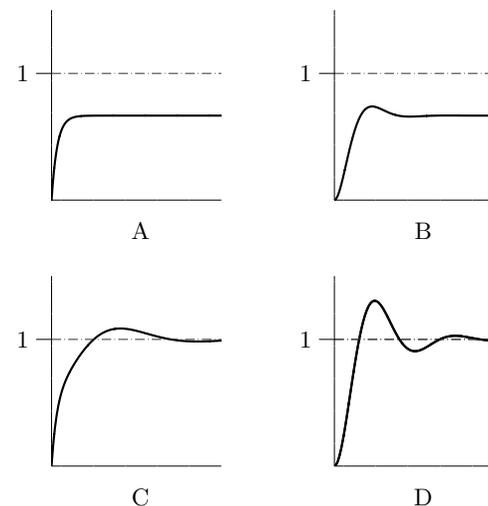


Figure 3.25a. Four step responses. All comparable axes have equal scaling.

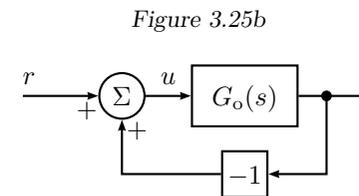


Figure 3.27a

- a) of the loop gain,
- b) of the closed loop system from $R(s)$ to $Y(s)$,
- c) from the disturbance $N(s)$ to the output $Y(s)$,
- d) from the reference signal $R(s)$ to the error signal $E(s)$.

3.29 Consider again the control system in Figure 3.28a, with $n = 0$ and

$$G(s) = \frac{1}{(s+1)(s+3)}$$

- a) Assume $F(s) = K$. Determine the steady state control error when $r(t)$ is a step.

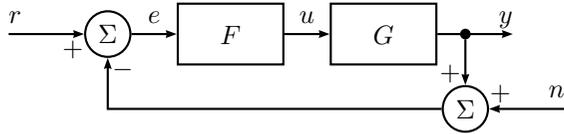


Figure 3.28a

- b) Determine a regulator $F(s)$ such that the steady state error is zero when $r(t)$ is a step.
- c) Assume $F(s) = 1$. Determine the poles and zeros of the closed loop system.

3.30 The system

$$Y(s) = \frac{1}{(s/0.6 + 1)(s + 1)}U(s)$$

is controlled using PID feedback

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)(R(s) - Y(s))$$

Figure 3.30a shows the step responses for the following four combinations of coefficient values. Combine the step responses and coefficients.

- (1) $K_P = 4 \quad K_I = 0 \quad K_D = 0$
- (2) $K_P = 4 \quad K_I = 3 \quad K_D = 0$
- (3) $K_P = 4 \quad K_I = 1 \quad K_D = 0$
- (4) $K_P = 4 \quad K_I = 0 \quad K_D = 4$

3.31 Continuously stirred tanks have an extensive use in chemical processes. They are often supplied with some sort of heat exchange system. If the tank is used for a chemical or biochemical reaction it is often important to keep a certain temperature to obtain desired productivity. A continuously stirred tank with a common type of heat exchange system is shown in Figure 3.31a. The tank is surrounded with a heating/cooling layer in which a liquid flows through in order to heat or chill the liquid in the tank.

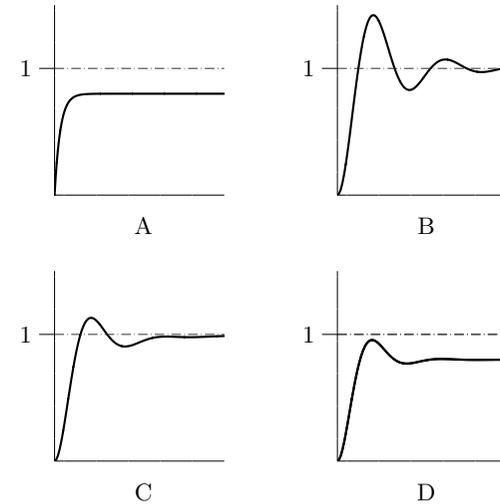


Figure 3.30a. Four step responses. All comparable axes have equal scaling.

- a) Determine the important signals of the system. Suggest a control strategy based on feedback and draw the block diagram.
- b) To be able to construct and evaluate different controllers it is necessary to have a process model. Determine a dynamical model for this system. Assume that the volumes in the tank and in the heat exchange system are constant.
- c) Linearize the model. (Stationary values: $T_t^* = 50.0 \text{ }^\circ\text{C}$, $T_c^* = 75.0 \text{ }^\circ\text{C}$.)
- d) The parameters of the model are as follows

$$\begin{aligned} \rho_t = \rho_c = 1000.0 \text{ kg/m}^3 \quad c_{t,\text{in}}^p = c_c^p = 4.2 \text{ kJ/kg}^\circ\text{C} \quad U = 672 \text{ kJ/}^\circ\text{Cmin} \\ F_t = 0.1 \text{ m}^3/\text{min} \quad F_c = 0.2 \text{ m}^3/\text{min} \quad V_t = 1.0 \text{ m}^3 \quad V_c = 1.0 \text{ m}^3 \\ T_{t,\text{in}} = 10.0 \text{ }^\circ\text{C} \quad T_{c,\text{in}} = 95.0 \text{ }^\circ\text{C} \end{aligned}$$

Determine the transfer function from flow in the heat exchange system to the temperature in the tank.

- e) Let the system be controlled by a P controller. Draw the root locus for the system.

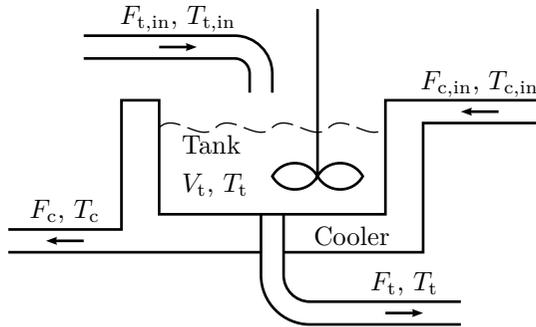


Figure 3.31a. Process consisting of a tank and a cooler. Input flows have a “in” subscript, while outputs have no such subscript since they are also the same as the quantities found inside the tank or cooler.

3.32 Consider a continuously stirred tank with a cooling system. In the tank a component A reacts to form component B in an exothermic reaction. This reaction is unstable, but possible to stabilize with feedback. A model for the purpose of control has been established

$$Y(s) = \frac{-1}{s^2 + 2s - 3} U(s)$$

where $y(t)$ is the temperature in the tank and $u(t)$ is the cooling flow.

- Show that the system is unstable.
- Prove that the system can be stabilized by a P controller.

3.33 Bacterial growth is described by the equation $\dot{y} = \mu y$ where y is the amount of bacteria and μ is a positive constant. Assume that we have a control signal u available that affects the speed of growth so that

$$\dot{y} = \mu y + u$$

One can then use a P controller $u = K(r - y)$ where r is a reference signal. For which K -values will the system approach an equilibrium?

3.34 While working in space an astronaut has to be able to move. Necessary force is obtained by letting out gas from thrusters. For such a positioning control system the control law

$$u = K_1(r - y) - K_1 K_2 \frac{dy}{dt}$$

is used, where u = thruster force, r = set point for the astronaut's position and y = the actual value of the astronaut's position. Draw a block diagram of the system, and use physics (Newton's law) to make a model of the astronaut. Also determine K_1 and K_2 such that

- If the set point $r(t) = t$ there has to be a time T_0 so that $|r(t) - y(t)| < 1$ for $t \geq T_0$.
- The damping ratio of the closed loop system will be 0.7. The mass of the astronaut is assumed to be 100 kg, equipment included.

4 Frequency Description

4.1 A mercury thermometer can be described with high accuracy as a first order linear time invariant dynamic system. The input is the real temperature and the output is the thermometer reading. In order to decide the transfer function in a thermometer it is placed in liquid where the temperature is varied as a sinusoid. The obtained result is shown in Figure 4.1a. Find the transfer function of the thermometer.

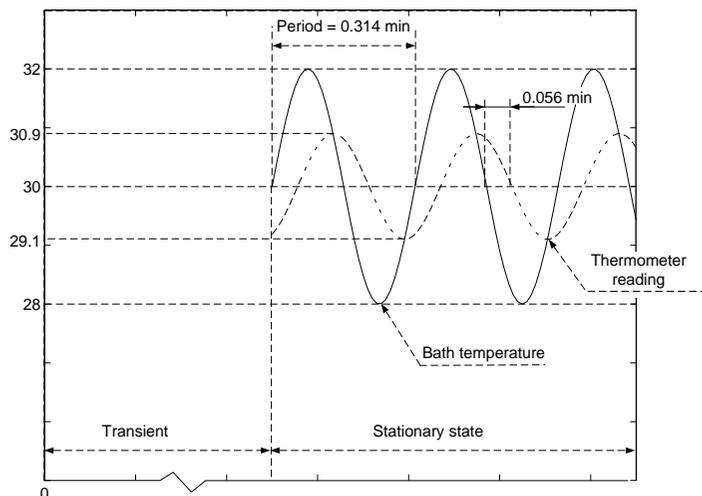


Figure 4.1a

4.2 We want to keep a ship on a given course, Ψ , with an automatic control system using the rudder angle δ . See Figure 4.2a. If ω denotes the angular velocity of the ship,

$$\omega = \dot{\Psi} \tag{4.1}$$

the following differential equation is valid for small values of ω and δ ,

$$T_1 \dot{\omega} = -\omega + K_1 \delta \tag{4.2}$$

where $T_1 = 100$ and $K_1 = 0.1$. The desired course, Ψ_{ref} , and the measured course, Ψ , are fed in to the auto pilot, which gives the signal u to the rudder

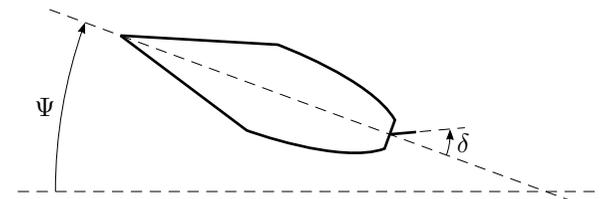


Figure 4.2a

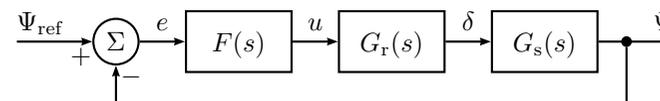


Figure 4.2b

engine. Figure 4.2b shows a block diagram of the auto pilot. The auto pilot has the transfer function

$$F(s) = K \frac{1 + \frac{s}{a}}{1 + \frac{s}{b}}, \quad a = 0.02, b = 0.05$$

while G_r is given by

$$G_r(s) = \frac{1}{1 + sT_2}, \quad T_2 = 10$$

and $G_s(s)$ is defined by (4.1) and (4.2).

- a) Make a Bode plot for the transfer function FG_rG_s , for $K = 0.5$.

- b) At the testing of the auto pilot we do the following experiment. The gain of the auto pilot K is increased until the control system oscillates with constant amplitude. At what value of K does this occur? What is the period time of the oscillation?
- c) Ψ_{ref} is allowed to vary as a sinusoid

$$\Psi_{\text{ref}}(t) = A \sin \alpha t$$

where $A = 5^\circ$ and $\alpha = 0.02$. When the movements of the ship have stabilized we have

$$\Psi(t) = B \sin(\beta t + \varphi)$$

What values do B , β , and φ have if $K = 0.5$?

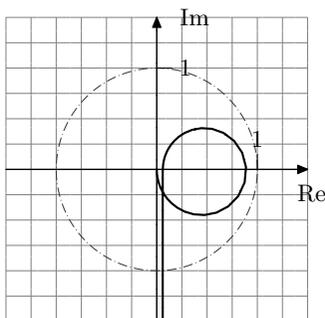


Figure 4.3a

- 4.3 a) In Figure 4.3a the Nyquist curve for a system is shown. Draw the Bode plot for the same system. The scale on the ω -axis is not important, as long as the amplitude and phase curve are in agreement.
- b) Draw a diagram for the poles and zeros for the system. The relative placement is important, not the scale.

- 4.4 Figure 4.4a shows the step responses (when the input is a unit step) and Bode gain plots of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each step response, find the corresponding Bode gain plot (amplitude curve). Motivate your answer by pointing out a set of unique features for each system.

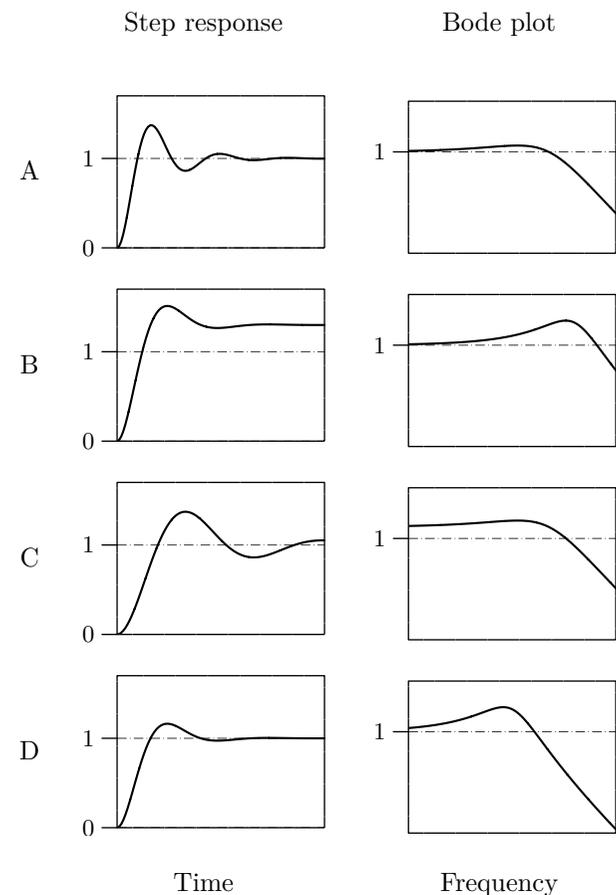


Figure 4.4a. All comparable diagrams have equal scaling.

- 4.5 a) Consider the transfer functions $G_A(s)$, $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ in Problem 2.6.

$$G_A(s) = \frac{1}{s^2 + 2s + 1}, \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1}, \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

Study the amplitude curves of the Bode plots for the systems and find the static gain and bandwidth of the systems. In cases when it is relevant find also the resonance frequency and resonance peak.

- b) Describe qualitatively (without formulas) the relationships between T_r (rise time) and ω_B (bandwidth) and between M (overshoot) and M_p (resonance peak) respectively?

- 4.6 A system has the transfer function

$$G(s) = \frac{e^{-2s}}{s(s+1)}$$

What is the output (after transients) when the input is

$$u(t) = 2 \sin(2t - 1/2)$$

- 4.7 For the systems below the input is chosen as $u(t) = \sin(2t)$. Determine the output signal $y(t)$ after transients have faded away, provided that it exists.

- a) $Y(s) = \frac{1}{s+1}U(s)$
 b) $Y(s) = \frac{1}{s-1}U(s)$
 c) $Y(s) = \frac{1}{(s+1)(2s+1)}U(s)$
 d) $Y(s) = \frac{e^{-0.5s}}{s+1}U(s)$

- 4.8 A system is described by $Y(s) = G(s)U(s)$. Figure 4.8a shows $u(t) = \sin(\omega t)$ and the corresponding output $y(t)$ (after all transients have faded away) for the frequencies $\omega = 1, 5, 10,$ and 20 rad/s (from top to bottom).

- a) Determine the gain ($|G(i\omega)|$) and phase ($\arg G(i\omega)$) for the system for each value of ω .
 b) Determine the gain values in dB₂₀ ($20 \log_{10}(|G(i\omega)|)$).
 c) Sketch the Bode plot using the values determined above.

- 4.9 Combine the transfer functions below with the Bode plots in Figure 4.9a.

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{6(s+1)}{(s+2)(s+3)}$$

$$G_3(s) = \frac{1}{s}, \quad G_4(s) = \frac{1}{s(s+1)}$$

$$G_5(s) = \frac{5}{s^2 + 2s + 5} \text{ (poles: } -1 \pm i2)$$

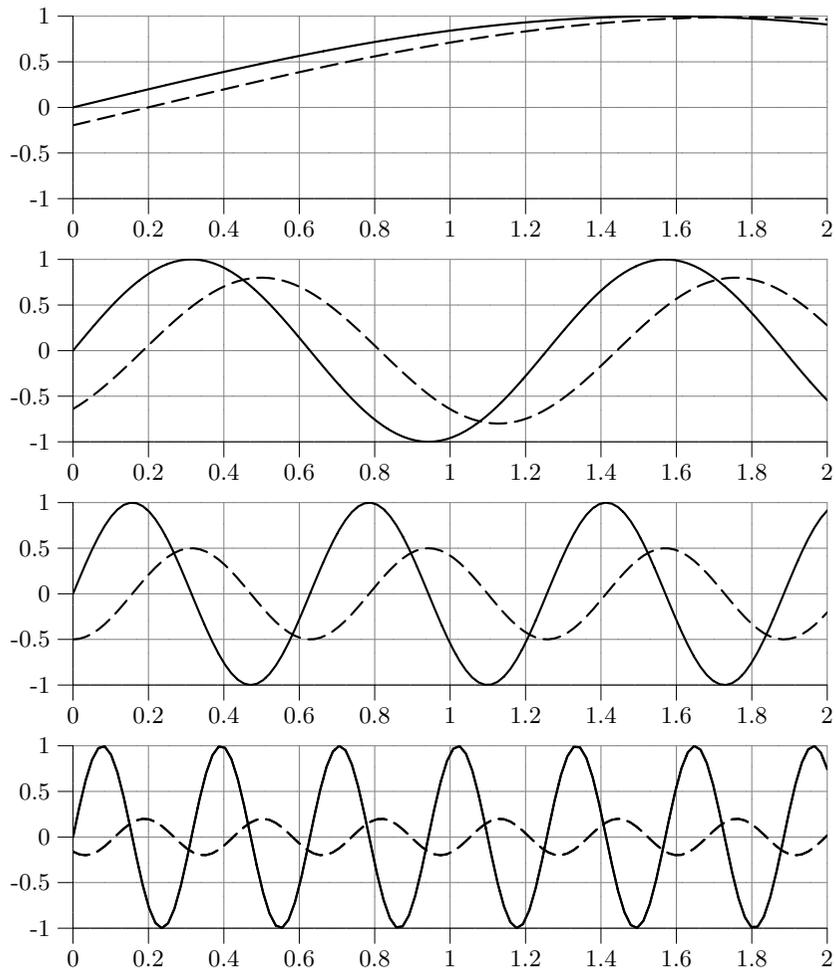


Figure 4.8a. $u(t) = \sin(\omega t)$ (solid) and $y(t)$ (dashed).

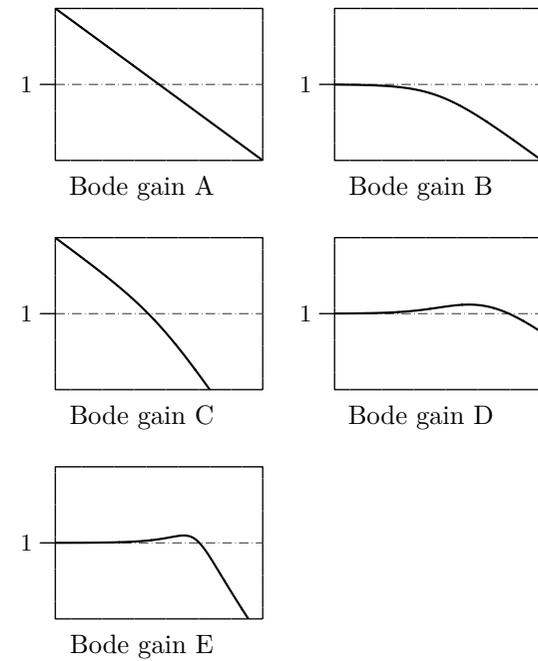


Figure 4.9a. All diagrams have equal scaling.

4.10 Figure 4.10a shows the Bode gain plots and step responses of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each Bode gain plot (amplitude curve), find the corresponding step response. Motivate your answer by pointing out a set of unique features for each system.

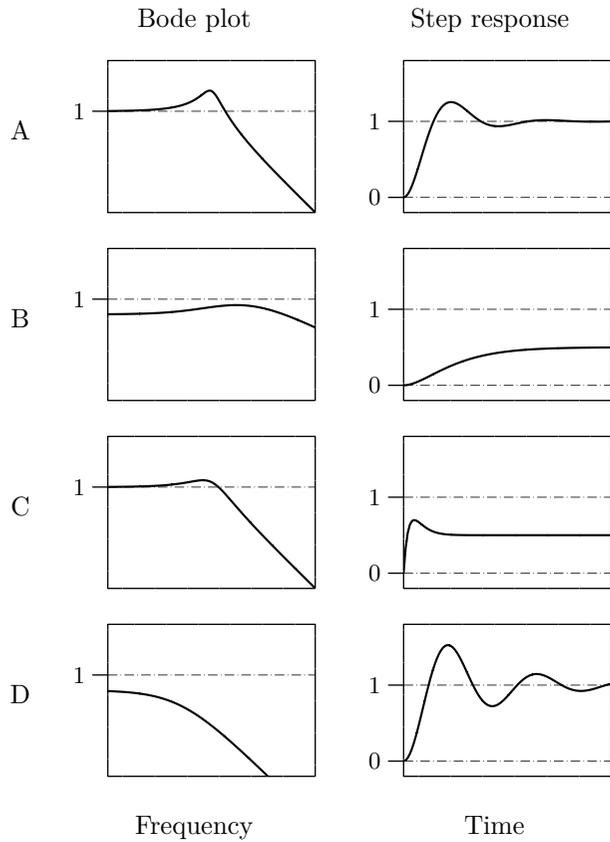


Figure 4.10a. All comparable diagrams have equal scaling.

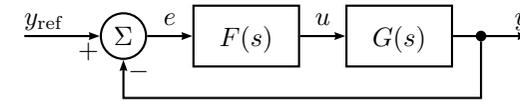


Figure 4.11a

4.11 The pH in a biochemical reactor is controlled by addition of a base. The transfer function $G(s)$ from added base to pH for the open system has been determined by experiments to be

$$G(s) = \frac{1.7}{(s+1)(0.7s+1)(0.5s+1)}$$

In an attempt to control the pH the control structure shown in Figure 4.11a is employed

- Make a Bode plot for the transfer function $G(s)$.
- Assume that a P controller is used ($F(s) = K$). At what value of K does the pH start to oscillate with constant amplitude?

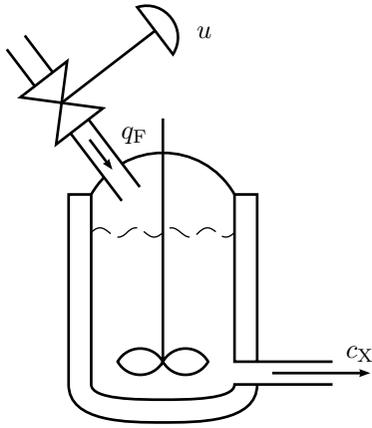


Figure 4.12a

4.12 Consider the biochemical reactor in Figure 4.12a. It is desirable to control the concentration of biochemical material c_X (output y) by manipulating the dilute flow q_F (input u). A model of this system can be described as

$$Y(s) = \frac{2e^{-5s}}{30s + 1}U(s)$$

where the time delay reflects the time it takes to measure the biochemical concentration. The bode digram of the system is shown in Figure 4.12b.

- For which values of K is a P controller going to stabilize the system?
- Construct a controller which has the crossover frequency $\omega_{c,d} = 0.1$ and no steady-state error.

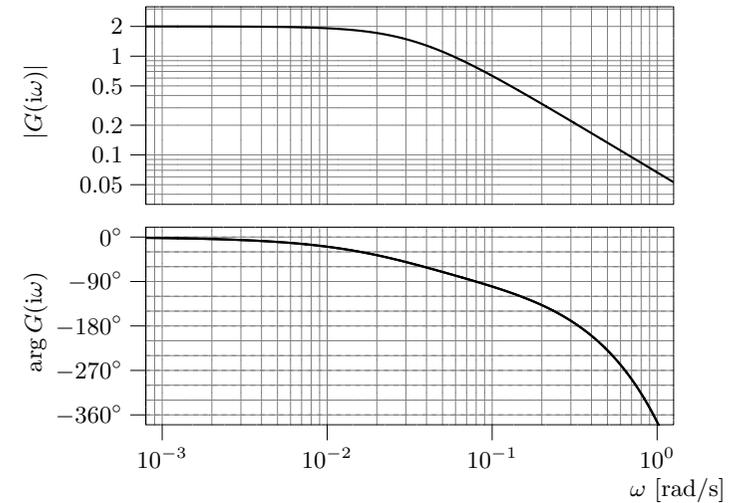


Figure 4.12b

5 Compensation

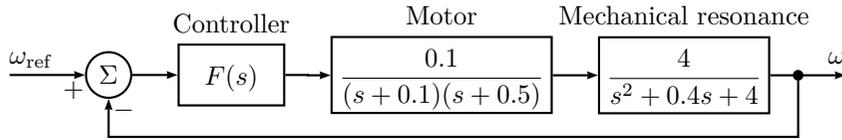


Figure 5.1a

5.1 The block diagram for speed control of a DC-motor is shown in Figure 5.1a. Find a compensator $F(s)$ such that the following specifications are fulfilled.

- The system should be twice as fast as for $F(s) = 1$, but with the same damping as for $F(s) = 1$.
- If ω_{ref} is constant, $|\omega_{\text{ref}} - \omega| / \omega_{\text{ref}}$ should be less than 5%.
- The controller should not be unnecessarily sensitive for high frequency disturbances and the open loop system's low frequency amplification should not be larger than necessary.

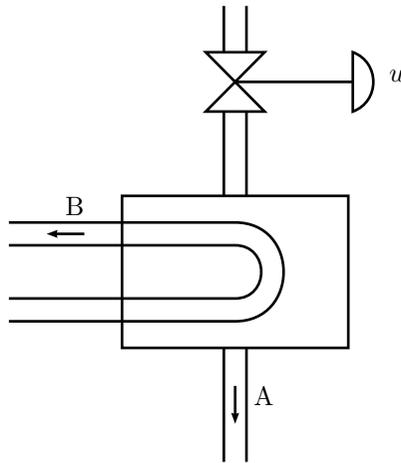


Figure 5.2a

5.2 The outflow temperature θ in the liquid A can be controlled in a heat exchanger by controlling the flow of the liquid B by a valve with the setting denoted u .

See Figure 5.2a. Measurements have been made using a sinusoidal input u and the gain and phase shift have been measured at different frequencies. The results are given in the following table.

Frequency [rad/s]	Gain	Phase shift
0.05	1.37	-67°
0.1	0.80	-106°
0.2	0.34	-153°
0.3	0.18	-185°
0.4	0.11	-210°

- Make a Bode plot for the system.
- What is the largest crossover frequency possible to achieve when using proportional control and wanting a phase margin of at least 50° ?
- Suggest a compensator that doubles the speed compared to b) and still keeps the phase margin.

5.3 A hydraulic system with a valve and a piston is described by the following linearized transfer function

$$G(s) = \frac{k_u/A}{s(\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1)}$$

where A is the area of the piston and k_u the hydraulic gain.

- Make a Bode plot for the system when $\omega_0 = 150$ rad/s, $\zeta = 0.1$, and $k_u/A = 20$.
- What is the smallest value of the ramp error that can be achieved using proportional control if we want an amplitude margin of 2? What is the crossover frequency in this case?
- Suggest a compensator such that the ramp error decreases 15 times at the same time as the crossover frequency, phase margin and amplitude margin will be the same as in b). Due to physical constraints in the implementation, this regulator has to have finite amplification at all frequencies.

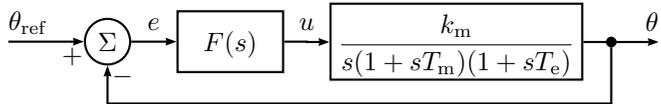


Figure 5.4a

5.4 Figure 5.4a shows a position servo including a DC-motor. The extra time constant T_e is due to the inductance in the winding of the motor, which is usually not taken into account. The parameter values are $k_m = 10$, $T_m = 0.1$ and $T_e = 0.01$. We want the servo to fulfill the following specifications:

- Rise time ≤ 0.1 s.
- Overshoot $\leq 10\%$.
- The steady state error at step in θ_{ref} should be zero.
- The steady state error when θ_{ref} is a ramp with slope $10^\circ/\text{s}$ should be less than 0.1° .

Suggest a compensator such that the specifications are fulfilled. (Clue: Suppose that the relationship between rise time, overshoot, and other specifications are the same as for a second order system, that is, according to Figures 5.4b and 5.4c in Solution 5.4 (the figures can also be found in Glad&Ljung).)

5.5 In Figure 5.5a we have arranged step responses and open loop and closed loop Bode plots for five different systems. Identify the three plots that belong to each of the five systems, one open loop and one closed loop Bode plot and one step response. Motivate your answer by pointing out one unique feature for each system.

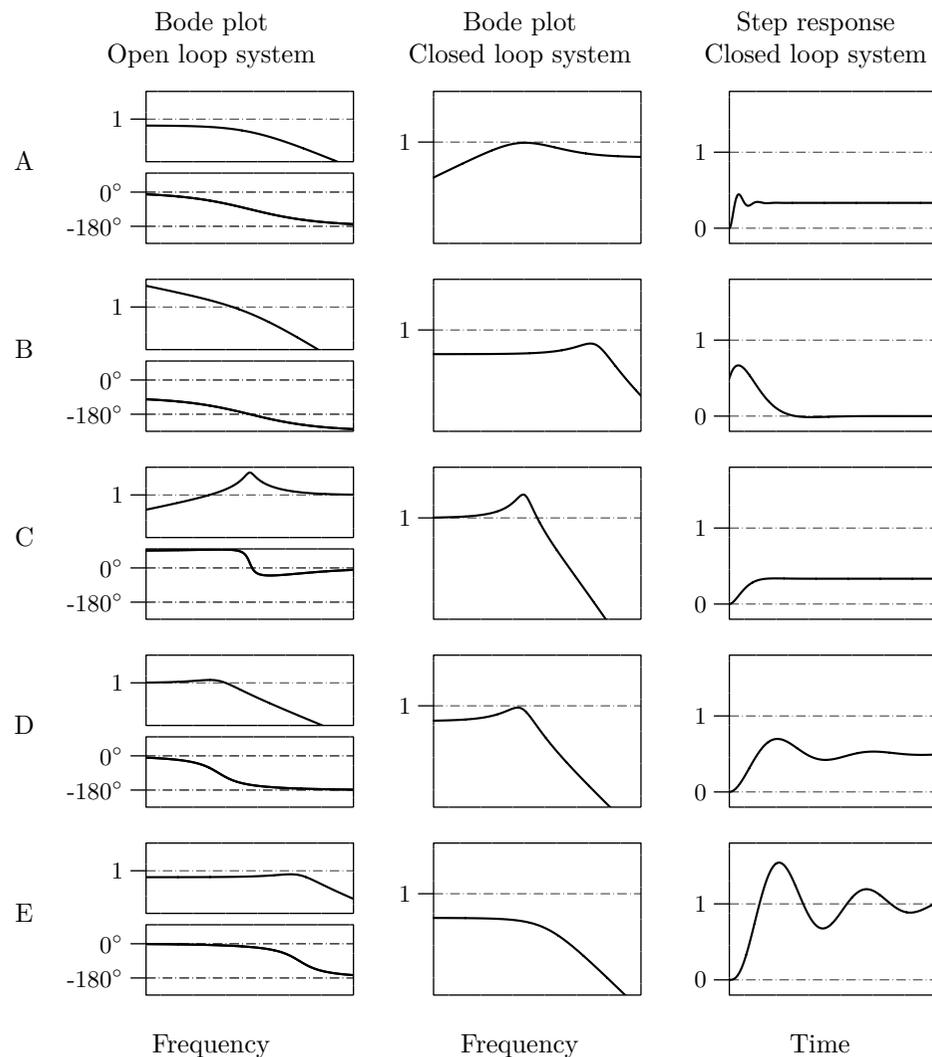


Figure 5.5a. All comparable diagrams have equal scaling.

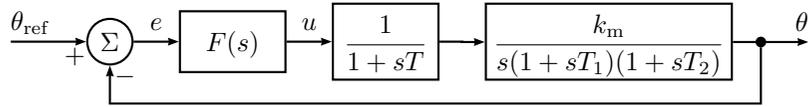


Figure 5.6a

5.6 A DC-servo is described by the block diagram in Figure 5.6a, where $T_1 = 50$ ms is a mechanical time constant, $k_m = 10$ is a proportional constant, $T_2 = 25$ ms is an electrical time constant, and $T = 10$ ms is an amplifier time constant. The system is tested with $F(s) = 1$ and we find that the dynamic properties are satisfactory but that the system is somewhat too slow. Find an $F(s)$ so that the closed loop system is twice as fast as for $F(s) = 1$, without increasing the overshoot. $F(s)$ should also give a closed loop system which fulfills the following accuracy demands:

- $|\theta - \theta_{\text{ref}}| \leq 0.001$ rad in steady state when θ_{ref} is constant.
- When θ_{ref} is a ramp with slope 10 rad/s we should have $|\theta - \theta_{\text{ref}}| \leq 0.01$ rad in steady state.

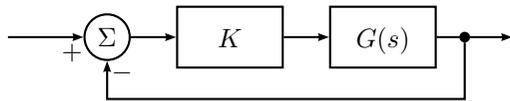


Figure 5.7a

5.7 The amplitude curves and the phase curves in Figure 5.7b have been measured for a system without poles in the right half plane. The system is controlled using feedback according to Figure 5.7a. Use the Nyquist criterion to decide for which values of K the closed loop system is stable ($K > 0$).

5.8 A block diagram for a control system with time delay is shown in Figure 5.8a. The system G_1 has no poles in the right half plane.

- G_1 has a Bode plot according to the plot in Figure 5.8b. Determine for what values of the time delay T the closed loop system is stable.
- The same as in a), but for the plot in Figure 5.8c.

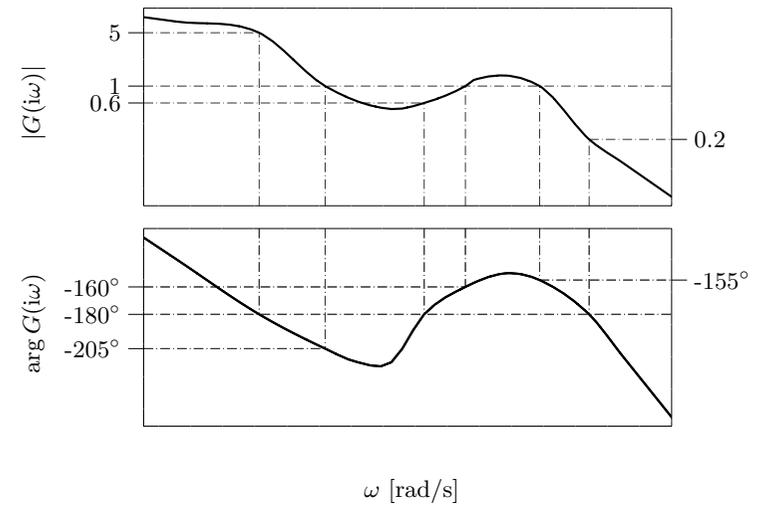


Figure 5.7b

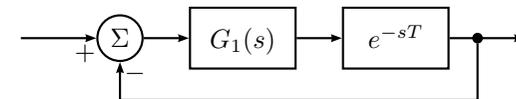


Figure 5.8a

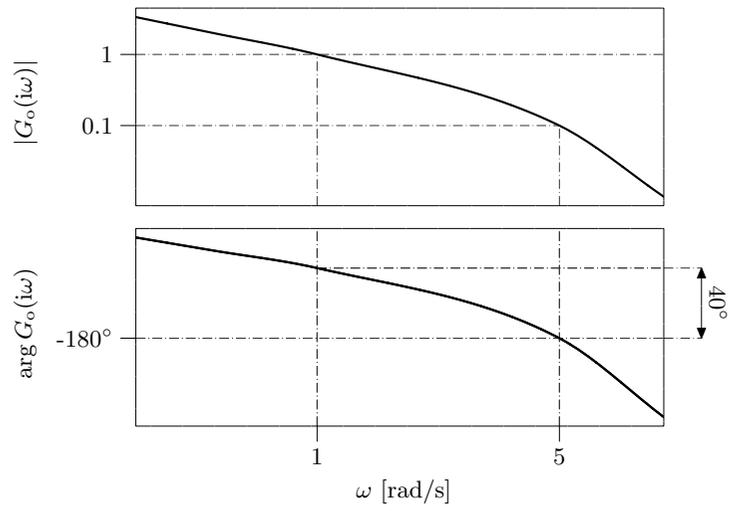


Figure 5.8b

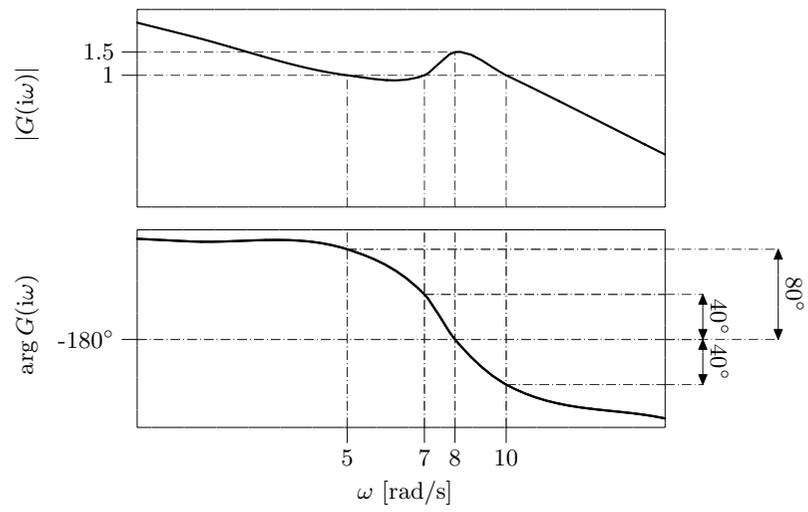


Figure 5.8c

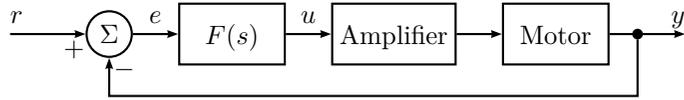


Figure 5.9a

5.9 A servo system based on a DC-motor has to be designed. A block diagram for the system is given in Figure 5.9a. By measuring the phase shift and the gain at different frequencies the Bode plot for the motor, see Figure 5.9b, has been determined. The amplification is a system of the first order, that is, it has the transfer function

$$G_A(s) = \frac{k_A}{s + a}.$$

In order to find the constants k_A and a , a unit step experiment has been carried out on the amplifier, giving the output shown in Figure 5.9c.

- a) Find the constants k_A and a from Figure 5.9c. Also draw the Bode plot for the open loop system, that is, the system from u to y .
- b) Find a compensator $F(s)$, such that the closed loop system fulfills the following demands:
 - ◇ The system has to be 5 times as fast as when using $F(s) = 1$.
 - ◇ The overshoot should not be larger than for $F(s) = 1$.

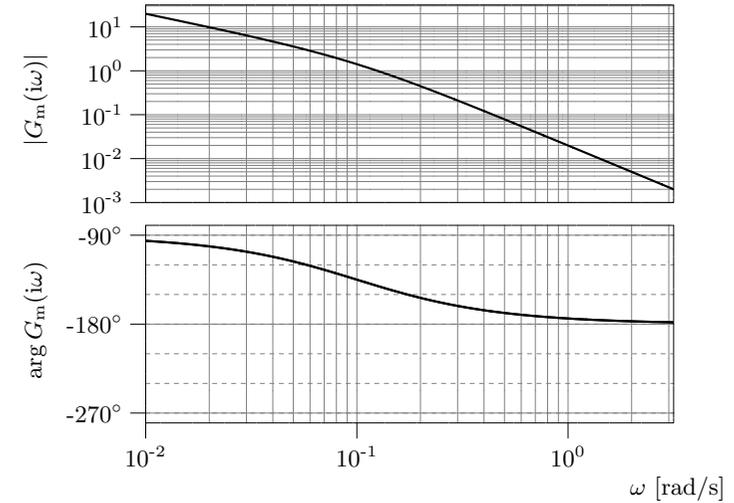


Figure 5.9b

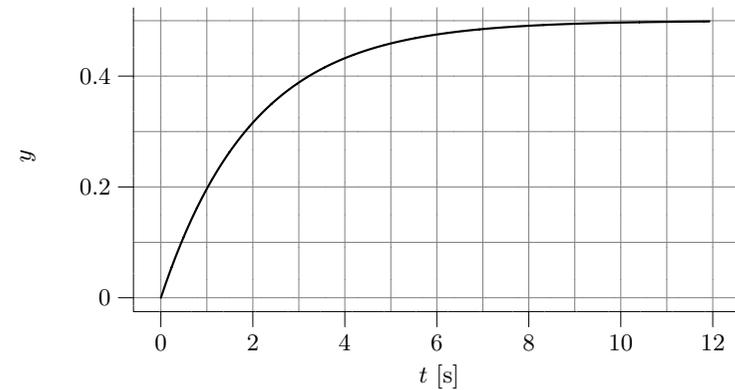


Figure 5.9c

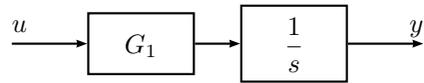


Figure 5.10a

5.10 A system $G(s)$ can be split into two sub-systems

$$G(s) = G_1(s) \frac{1}{s}$$

according to Figure 5.10a. The Bode plot for $G_1(s)$ is given in Figure 5.10b. Find a compensator for the system $G(s)$ such that the following is fulfilled:

- The phase margin for the compensated system is 40° .
- The closed loop system is twice as fast as what is possible to achieve using proportional control with a 40° phase margin.
- The steady state error when the reference signal is a ramp is 1% of the corresponding error with proportional control and 40° phase margin.

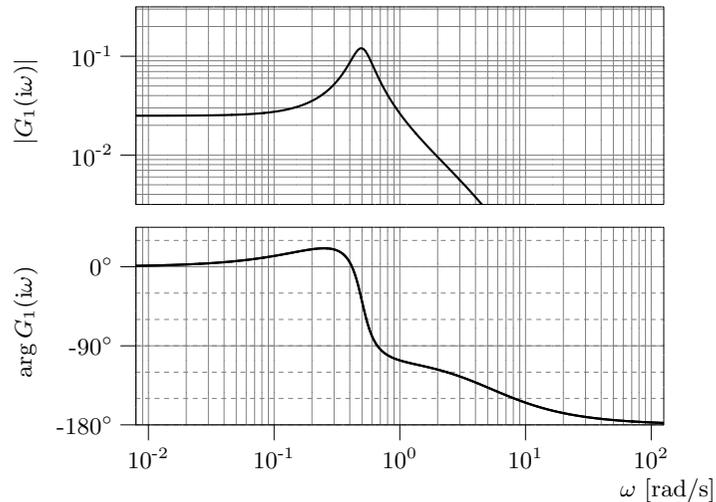


Figure 5.10b

5.11 The Bode plot for a system is given in Figure 5.11a.

- Draw the Nyquist curve of the system.
- Assume that the system is controlled using the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

For which $K > 0$ is the closed loop system asymptotically stable?

- Assume that we choose $K = 2$ in the proportional controller in problem b). What will the steady state error be when $r(t) = 10t$?
- Assume that $y(t)$ is delayed T seconds. How large is T allowed to be in order for the system to still be asymptotically stable with $K = 2$?

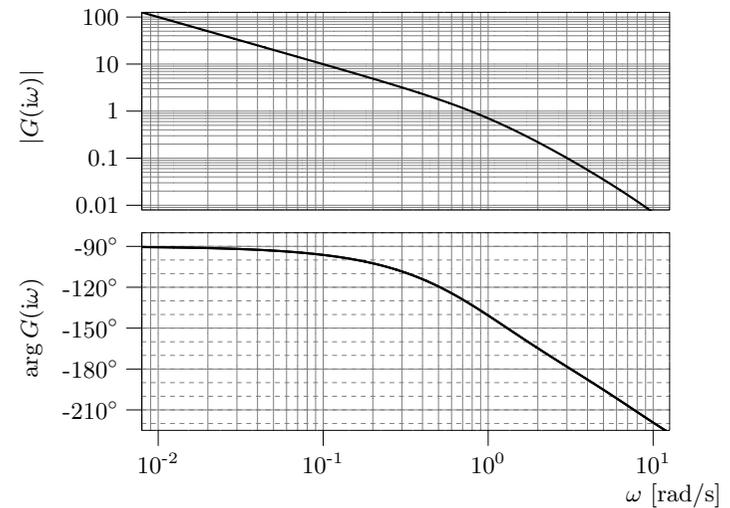


Figure 5.11a

5.12 a) A plot of the amplitude curve of a stable transfer function $G_o(s)$ is given in Figure 5.12a. Choose one of the following alternatives regarding the stability of the closed loop system $\frac{G_o}{1 + G_o}$:

1. It is stable.
2. It is not stable.
3. Impossible to determine given these facts only.

b) Repeat for the transfer function whose amplitude curve is given in Figure 5.12b. Justify your answers carefully.

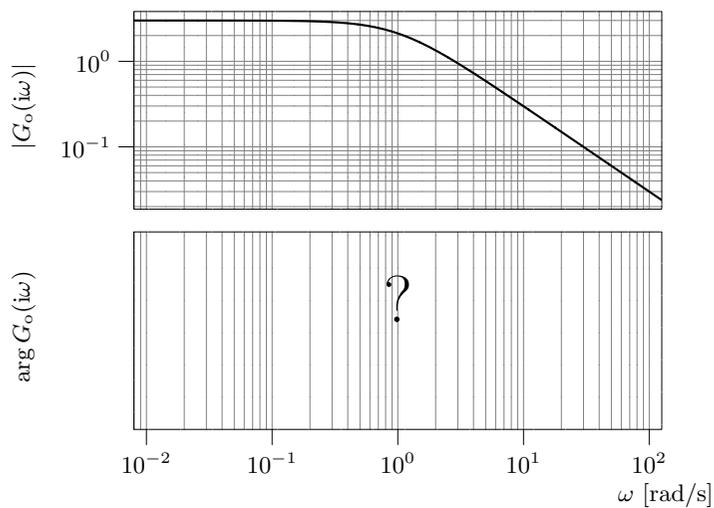


Figure 5.12a

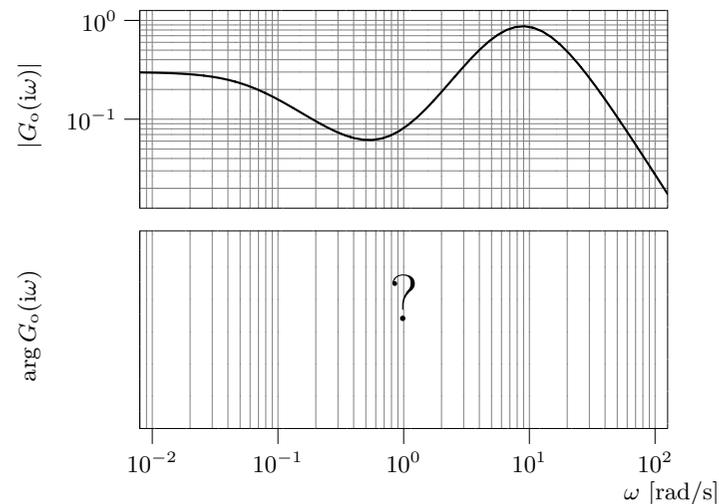


Figure 5.12b

5.13 Consider the relation

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

a) Assume that the system is controlled by

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 1$. Find ω_c , ω_p , φ_m , and A_m for the loop gain.

b) Compute a regulator such that the open loop system fulfills the following requirements:

- (i) $\omega_c = 5$
- (ii) $\varphi_m \geq 60^\circ$

and the closed loop system fulfills:

- (iii) $e_0 = 0$

Draw the Bode plot of the compensated open loop system and check that the requirements are satisfied. Simulate the closed loop system for a step in the reference signal and plot the step response. Check that the requirement on the steady state error is satisfied.

c) Draw the amplitude curve of the Bode plot of the closed loop system with and without the compensator. Describe how the properties of the closed loop system have been changed by the compensation.

d) Simulate the control error when the reference signal is a ramp and the regulator designed in b) is used. Is the stationary error zero?

5.14 When using microorganisms in production it is important to keep the oxygen concentration at a certain level to get maximum productivity. There are many ways to control the amount of dissolved oxygen, in this example we will use the speed of stirring as the controlled signal. The transfer function from the stirrer speed N_Δ to the oxygen measurement $O_{p\Delta}$ becomes (linearized model)

$$G(s) = \frac{b}{s + T_1} \frac{e^{-s\tau}}{1 + sT_2}$$

The parameters $\tau = 2$ s, $T_2 = 20$ s and $b = 0.02$ remain constant with change in stirrer speed while T_1 can vary from 0.02 s⁻¹ to 0.224 s⁻¹ as the stirrer speed increases from 400 r/min to 1200 r/min. A Bode plot for $G(s)$ is given in Figure 5.14a. Construct a controller, for the 1200 r/min case, which has a crossover frequency $\omega_c = 0.2$ rad/s, a phase margin $\varphi_m = 60^\circ$ and no steady-state error.

5.15 Earlier in the history of the basic automatic control courses at LiTH, the lead and lag compensators were parameterized in a different way. This parameterization is used in this problem, which is not only meant to be a crash course in understanding exams with solutions dating back to those days, but should be

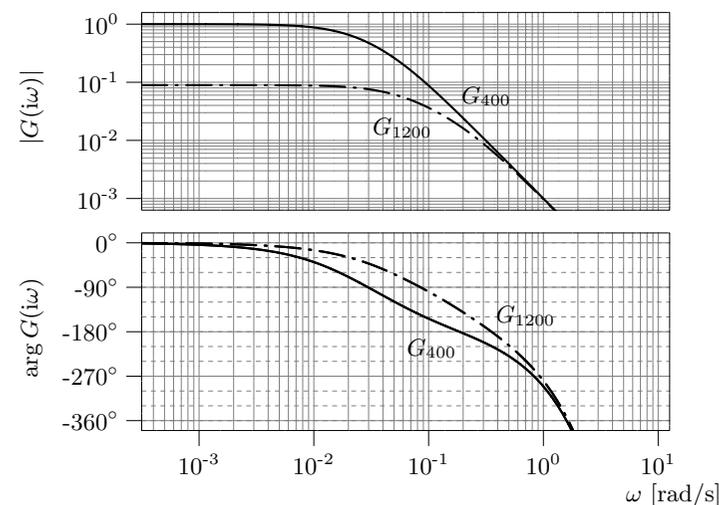


Figure 5.14a. Solid line: 400 r/min. Dash-dotted line: 1200 r/min.

seen as a chance of practicing the *ideas* of loop shaping rather than exercising *recipe* knowledge.

The old parameterization of a lead compensator looked as follows:

$$F_{\text{lead}}(s) = N \frac{s + b}{s + bN}$$

The schematic Bode plot of such a lead compensator is shown in Figure 5.15a, and it can be shown that the maximum phase lead is obtained at the frequency $b\sqrt{N}$, where the gain is \sqrt{N} , and the phase lead is given by

$$\varphi_{\text{max}} = \arctan \frac{\sqrt{N} - 1/\sqrt{N}}{2}$$

(shown in Figure 5.15b).

The old parameterization of a lag compensator looked as follows:

$$F_{\text{lag}}(s) = \frac{s + a}{s + a/M}$$

The schematic Bode plot of such a lag compensator is shown in Figure 5.15c. Just like the parameter τ_1 of the new parameterization, the parameter a used

here shall be determined as a trade-off between the undesirable phase lag and the bandwidth of the desirable low frequency gain. As a rule of thumb, one may use $a = 0.1 \omega_{c,d}$ as an initial guess, and iteratively improve from there. The worst phase lag for this choice of a is about 5.7° .

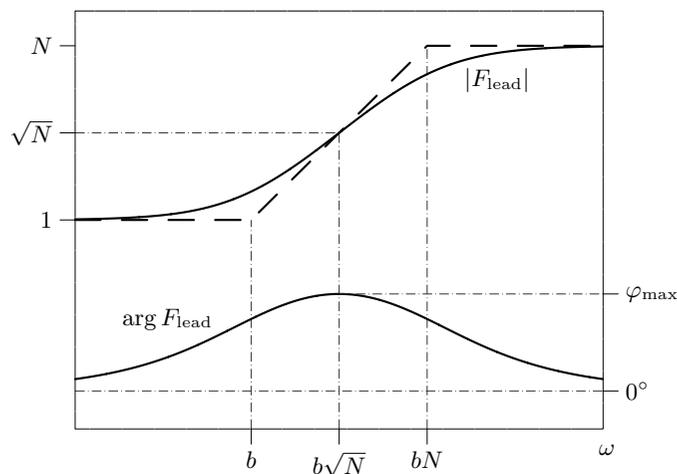


Figure 5.15a. Schematic Bode plot for the old parameterization of a lead compensator.

A read/write head of a hard disk is mounted on a mechanical arm which is moved by a motor. The system from motor input voltage to the angle of the arm is modelled by

$$Y(s) = \frac{5}{\tau_1 s + 1} \cdot \frac{0.05}{s(\tau_2 s + 1)} U(s)$$

where Y and U are the Laplace transforms of the output and input respectively. The numerical values of the constants are $\tau_1 = 10^{-3}$ och $\tau_2 = 0.05$. The Bode plot of the system is given in Figure 5.15d.

- a) To begin with, assume that the arm is controlled using proportional feedback,

$$U(s) = K(R(s) - Y(s))$$

What are the step and ramp error coefficients (often referred to as e_0 and e_1)? For what values of K are they defined?

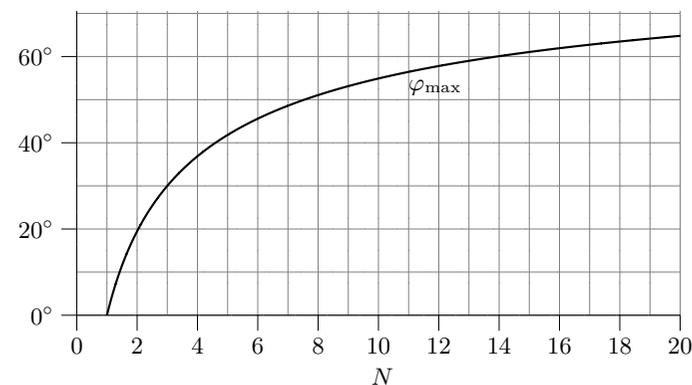


Figure 5.15b. Maximum phase advance as a function of N in a lead compensator parameterized the old way.

- b) Compute a controller,

$$U(s) = F(s)(R(s) - Y(s))$$

for the same system, such that the resulting system fulfills the following requirements:

- ◇ $e_0 = 0$
- ◇ $e_1 \leq 0.001$
- ◇ $\omega_c = 100$ rad/s
- ◇ $\varphi_m \geq 50^\circ$

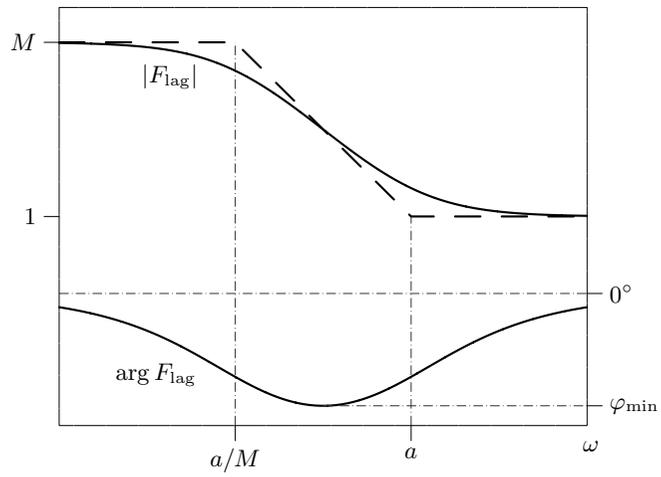


Figure 5.15c. Schematic Bode plot for the old parameterization of a lag compensator.

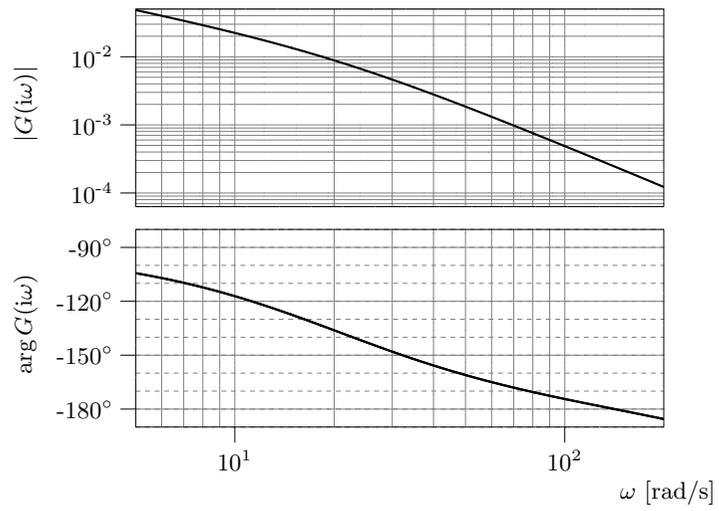


Figure 5.15d

6 Sensitivity and Robustness

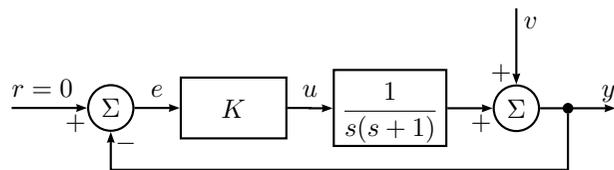


Figure 6.1a

- 6.1 Consider the control system in Figure 6.1a where $v(t)$ is a sinusoidal disturbance, $v(t) = \sin(t)$. Compute the absolute value of the sensitivity function at $\omega = 1$ rad/s as a function of K . How must K be selected if the amplitude of $y(t)$ shall be less than the amplitude of $v(t)$ at this frequency?

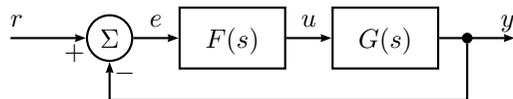


Figure 6.2a

- 6.2 Assume that we have constructed a controller $F(s)$ for the model $G(s)$, see Figure 6.2a, such that there is no steady state error when the reference signal is a step. Let the real system be given by

$$G^0(s) = (s + 1)G(s)$$

and assume that $G^0(s) \rightarrow 0$, $s \rightarrow \infty$. Also assume that the amplitude curve of the closed loop system has no resonance peaks and decreases, at least and asymptotically, with 20 dB₂₀/decade for frequencies over the bandwidth. What is the highest possible bandwidth we can use for the closed loop system in Figure 6.2a, while at the same time guaranteeing stability?

- 6.3 Figure 6.3a shows a Nyquist diagram for the loop gain G_o . Show in a figure for what frequencies (that is, for what part of the Nyquist curve above) additive disturbances on the output are amplified in the sense that the output amplitude of the control system in Figure 6.3b is larger than the disturbance amplitude.

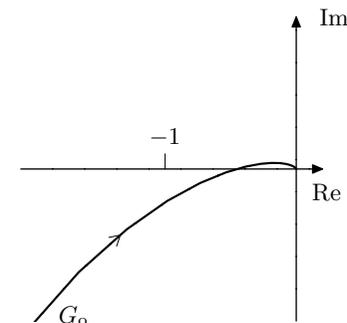


Figure 6.3a

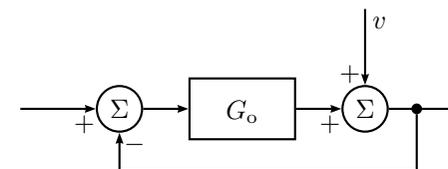


Figure 6.3b

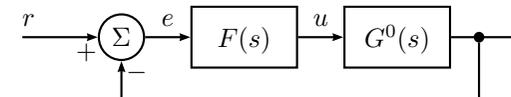


Figure 6.4a

- 6.4 Consider the control system in Figure 6.4a. The true system, denoted $G^0(s)$, is modeled as

$$G(s) = \frac{1}{s + 10}$$

The controller

$$F(s) = \frac{s + 10}{s}$$

gives an asymptotically stable closed loop system with the model $G(s)$. Now

assume that the system is given by

$$G^0(s) = G(s)(1 + G_\Delta(s))$$

where it is known that $G_\Delta(s)$ has no poles in the right half plane, and that

$$|G_\Delta(i\omega)| < \frac{0.9}{\sqrt{1 + \omega^2}}$$

Can we be sure that the closed loop system is asymptotically stable?

6.5 A process is described by the model $G(s)$, while the process in reality has the transfer function

$$G^0(s) = e^{-sT} G(s)$$

a) Draw the absolute value of the inverse of the relative model error, that is,

$$\frac{1}{|G_\Delta(i\omega)|}$$

b) Assume that we design a controller $F(s)$ starting with the model $G(s)$. How large may

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

be at most, in order to guarantee asymptotic stability of the closed loop system for all values of T , when the controller $F(s)$ is used on the system $G^0(s)$?

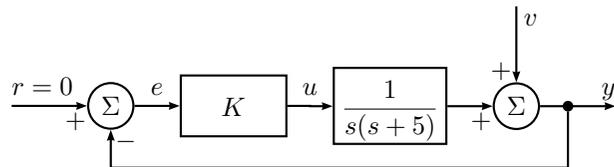


Figure 6.6a

6.6 Consider the control system in Figure 6.6a.

a) Assume that the real system is given by

$$G^0(s) = G(s) + \tilde{G}(s),$$

where

$$G(s) = \frac{1}{s(s+5)}$$

and let $K = 25/2$. Use the robustness criterion to obtain a condition on $|\tilde{G}(i\omega)|$ that guarantees stability in the closed loop system. Does $\tilde{G}(s) = 1$ fulfill the conditions?

b) Now let $\tilde{G}(s) = \alpha$ where α is a scalar. Calculate the characteristic equation for the closed loop system and decide for which α the system is stable. Does this contradict the condition from the robustness criterion?

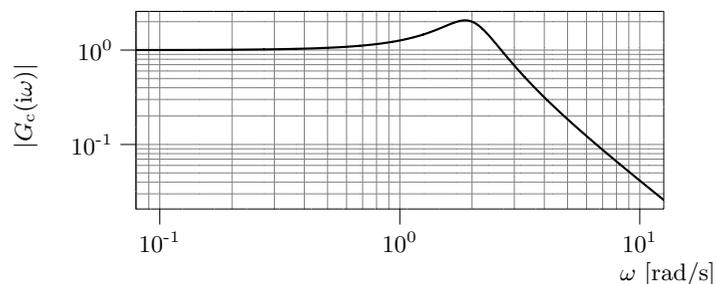


Figure 6.7a

6.7 A DC-motor is assumed to have the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

and it is controlled using proportional feedback,

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 4$. The amplitude curve of the feedback system

$$|G_c(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

is given in Figure 6.7a. Assume that the real system is given by

$$G^0(s) = G(s) \frac{\alpha}{s + \alpha}, \quad \alpha > 0$$

and the controller $F(s)$ is used on the system $G^0(s)$.

- Draw a root locus with respect to α for the characteristic equation of the closed loop system and determine for which α the closed loop system is asymptotically stable.
- Use the robustness criterion to decide for which α the closed loop system is asymptotically stable.
- Comment on the possible differences in the demands on α in a) and b).

6.8 A system $G^0(s)$ is controlled using a regulator $F(s)$. In Figure 6.8a the amplitude part the Bode plot of the nominal closed loop system,

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

is shown. It is known that G_c is stable, and it is assumed that G and G^0 have the same number of poles in the right half plane. The model uncertainty $G_\Delta(s)$, defined by

$$G_\Delta = \frac{G^0 - G}{G}$$

is assumed bounded by $|G_\Delta(i\omega)| \leq \gamma\omega$. In what interval must γ lie to guarantee stability of the closed loop system?

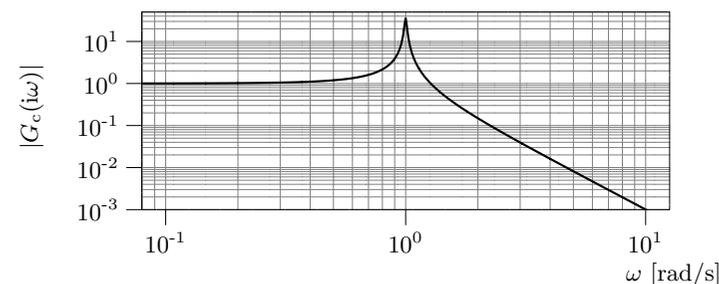


Figure 6.8a

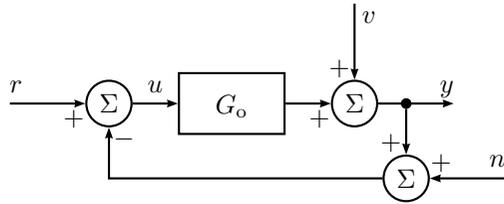


Figure 6.9a

6.9 Consider the system in Figure 6.9a. For $r(t) = 0$, $n(t) = 0$ and $v(t) = \sin t$ the steady-state output is given by

$$y(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)$$

Determine the steady-state output $y(t)$ when $r(t) = 0$, $v(t) = 0$ and $n(t) = \sin t$.

6.10 Recall the model that was used in the design of a lead-lag controller using MATLAB in Problem 5.13. Assume that the true system contains a time constant that was neglected, and that the transfer function of the system is given by

$$G^0(s) = G(s) \frac{1}{s+1}$$

a) Determine the relative model error $G_\Delta(s)$.

b) Draw $\frac{1}{|G_\Delta(i\omega)|}$ and $\left| \frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)} \right|$ in a Bode plot, when $G(s)$ is given by

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

for the two cases $F(s) = 1$ and $F(s)$ being the controller designed in Problem 5.13. What can be said about the robustness of the closed loop system in these two cases when $F(s)$ is used for control of the “true” system $G^0(s)$?

One possible solution to the design problem in Problem 5.13 was the controller

$$F(s) = 0.46 \cdot \frac{0.43s+1}{0.090s+1} \cdot \frac{2.0s+1}{2.0s}$$

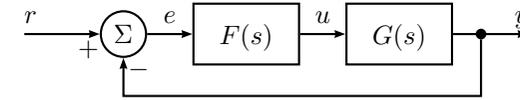


Figure 6.11a

6.11 Consider the control system in Figure 6.11a. The controller $F(s) = 2$ gives the Bode plot of the loop gain $F(i\omega)G(i\omega)$ shown in Figure 6.11b. The Bode plot of the sensitivity function is shown in Figure 6.11c. The reference signal is $r(t) = 2 \sin 0.1t$. Determine the amplitude of the error in steady state.

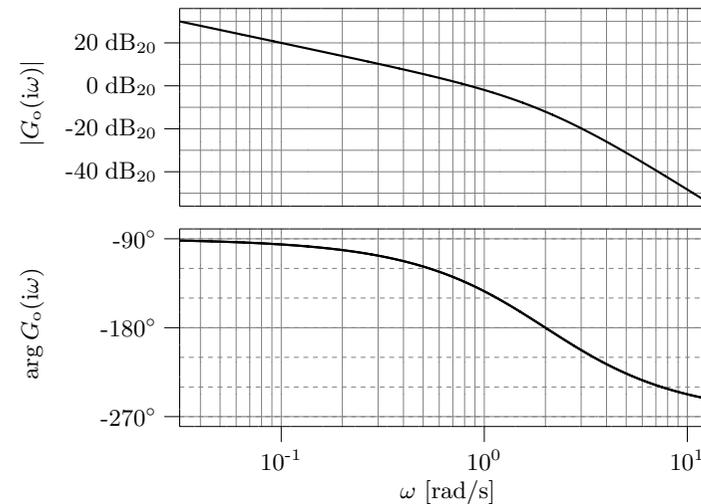


Figure 6.11b

6.12 In Problem 5.14 we saw how the amount of dissolved oxygen depends on the stirring speed. A lead-lag controller was designed for the model linearized around 1200 r/min. Check if this controller also stabilizes the system when the stirring speed is 400 r/min.

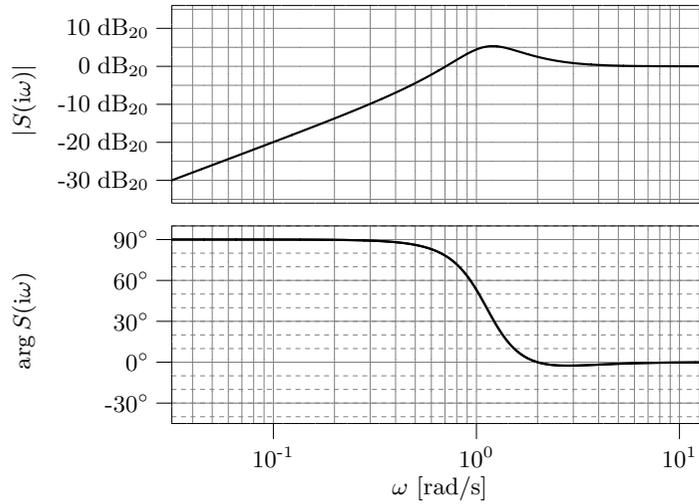


Figure 6.11c

6.13 Consider the connected tank system in Figure 6.13a where u is the inflow to the upper tank and y is the level in the lower tank. The system can approximately be described by the following transfer function

$$G(s) = \frac{1}{(s + 1)^2}$$

The level in the lower tank is controlled by a P controller;

$$U(s) = K(R(s) - Y(s))$$

The goal of the control is to minimize the influence of the disturbance v . This has been formalized as the following demand on the system:

$$|S(i\omega)| \leq 0.1$$

at $\omega = 1$, where $S(s)$ denotes the sensitivity function. How must K be selected if the demand should be satisfied?

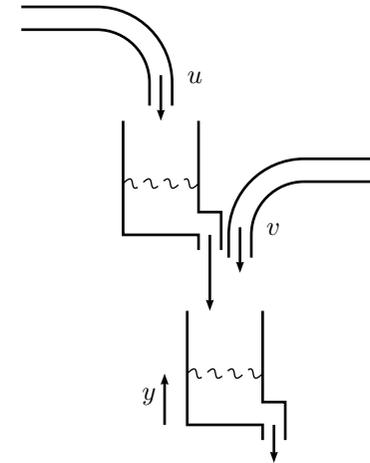


Figure 6.13a

7 Special Controller Structures

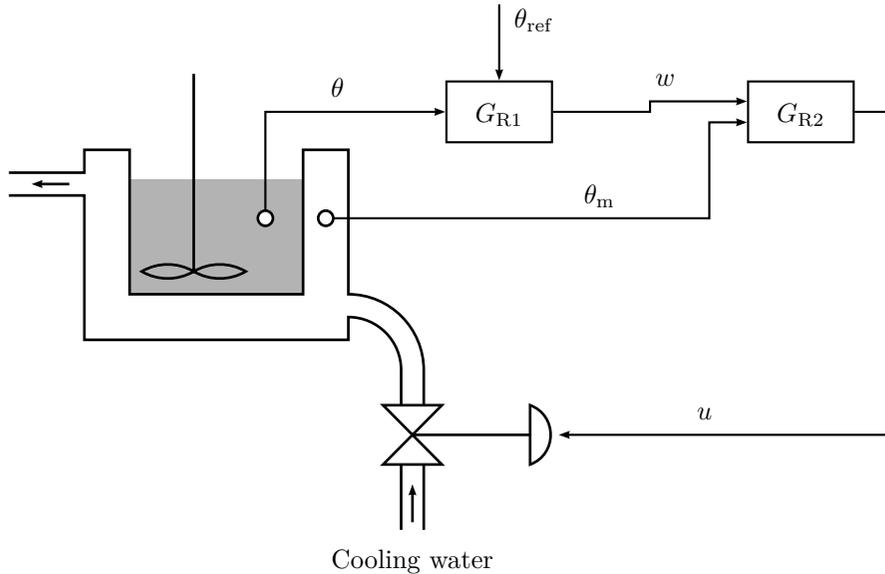


Figure 7.1a

7.1 To control the temperature θ in a chemical reactor, the control system in Figure 7.1a is used, where θ_{ref} is the desired (reference) temperature. The temperatures θ and θ_m in the reactor and the cooler, respectively, are measurable and can be used to control the valve u . The structure of the control system is given by Figure 7.1b. It is here assumed that both G_{R1} and G_{R2} are P controllers.

- a) Let $K_2 = 9$ and draw the Bode plot of the transfer function from w to θ . Then choose K_1 so that the gain margin $A_m = 2$. What are the gain crossover frequency ω_c and the steady-state error, if we assume that θ_{ref} is changed stepwise?

- b) Suppose that we make a simple feedback loop instead, see Figure 7.1c. How is the Bode plot affected? Again, choose K_1 so that the gain margin $A_m = 2$, and determine the gain crossover frequency ω_c and the steady-state error. Compare with a) with respect to steady-state errors and response times. Conclusions?

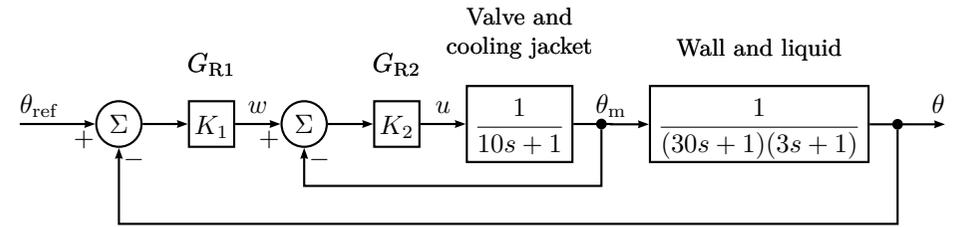


Figure 7.1b

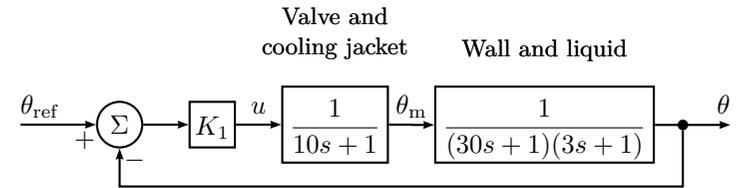


Figure 7.1c

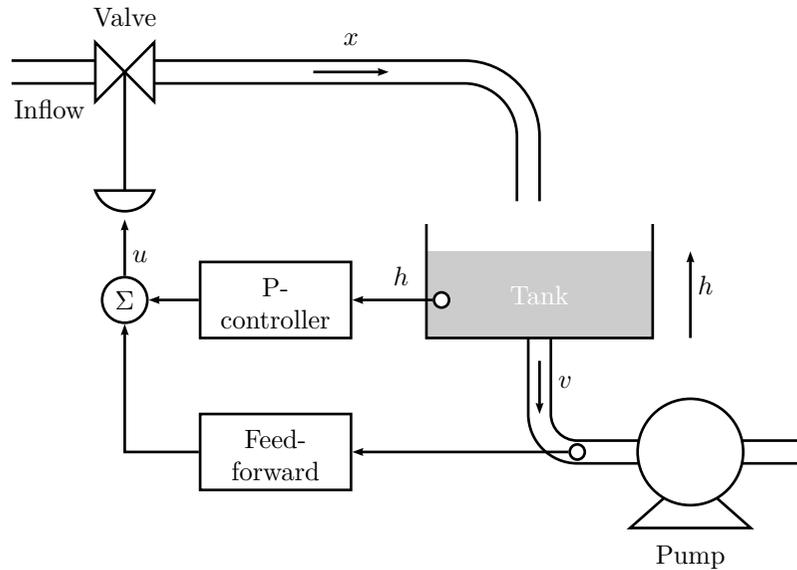


Figure 7.2a

7.2 A level control system for a water tank is shown in Figure 7.2a, where all variables denote offsets from an operation point. The inflow, $x(t)$, to the tank is determined by the valve, and the outflow, $v(t)$, is determined by the pump. Stu Dent has got the assignment to keep the water level in the tank constant, in spite of variations in the outflow $v(t)$. First, Stu determines the transfer function $G_v(s)$ from the valve to $x(t)$. By step response experiments, he obtains the following result:

$$G_v(s) = \frac{1}{0.5s + 1}$$

- a) Because the disturbance $v(t)$ is measurable, Stu first considers a feedforward compensator to completely eliminate it. Stu, who also knows that it is dangerous to differentiate the disturbance, cancels all the derivative terms in the compensator. Compute the feedforward compensator, and determine the response $h(t)$ Stu will get, if the outflow $v(t)$ is changed stepwise with an amplitude of 0.1.

- b) To improve the control system, Stu also introduces a proportional feedback of the water level h . What is the steady-state error in the level h now, if the outflow is changed in the same way as in a)?

7.3 Consider the following system

$$Y(s) = \frac{2}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where u is the control signal, y is the output and v is a disturbance. It is desired that y should be as small as possible despite the disturbance v .

- a) Design a feedforward controller from v to u that eliminates the influence of v on y .
- b) Assume that v is a pure sinusoid with amplitude 2. How large will the control signal be?
- c) The real system is described by

$$Y(s) = \frac{b}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where b value is not exactly known but has its value close to 2. To solve this problem a P controller is added to the feedforward controller that was designed in a). The full controller looks like

$$U(s) = -KY(s) + F_f(s)V(s)$$

where $F_f(s)$ is the feedforward controller. What is the stationary error if $v = 1$?

7.4 The transfer function for a temperature control system is given by

$$Y(s) = \frac{3}{s+1}U(s) + \frac{4}{(s+2)(s+5)}V(s)$$

where y is the controlled temperature, u is the supplied power and v is the temperature of the surroundings. Assume that the desired temperature is zero.

- a) Design a feedforward controller $U(s) = F_f(s)V(s)$ which eliminates the influence of the disturbance v on y .
- b) To simplify implementation $F_f(s)$ is replaced with a constant, $\tilde{F}_f = F_f(0)$. Assume that v is given by $v(t) = -1 - 0.1t$ and that $U(s) = \tilde{F}_f V(s)$ is used. What will $y(t)$ be in steady state?
- c) The previous controller is now extended with a P controller:

$$U(s) = \tilde{F}_f V(s) - KY(s)$$

What will now $y(t)$ be in steady state?

- d) Assume that one only uses the P controller

$$U(s) = -KY(s)$$

What will now $y(t)$ be in steady state?

8 State Space Description

8.1 Define suitable state space variables for the DC motor discussed in Problem 2.1, and write the system in state space form.

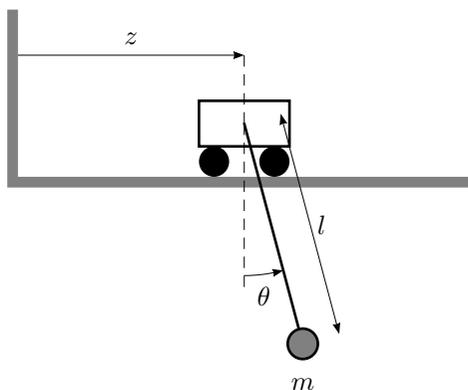


Figure 8.2a

8.2 Consider the system illustrated in Figure 8.2a. It consists of a hinge that can move in the direction marked “z”, and a thereto attached pendulum. The system is described by the equation

$$\ell \ddot{\theta} + g \sin \theta + \ddot{z} \cos \theta = 0$$

Define state space variables, input, and output as

$$x_1 = \theta \quad x_2 = \dot{\theta} \quad u = \ddot{z}/\ell \quad y = \theta$$

and

$$\omega_0^2 = g/\ell$$

Linearize the system around the equilibrium point given by

$$x_1 = \pi \quad x_2 = 0 \quad u = 0$$

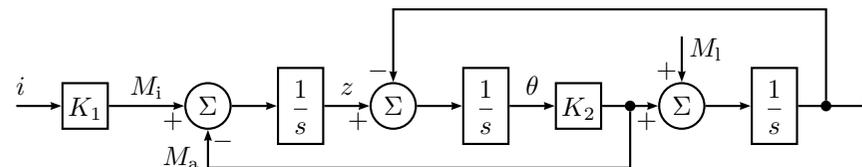


Figure 8.3a

8.3 The block diagram in Figure 8.3a describes an electric motor that drives a load via an elastic axis. Here i is the driving current to the motor, which gives the torque M_i . z is the turning rate of the motor and y is the turning rate of the load. θ is the angle of the transmission axis. $M_a = K_2\theta$ is the torque this angle causes. M_1 is the torque from the load. Give a state space description for the system with M_1 and i as inputs and y as output. (There are at least two different ways to solve this problem.)

8.4 Write the following systems in state space form.

a)

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 6u$$

b)

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 4\frac{d^2u}{dt^2} + \frac{du}{dt} + 2u$$

c)

$$G(s) = \frac{2s + 3}{s^2 + 5s + 6}$$

Use for example controllable or observable canonical form or diagonal form.

8.5 A system has the impulse response (weight function)

$$g(t) = 2e^{-t} + 3e^{-4t}$$

Write the system in state space form.

8.6 Consider the system

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (-1 \quad 2) x$$

Compute the transfer function of the system.

8.7 Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

The input is being held constant, $u = u_0$, for the time $t_0 \leq t \leq t_0 + T$.

Give a relation between $x(t_0)$, $x(t_0 + T)$, $y(t_0)$, $y(t_0 + T)$ and u_0 .

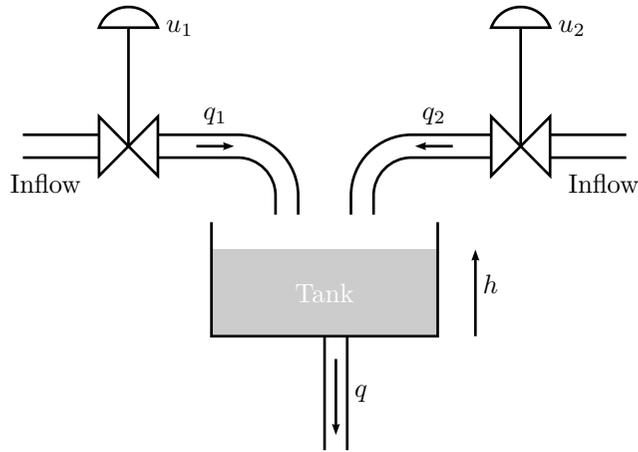


Figure 8.8a

8.8 Consider the tank in Figure 8.8a. The tank can be filled from two different pipes, where the flows q_1 and q_2 are determined by the valve settings u_1 and u_2 . If q_1 , q_2 , u_1 , u_2 , and h denote the deviation from a nominal value, we get the linearized equation

$$\dot{h} + \frac{1}{\tau} h = u_1 + u_2$$

where $\tau = 1$. It is desired that the level should follow a reference value h_{ref} and that q_1 and q_2 should be of approximately the same size. Therefore two PI controllers are used so that

$$u_1 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.1)$$

$$u_2 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.2)$$

- Introduce the state variable $x_1 = h$, and let x_2 and x_3 represent the integrals in (8.1) and (8.2) respectively. Derive a state space description of the closed loop system with h_{ref} as input and h as output.
- Verify that the closed loop system is unobservable and that the unobservable subspace is spanned by the vector

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Give a practical interpretation of this phenomenon.

- The level is measured by two different sensors, and due to the poor accuracy in the first sensor it delivers the level signal together with an error. The equation of the first regulator can hence be written

$$u_1 = -(h + n) + \int_0^t -(h + n) d\tau \quad (8.3)$$

where it for simplicity has been assumed that $h_{\text{ref}} = 0$. The second regulator is then given by

$$u_2 = -h + \int_0^t -h d\tau \quad (8.4)$$

Modify the state space model by letting the measurement disturbance be the input to the state space model of the closed loop system.

8.9 Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

Is it possible to control the system from the origin to $x = (1 \ 3)^T$ within 4 seconds?

8.10 Give the dimensions of the controllable and unobservable subspaces to the systems below. Give also the controllable and unobservable subspaces.

a)

$$\dot{x} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \ 3 \ 1.5) x$$

b)

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} u$$

$$y = (0 \ 3 \ 0) x$$

8.11 A state space representation of

$$G(s) = \frac{1}{s+1}$$

is given by

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0) x$$

a) Compute $x_1(t)$, $x_2(t)$ and $y(t)$ if $x(0) = 0$ and

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

b) Is the system asymptotically stable? Input-output-stable?

c) Examine the controllability and observability for the system.

d) Explain why the realization is not suitable for simulating a system whose transfer function is $G(s)$.

8.12 Compute the poles and zeros of the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = (1 \ 1) x$$

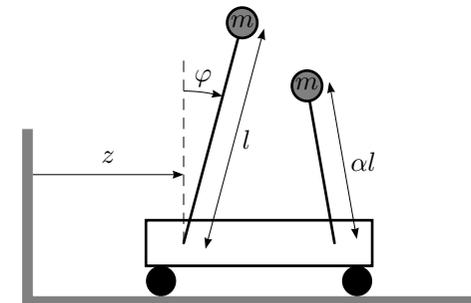


Figure 8.13a

8.13 Two mathematical pendulums are mounted on a trolley. They are mounted so that they can move without friction in a plane coinciding with the direction of movement for the trolley. The lengths of the pendulums are l and αl and their masses are m . For one pendulum we have

$$\ddot{z} \cos \varphi + \ddot{\varphi} l = g \sin \varphi$$

a) Linearize the system to the left in Figure 8.13a around $\varphi = 0$ and put the constants l , m , and g to 1 and write the equations in the form $\dot{x} = Ax + Bu$.

b) Give the values on α for which the system is controllable. Give a practical motivation.

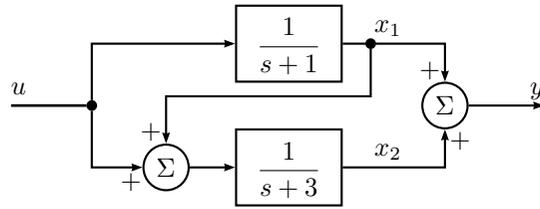


Figure 8.14a

- 8.14 A system is given by the block diagram in Figure 8.14a. Derive a state space model of the system, with the state space variables given in the figure.
- 8.15 The substances A and B react according to, $3A \rightarrow B$, in a tank. The reaction speed is given by $r_A = -k_1 c_A^3$. The inflow, q to the tank has concentration $c_{A,in}$. The tank volume V and the in- and outflow can be considered constant.
- Determine the dynamical mass balance for the components A and B in the form of differential equations.
 - Linearize the differential equations around a stationary point, c_A^* , c_B^* , $c_{A,in}^*$, and use the state space representation

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

where the state x consists of the deviations $c_{A,\Delta}$ and $c_{B,\Delta}$ of the concentrations. The input signal u is the deviation $c_{A,in,\Delta}$ in the inflow concentration and the output signal y is the deviation $B c_{B,\Delta}$ in concentration of component.

- 8.16 Figure 8.16a shows a protein that changes between the conformations x_1 and x_2 . The kinetic parameters k_{ij} determine the rate of the conformational changes. The input signal, u , together with the constant, K , represent introduction of x_2 to the system. We assume that we can measure x_2 , i.e. $y = x_2$. All reaction rates are assumed to be given by expressions proportional to the concentration of the states they emanate from. That is, for a state i the differential equation will be $\dot{x}_i = \sum_p k_{ip} x_p - \sum_q k_{qi} x_i$, where p is the number of incoming and q the number of outgoing flows from the state i . This corresponds to mass action kinetics for a regular biochemical reaction network.

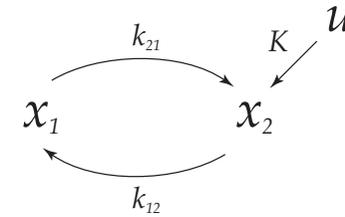


Figure 8.16a. Protein in two possible conformations.

- Write down the state space equations that describe the system.
- Compute the poles of the system. Is the system stable?
- What are the conditions on the kinetic parameters k_{ij} for the system to be observable? Give an interpretation of this result.
- Assume that the conformation change from x_2 to x_1 is catalyzed by an enzyme. To describe this, include Michaelis-Menten saturation in the flow from x_2 to x_1 , i.e. replace the expression $k_{12}x_2$ with $\frac{V_{max}x_2}{K_M + x_2}$, where V_{max} is the maximal rate and K_M the saturation constant. Write down the differential equations for the states x_1 and x_2 .
- Sketch the relation between x_2 and the saturated reaction rate in d), i.e. \dot{x}_2 as a function of x_2 .
- Linearize the system around the stationary point $x_1 = \frac{V_{max}}{2k_{21}}$, $x_2 = K_M$, $u = 0$.

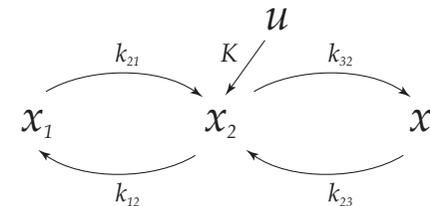


Figure 8.16b. Protein in three possible conformations.

Now, assume that the protein can exist in a third conformation x_3 , according to Figure 8.16b. We measure $y = x_2$ as before.

- g) Write down the state space equations that describe the system.
- h) What are the conditions on the kinetic parameters k_{ij} for the system to be observable? Give an interpretation of this result.

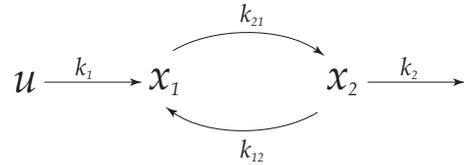


Figure 8.17a

8.17 The model in Figure 8.17a describes the interaction between two proteins with concentrations x_1 and x_2 , where the kinetic parameters k_1, k_{21}, k_{12} and k_2 determine the reaction rate.

The law of mass actions for biochemical reaction networks leads to the following state space model for the system.

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} - k_2 \end{pmatrix} x + \begin{pmatrix} k_1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

The input u is a unit step.

- a) What are the steady state values of x_1 and x_2 , when the input u is a unit step?
- b) Compute the transfer function of the system, from u to y .
- c) Use the final value theorem to find the steady state gain of y . Compare with the result in a). What conclusions can be drawn?

9 State Feedback

9.1 Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \ 0) x \end{aligned}$$

- Calculate a state feedback that places the poles in I) $\{-3, -5\}$, II) $\{-10, -15\}$. What limits the possibility to achieve arbitrary dynamics of the closed loop system?
- Suppose only the output is measured. Calculate an observer that makes the transfer function from the reference signal to the output the same as in a). Discuss the influence of the poles of the observer.

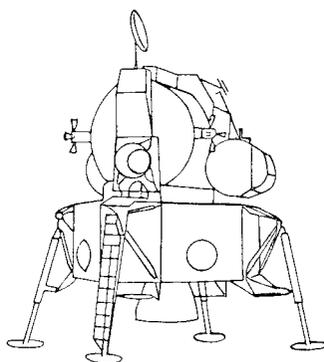


Figure 9.2a

9.2 Figure 9.2a shows the Lunar Excursion Module from the Apollo project. Consider the module hovering a short distance above the surface of the moon using its main engine. If the pitch angle of the module (angle between the vertical line and the direction of movement) differs from zero, a horizontal component of the force is obtained and the module is accelerating along the surface.

We will study a block diagram which shows the connection between the input u (the control signal to the attitude thrusters), the pitch angle θ and the position coordinate z . See Figures 9.2b and 9.2c.

The module is both in the θ -direction and in the z -direction obeying Newton's law of motion without any kind of damping. The transfer function from the control signal of the astronaut (y_{ref}) to velocity \dot{z} is

$$\frac{K_1 K_2}{s^3}$$

which is very difficult to control by hand.

- Write the system in state space form.

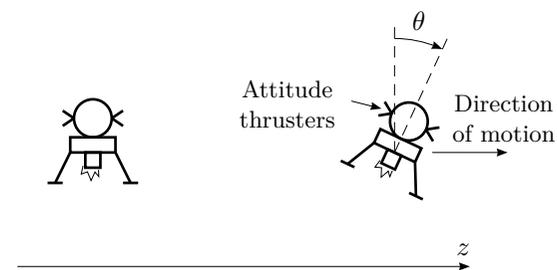


Figure 9.2b

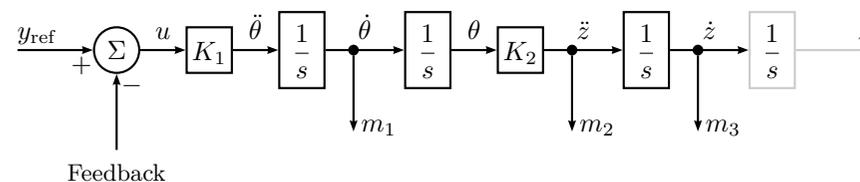


Figure 9.2c

b) In order to make the control duty of the astronaut easier we change the dynamics of the module by making internal feedback. The following signals are measurable:

m_1 , the attitude angular velocity measured using rate gyro.

m_2 , the acceleration in z -direction measured using accelerometers positioned on gyro-stabilized platforms.

m_3 , the velocity in z -direction measured using doppler-radar.

Calculate a state-feedback using these signals such that the closed loop system obtains its poles in $s = -\frac{1}{2}$ and the control signal of the astronaut becomes the reference signal of the velocity in z -direction.

c) Suppose we by safety reasons are interested in the possibility of controlling the module even if the sensors measuring m_1 and m_2 are not working. Design a controller that can handle this and has approximately the same behavior as in a).

9.3 A DC motor with an external load, T , is described by

$$\begin{aligned}\omega &= \dot{\theta} \\ \dot{\omega} &= -\frac{1}{\tau}\omega + c_1 u + c_2 T\end{aligned}$$

where θ is the angle, ω the angular velocity, u the control signal, T the torque of the load, and c_1 , c_2 , and τ are constants.

a) Introduce a controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega$$

such that the poles of the closed loop system becomes $\frac{1}{\tau}(-1 \pm i)$ and $\theta = \theta_{\text{ref}}$ in steady-state if $T = 0$ and θ_{ref} is constant.

b) Introduce a modified controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega + u'$$

such that $\theta = \theta_{\text{ref}}$ in steady-state even for constant non-zero T and constant θ_{ref} .

9.4 A system can be described in state space form as

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (1 \quad -1) x\end{aligned}$$

We want to place the poles in $\{-2, -3\}$. Suggest an observer, and use a linear state feedback controller. Which are the poles of the closed loop system?

9.5 Is it possible to design an observer with poles in $\{-5, -6, -7, -8\}$ for the system below? Motivate your answer.

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 10 \\ -3 \\ 2 \end{pmatrix} u \\ y &= (1 \quad 0 \quad 0 \quad 0) x\end{aligned}$$

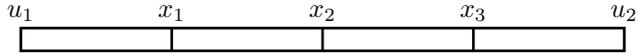


Figure 9.6a

9.6 We want to control the temperature in a long copper rod by heating or cooling its endpoints. Principally, this problem is described by a partial differential equation. To simplify the problem we assume that the temperature profile in the rod can be approximated by the temperatures x_1 , x_2 , and x_3 at three points. The temperatures in the end points are the inputs, u_1 and u_2 . All temperatures are relative to the temperature of the surroundings.

We get the following ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= \alpha(u_1 - x_1) + \alpha(x_2 - x_1) \\ \dot{x}_2 &= \alpha(x_1 - x_2) + \alpha(x_3 - x_2) \\ \dot{x}_3 &= \alpha(x_2 - x_3) + \alpha(u_2 - x_3)\end{aligned}$$

where α is a constant that depends on the coefficient of thermal conductivity and the specific heat of the rod. For simplicity, let $\alpha = 1$. Consider the problem of controlling the temperature in x_1 , x_2 , and x_3 with u_1 only, assuming $u_2 = 0$.

- Assume that we want to have an arbitrary temperature profile, that is, arbitrary values of x_1 , x_2 , and x_3 . Is this possible? Why/why not?
- Assume that all the temperatures x_1 , x_2 and x_3 are measurable. Find a state feedback that brings any initial state to zero as e^{-3t} .
- Assume that only one of the temperatures x_1 , x_2 , or x_3 is measurable, and that we still want a controller which damps a disturbance as e^{-3t} by using an observer. The sensor can be placed so that any of the three values x_1 , x_2 , or x_3 is measured. Which choices of measure point make it possible to control the system as desired? Give a motivation. Choose one of the points making the wanted design possible and design a controller, that is, an observer and a state feedback, giving the desired error damping.

9.7 Consider the lunar excursion module in Problem 9.2. Suppose that there are no rate-gyro measurements available but that the sensors measuring m_2 and m_3 are still working.

Show how m_1 can be reconstructed from u and measurements of m_2 such that the reconstruction error decreases arbitrarily fast without differentiation of any of the measured signals.

Propose a filter and describe the resulting controller when the feedback consists of both measured and reconstructed states as in Problem 9.2.

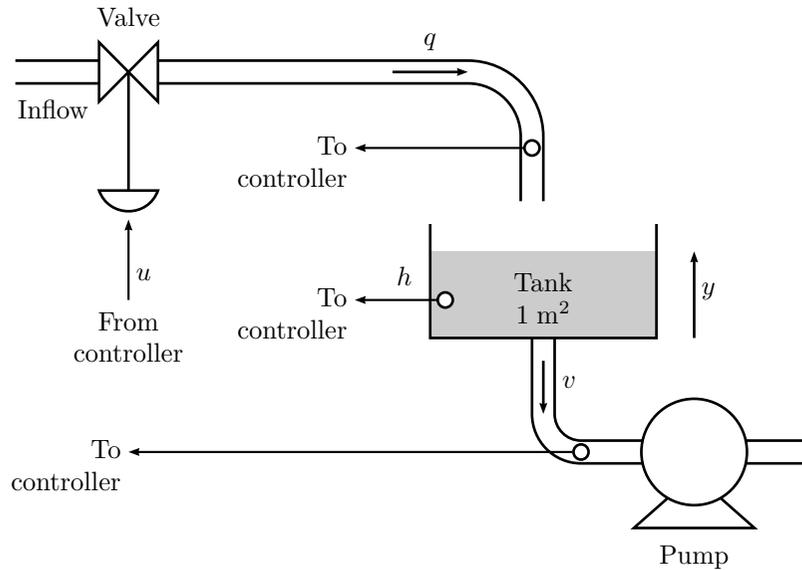


Figure 9.8a

9.8 In Figure 9.8a a level control system for a tank is shown. The objective is to keep the level at a desired value. Let u , h , q , and v denote small variations around the desired working point. The inflow, q , to the tank is determined by the valve, u , calculated by the controller. The outflow v is determined by the pump and deviations from zero is considered as process noise. The valve has some dynamics, which is modeled with the transfer function

$$Q(s) = \frac{k_1}{1 + Ts} U(s)$$

where $k_1 = 1$ and $T = 0.5$. The level is given by

$$A\dot{h} = q - v$$

where the tank cross-section area is $A = 1\text{m}^2$.

- Let q and h be state variables and give a corresponding state space model of the process. Compute a state feedback $u = -l_1q - l_2h + r$, such that the closed loop system poles both are at -2 .
- How large is the steady-state level error for a constant disturbance $v = 0.1$ if $r = 0$?

- Consider the closed loop system in a) and compute a feedforward control law from v to r such that the influence from v is completely eliminated. Exclude all terms in the control law in which v is differentiated to make it implementable. How does this modified feedforward control law work? Steady-state level error?
- Suppose that k_1 differs slightly from 1, but that the same control law as in c) is used (the control law derived under the assumption $k_1 = 1$). What happens with the steady-state level error?
- Propose a modified control law such that the stationary level error is zero for constant disturbances regardless of small deviations from the nominal value of k_1 .

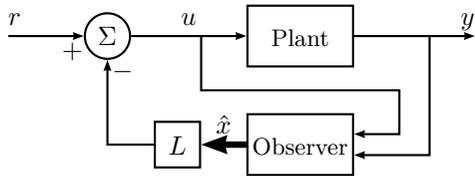


Figure 9.9a

9.9 We want to control the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

with a state feedback. (This can be interpreted as a moving vehicle in one dimension, where x_1 is the position, x_2 is the velocity, and the acceleration is the control signal.) We introduce the control law

$$u(t) = -L\hat{x}(t) + r(t)$$

where \hat{x} is constructed by an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

We choose the vectors K and L as $L = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $K^T = \begin{pmatrix} 4 & 4 \end{pmatrix}$. These choices put the eigenvalues of $A - BL$ in -1 and the eigenvalues of $A - KC$ in -2 . A block diagram of the closed loop system is shown in Figure 9.9a. Due to a time delay, the real input is given by the equation

$$u(t) = -L\hat{x}(t - T) + r$$

What is the largest possible time delay T without the closed loop system getting unstable?

9.10 One wants to construct an observer for the system

$$\dot{x}(t) = \begin{pmatrix} -1 & 1 \\ a & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t)$$

- Suppose $a = 1$. Construct an observer with the poles in $\{-5, -10\}$. For which values of a is this possible?
- Suppose that the measured signal y is given by

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t) + v(t)$$

Here $v(t)$ is the measurement noise. Compute the transfer function from v to $\tilde{x}_1(t)$, that is, the first element in the state vector for the observer error $\tilde{x}(t) = x(t) - \hat{x}(t)$.

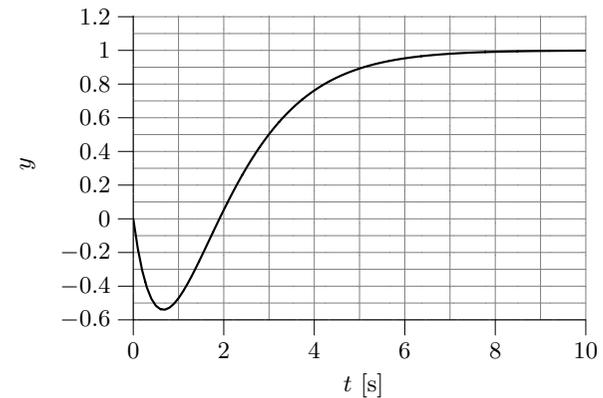


Figure 9.11a

9.11 Consider the system

$$G(s) = \frac{1 - s/\alpha}{(1 + s/\beta)^2}, \quad \alpha > 0, \beta > 0$$

Systems of this kind, that is, with a zero in the RHP have the property that the step response goes in the “wrong direction” initially, see Figure 9.11a.

- Show that the derivative of the step response at $t = 0$, that is, $\dot{y}(0)$, decreases as the zero of the system approaches the origin.
- Is it possible to use state feedback to eliminate the problem that the step response goes in the wrong direction initially? Justify your answer.

9.12 Theoretically one can place the poles of a controllable system arbitrarily. Which practical difficulties limit the performance that one can actually achieve?

9.13 Consider the system

$$\dot{x} = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \quad 1) x$$

- Determine L of a state feedback $u = r - Lx$, that places the poles at $-2 \pm i$.
- The state feedback of a) is used. It is observed that the output $y(t) = 0$ for all t is obtained for a reference signal of the form $r(t) = e^{\alpha t}$. For what value(s) of α does this occur?

9.14 Consider the model of a DC-motor

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{1}{s(s+1)}$$

- Generate a state space representation using MATLAB. Which physical signals are represented by the states?
- Suppose that the system is going to be controlled using state feedback

$$u(t) = -Lx(t) + l_0 r(t)$$

Compute the gain vector L and simulate the closed loop system for the following two choices of closed loop poles:

- ◊ Real poles at $\{-2.2, -2.1\}$
- ◊ Poles at $-1 \pm i$

Also compute l_0 such that the closed loop system gets static gain one. In particular look at the properties of the step response and the magnitude of the control signal in the two cases. Which pole locations give the best trade off between response speed and control signal magnitude?

- Now let L be computed using linear quadratic optimization (LQ) in order to minimize the cost function

$$\int_0^{\infty} x(t)^T Q x(t) + u(t)^2 dt$$

for the three choices of weight matrices given below. Compute the closed loop poles and the step responses of the closed loop system for the three cases. Describe how the properties of the step responses in the different cases.

$$(i) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(ii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$$

$$(iii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

- Start from case (ii) and increase the weight on the control signal gradually until the cost function becomes

$$\int_0^{\infty} x(t)^T Q x(t) + 10 u(t)^2 dt$$

Compare the result with the result obtained for case (i).

- Start from case (i) and introduce a weight on the velocity $\dot{y}(t)$. Increase the weight gradually and study how the poles and the step response of the closed loop system change.

9.15 The ingestion and metabolism of a drug in a human body can be described by the following equations:

$$\begin{aligned}\frac{dq(t)}{dt} &= -k_1q(t) + u(t) \\ \frac{dy(t)}{dt} &= k_1q(t) - k_2y(t)\end{aligned}$$

where the input signal $u(t)$ is the ingestion rate of the drug, the output $y(t)$ is the mass of the drug in the blood, and $q(t)$ is the mass of the drug in the gastrointestinal tract. The constants k_1 and k_2 are metabolism rates, satisfying $k_1 > k_2 > 0$. k_2 characterizes the excretory process of the individual. In this example, $k_1 = 0.05$ and $k_2 = 0.02$.

- Is the system controllable?
- Design a state feedback that places the closed loop poles in -0.1 .

$q(t)$ (the mass of the drug in the gastrointestinal tract) cannot be measured, so to be able to use the state feedback in b) we need an observer.

- How should the poles of the observer be selected?
- Design an observer with poles in -0.2 .

9.16 A system is described by the state space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t - \tau) \\ y(t) &= \kappa Cx(t) + e(t)\end{aligned}\tag{9.1}$$

where it is known that

$$\begin{aligned}\tau &< 0.3 \\ 0.9 &\leq \kappa \leq 1.1\end{aligned}$$

It is also known that e is a sinusoidal disturbance with angular frequency 10 rad/s. The control design is based on the simplified model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{9.2}$$

The specifications for the control system are given by:

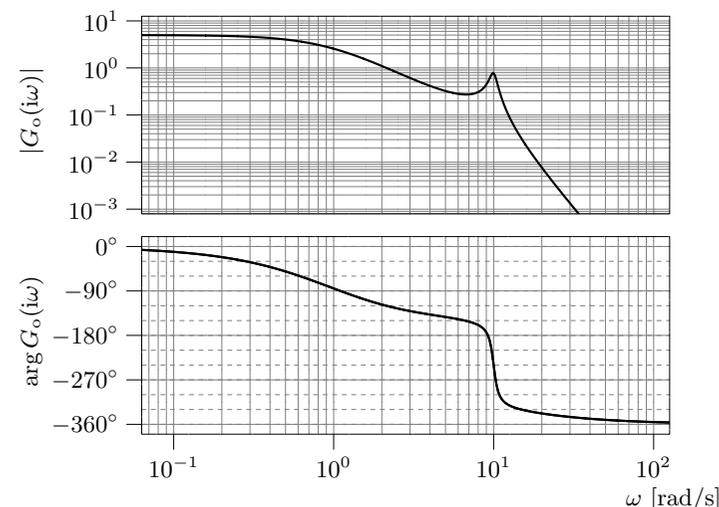


Figure 9.16a. Bode plot of the loop gain.

- The bandwidth of the closed loop system must fulfill $\omega_B \geq 5$ rad/s.
- The closed loop system shall be stable despite the disturbance and the uncertainties in the parameters τ and κ .
- The static gain of the closed loop system shall be 1 despite the disturbance and the uncertainties in the parameters τ and κ .
- The closed loop system shall handle the measurement disturbance sufficiently well.

The regulator design is carried out using state space methods. The poles of the closed loop system, that is, eigenvalues of $A - BL$, are placed in $\{-4, -2 \pm 2i\}$ and the poles of the observer, that is, the eigenvalues of $A - KC$, are placed in $\{-17, -0.2 \pm 10i\}$.

The figures below show different aspects of the control system. It is important to note that the diagrams are obtained using the model (9.2) and the designed regulator. Determine if the requirements 1–4 are fulfilled when controlling the system given by (9.1).

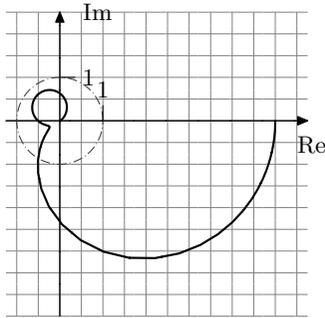


Figure 9.16b. Nyquist curve of the loop gain.

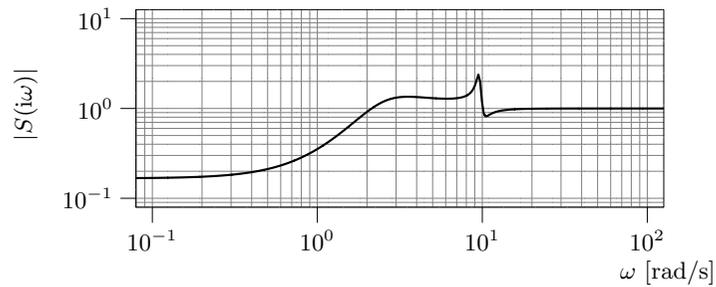


Figure 9.16c. Gain curve of the sensitivity function.

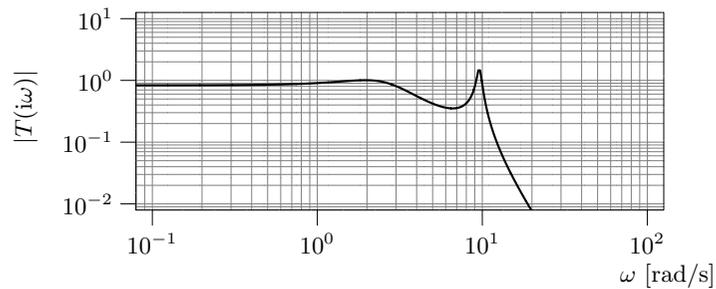


Figure 9.16d. Gain curve of the complementary sensitivity function.

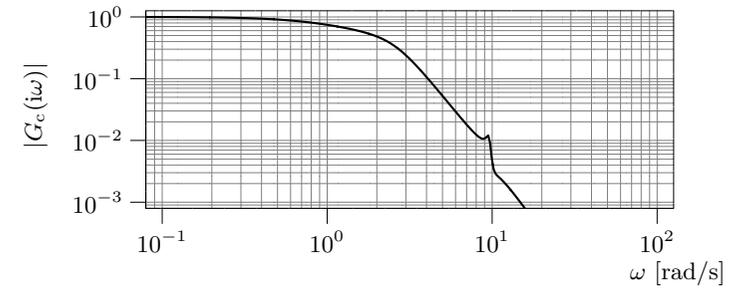


Figure 9.16e. Gain curve of the transfer function of the closed loop system.

9.17 In purification processes sometimes bacteria are used to consume the unwanted substance (possibly converting it to something more useful). Let ξ denote the amount of bacteria, η the amount of substance to be removed, and q the input flow (that contains the substance to be removed). The system is then described by a set of nonlinear differential equations

$$\begin{aligned}\dot{\xi} &= f_1(\xi, \eta, q) \\ \dot{\eta} &= f_2(\xi, \eta, q)\end{aligned}$$

When considering small deviations from an equilibrium the equations can be approximated by a linear system. In this example, the numeric values of the linearization are given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$

where x_1 , x_2 , and u denote the deviations from the equilibrium values of ξ , η , and q respectively.

- Assume that both x_1 and x_2 can be measured. Determine a state feedback placing the closed loop poles in $\{-2, -4\}$.
- Assume that only x_2 is measured. Is it possible to calculate the amount of bacteria x_1 from this measurement if u is known? If the answer is yes: Why is it important that u is known?
- Assume that the value of u is unknown, but let it be known that it is constant. Is it then possible to calculate the amount of bacteria, x_1 from a measurement of x_2 ? In case it is, show some way of doing the computation.

9.18 A certain species of bacteria grows by consuming glucose, whose inflow is controlled. The following model is used

$$\dot{m} = (f - 1)m \quad \dot{f} = -m + q$$

where m is the amount of bacteria, f the amount of glucose and q the inflow of glucose. One wants the system to operate in the neighborhood of the operating point $m = 1, f = 1, q = 1$. Using the notation $x_1 = m - 1, x_2 = f - 1, u = q - 1$, an approximate model ($x_1 x_2$ neglected) is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

Sometimes there are disturbances that are modeled as a constant, unknown external signal w :

$$\begin{aligned}\dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

One wants to drive x_1 to a reference value r using u as control variable.

- a) x_1 is measured. What performance (stationary error, speed of response) can be obtained using a P controller?
- b) x_2 is measured. What performance (stationary error, speed of response) can be obtained by using a control law where u depends linearly on r and x_2 if $w = 0$? What happens when $w \neq 0$?
- c) x_2 is measured. Determine a controller that does not differentiate x_2 , gives an asymptotically stable closed loop system and makes x_1 converge to r asymptotically for an arbitrary constant but unknown w .

11 Implementation

11.1 If you “translate” the compensator

$$U(s) = KN\left(\frac{s+b}{s+bN}\right)E(s)$$

with Tustin’s formula you get a controller of the form

$$u(t) = \beta_1 u(t-T) + \alpha_1 e(t) + \alpha_2 e(t-T)$$

What are the values of α_1 , α_2 , and β_2 , if $T = 0.1$, $N = 10$, $b = 0.1$, and $K = 2$?

11.2 Consider the system

$$\dot{y}(t) = u(t)$$

Suppose it is controlled with a computer, so that the control signal is constant over the sampling interval, that is,

$$u(t) = u_k, \quad kT \leq t < (k+1)T$$

- Introduce the notation $y_k = y(kT)$ and derive a relation between y_{k+1} , y_k , and u_k .
- Suppose we use the proportional feedback

$$u_k = -Ky_k$$

and that $y(0) = y_0$. What are the values of K , for which the closed loop system is stable?

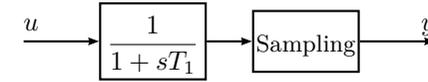


Figure 11.3a

11.3 Consider the system in Figure 11.3a, which illustrates sampling with prefiltering. Suppose we are sampling with the sampling period T and that $u = u_0 + u_1$, where u_0 is an “interesting” low frequency signal in the frequency interval $0 < \omega < \pi/T$ and that u_1 is a sinusoidal control signal

$$u_1(t) = \sin \omega_2 t, \quad \frac{\pi}{T} < \omega_2 < \frac{2\pi}{T}$$

Since the sampling causes aliasing, the output will be

$$y(t) = y_0 + y_1$$

where y_0 is interesting and y_1 is a disturbance signal

$$y_1(kT) = A \sin(\omega_1 kT + \varphi), \quad \omega_1 < \pi/T$$

- What are A , ω_1 and φ ?
- It is clear from a) that the choice of T affects the amplitude of the disturbance signal y_1 . What is the smallest amplitude you can get if you do not want to damp any frequencies in u_0 more than $\sqrt{2}$ times?

Hints

This version: January 2011

2 Dynamic Systems

2.1 Start with $J\ddot{\theta} = -f\dot{\theta} + M$ and try to write M as a function of θ and u using Kirchhoff's voltage law.

2.2 What is the relationship between the response of the system and the pole locations?

2.3 Separate the pure delay and the dynamic response. Use the final value theorem to find the steady state gain and calculate the time constant by estimating the time to reach 63% of the final value (neglecting the time delay).

2.4 Identify the coefficients ω_0 and ζ in the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}$$

2.9 See Glad&Ljung.

2.11 a) Consider what you can control, what is uncontrollable and what is desired.

b) Consider the relationship between the signals.

2.12 a) Use mass balance and assume that the densities are equal.

b) Consider the change in mass and change in component A.

c) Assume that all the other independent variables (q_1 , q_2 , $c_{A,2}$) are constant.

2.13 a) Use mass and component balance.

3 Feedback Systems

- 3.1 b) Consider the final value and the time constant.
 c) The outflow caused by the pump enters in the block diagram as an additive disturbance on the input of the tank.

3.4 Start with deriving an expression for the transfer function from the disturbance f_c to the error e .

- a) Use $F(s) = K$ and the final value theorem.
 b) Use $F(s) = K_1 + K_2/s$ and the final value theorem.

3.6 a) The characteristic equation is

$$s(s + 1)(s + 3) + K(s + 2) = 0$$

which gives $P(s) = s(s + 1)(s + 2)$ and $Q(s) = s + 2$.

b) Characteristic equation:

$$s(s^2 + 2s + 2) + K = 0$$

$$P(s) = s(s^2 + 2s + 2), Q(s) = 1.$$

c) Characteristic equation:

$$s(s - 1)(s + 6) + K(s + 1) = 0$$

$$P(s) = s(s - 1)(s + 6), Q(s) = s + 1.$$

3.7 Derive the general closed loop transfer function by first deriving the transfer function for the inner loop.

a) Let $\alpha = 0$. The characteristic equation is then

$$s(s + 2) + 4K = 0$$

Compute the poles explicitly as a function of K .

b) The characteristic equation is

$$s(s + 2) + 4K(1 + s) = 0$$

c) Characteristic equation:

$$s(s + 2) + 4K(1 + s/3) = 0$$

d) Characteristic equation:

$$s^2 + 2s + 4 + 4\alpha s = 0$$

3.8 a) Derive the transfer function from ω_{ref} to ω .

b) The characteristic equation is

$$(s + 10)(s + 4)(s - 3) + 10K(s + 1) = 0$$

3.9 a) Derive the closed loop transfer function by first deriving the transfer function for the inner loop. The characteristic function is

$$s((s + 1)(s + 10) + K_1) + K_2 = 0$$

We get two principally different root loci when there are complex starting points, and when all starting points are equal. Treat the cases separately.

3.10 a) The characteristic equation is

$$(s + 1)(s - 1)(s + 5) + K = 0$$

b) Characteristic equation:

$$(s + 1)(s - 1)(s + 5) + K(1 + 0.5s) = 0$$

3.11 a) Characteristic equation:

$$s^3 + 2s^2 + a(s^2 + 2s + 6) = 0$$

- b) First check for which a the system is stable and the steady state requirement is fulfilled. Then use that “sinusoid in” gives “sinusoid out” after transients.
- 3.12 Check the root locus to find which K -values gives a stable/unstable system, more/less oscillative system.
- 3.13 Investigate the starting points and end points of the root locus.
- 3.14 Find the open loop transfer function and use the Nyquist criterion.
- 3.15 Since $G(s)$ has no poles in the RHP, the closed loop system is stable if the Nyquist path of KG_o does not encircle -1 . Note that the K will only modify the distance to the origin, not the shape of the curve.
- 3.16 Study the amplitude and phase of $G(i\omega)$.
- 3.17 a) Draw the complete Nyquist path and use the Nyquist criterion. (Note that $G_o(-i\omega)$ is the mirror image of $G_o(i\omega)$, mirrored in the real axis.)
 b) Use the final value theorem, and that $G_o(0)$ is known from the Nyquist path.
 c) Apply the Nyquist criterion to $\frac{K}{s}G(s)$
- 3.18 The system oscillates when the open-loop gain is equal to -1 (check $Ke^{-i\omega T}G(i\omega)$).
- 3.19 Try to find the ω that gives $\arg F(i\omega)G(i\omega) = -180^\circ$.
- 3.20 The Nyquist curve for small ω determines if the system may have an integrator or not. Also check if the system is unstable for some K (from the Nyquist diagram).
- 3.25 Check the steady state error, the relative damping, etc.

- 3.26 a) To compute the closed loop transfer function combine

$$\theta(s) = G(s)U(s)$$

and

$$U(s) = F(s)(\theta_{\text{ref}}(s) - \theta(s))$$

- b) The control error can be computed using

$$E(s) = \frac{1}{1 + F(s)G(s)}\theta_{\text{ref}}(s)$$

To find the steady-state error, use the final value theorem.

- c) See b).

- 3.31 a) Consider what you can control, what is uncontrollable and what is desired.
 b) Use mass and energy balance for both the tank and the heating system.
- 3.32 a) Compute the poles of the system.
 b) Note that the system has negative sign in the numerator.
- 3.33 Check the pole for the closed loop system.
- 3.34 Use Newton's force equation $F = ma$ to derive the transfer function for the astronaut.

4 Frequency Description

- 4.1 Determine the angular frequency ω of the signals using the figure. Use the relationship saying that when $u(t) = A \sin \omega t$ the output becomes

$$y(t) = |G(i\omega)| A \sin(\omega t + \arg G(i\omega))$$

to determine $|G(i\omega)|$ and $\arg G(i\omega)$.

- 4.2 a) For $K = 0.5$ the open loop system is given by

$$G_o(s) = F(s)G_r(s)G_s(s) = \frac{0.05(1 + s/0.02)}{s(1 + s/0.01)(1 + s/0.05)(1 + s/0.1)}$$

Use the rules in Glad&Ljung to make the Bode plot.

- b) What can be said about the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?
c) When the reference signal is $A \sin \alpha t$ the output signal becomes

$$y(t) = |G_c(i\omega)| A \sin(\alpha t + \arg G_c(i\omega))$$

The Bode plot of the open loop system can be used to compute $G_c(i\omega)$.

- 4.3 a) Check the behavior of $G(i\omega)$ when $\omega \rightarrow 0$ and $\omega \rightarrow \infty$ respectively. See also if the absolute value and the argument decrease monotoneously or not.
b) Translate the behavior of the amplitude and phase curves to a pole-zero diagram.
- 4.4 Check the final values of $y(t)$ against the static gain $G(0)$. Check also the overshoots of $y(t)$ against the height of the resonance peaks in $G(i\omega)$. Check the frequency of the oscillation in $y(t)$ against the resonance frequency in $G(i\omega)$.

- 4.5 Use MATLAB, in particular the command `bode`.

- 4.6 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

- 4.7 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

- 4.11 a) Use the rules in Glad&Ljung to make the Bode plot.

- b) What can be said of the phase and gain margin when the output of the closed loop system oscillates with constant amplitude?

- 4.12 a) What is the stability criterion in the Bode plot?

- b) What is the current phase margin? Is a lead really necessary?

5 Compensation

- 5.1 Try a lead-lag compensator. A table of phase advance versus “the N -parameter” is found in Glad&Ljung.
- 5.2 a) Glad&Ljung gives a good description of how Bode plots can be drawn by hand.
b) A proportional controller does not affect the phase curve.
c) Try lead compensator.
- 5.3 a) Draw asymptotic Bode plot (see 5.1) by hand or use MATLAB.
b) Start with calculating the controller and then use the final value theorem.
c) Try a lag compensator.
- 5.4 Start with drawing a Bode plot for the open loop transfer function. The final value theorem is a good tool in this exercise.
- 5.5 Check for signs of dominating poles, pure integrations, resonance frequencies...
- 5.6 Draw asymptotic Bode plot using the guidelines in Glad&Ljung. See the discussion on lead-lag compensators in Glad&Ljung.
- 5.7 Use values of $|G(i\omega)|$ and $\arg G(i\omega)$ to plot the Nyquist curve $G(s)$.
- 5.8 The time delay alters the phase curve but not the amplitude curve. Use the Nyquist stability criterion.
- 5.9 Check steady state level and rise time. Modify Figure 5.9b using $G_A(s)$ and adapt a lead-lag compensator. You can use two lead compensators to achieve a big phase advance.
- 5.10 Start by adjusting Figure 5.10b to obtain the Bode plot of G .
- 5.11 b) It is possible to derive limits on K using either the Bode plot or the Nyquist curve.
c) Use the final value theorem.
d) A time delay is described by the transfer function e^{-sT} .
- 5.12 Think of all possible phase curves, for example originating from time delays, and think about the corresponding Nyquist curves or Bode plots.
- 5.13 See “Introduktion till CSTB” and previous exercises in this section.
- 5.14 Try a lead-lag compensator.

6 Sensitivity and Robustness

6.1 The sensitivity function is the transfer function from v to y .

6.2 Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Make a simple plot of $G_c(i\omega)$ using the information in the problem formulation. Compare with the inverse of the relative model error.

6.3 Convert the condition that the amplitude of y is larger than the amplitude of v to the condition

$$|1 + G_o(i\omega)| < 1$$

What does this inequality say about the distance between the Nyquist curve and the origin?

6.4 Compute the transfer function of the closed loop system. Apply the robustness criterion using the given upper bound of the relative model error.

6.5 a) Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

and plot $1/|G_{\Delta}(i\omega)|$.

b) Determine the level that $|G_c(i\omega)|$ cannot exceed.

6.6 a) Use the robustness criterion and check the condition for $\tilde{G}(i\omega)$ when $\omega \rightarrow \infty$.

b) The characteristic equation of the closed loop system is

$$s^2(2 + 25\alpha) + 5s(2 + 25\alpha) + 25 = 0$$

6.7 a) The characteristic equation becomes

$$s^2(s + 1) + \alpha(s^2 + s + 4) = 0$$

b) Derive the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

Check where the absolute value of the inverse of the relative model error intersects $|G_c(i\omega)|$ given in the figure. It is sufficient to check the low frequency asymptote.

c) What can be said about the necessity and sufficiency of the stability conditions in a) and b)?

6.8 Check where the absolute value of the relative model error intersects $|G_c(i\omega)|$ given in the figure.

6.9 Derive the closed loop equation relating $y(t)$, $r(t)$, $v(t)$, and $n(t)$ using $Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s))$. Then use the fact that the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ are related as $S(s) + T(s) = 1$. (Here $T(s)$ coincides with the closed loop system.)

6.10 a) The relative model error is given by

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)}$$

b) Use MATLAB and results from previous exercises.

6.11 Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

6.12 Create the loop gain transfer function and use its Bode plot to check stability.

6.13 The sensitivity function is the transfer function from v to y .

7 Special Controller Structures

- 7.1 a) Derive the transfer function from w to θ_m which then implies the open loop transfer function

$$\Theta(s) = \frac{0.9}{(1 + s/0.033)(1 + s/0.33)(1 + s)}W(s)$$

Draw the Bode plot using the rules from Glad&Ljung.

- b) Draw the Bode plot using the rules from Glad&Ljung.

- 7.2 a) Use the relationship

$$H(s) = \frac{1}{As} \left(\frac{1}{1 + s/2} U(s) - V(s) \right)$$

- b) Derive the transfer function from V to H when both feedforward and feedback are used.

- 7.3 a) Use $Y(s) = (G_u(s)F_f(s) + G_v(s))V(s)$.

- b) Recall that for stable, linear systems “a sinusoid in gives a sinusoid out” after initial transients.

8 State Space Description

8.1 Define $x_1 = \theta$, $x_2 = \dot{\theta}$, and utilize the differential equation for the motor.

8.2 For the nonlinear equation $\dot{x}_2 = f_2(x_1, x_2, u)$, the linearized equation is given by

$$\begin{aligned} \dot{x}_2 &= f_2(x_{1,0}, x_{2,0}, u_0) \\ &+ \frac{\partial f_2}{\partial x_1}(x_{1,0}, x_{2,0}, u_0) \cdot (x_1 - x_{1,0}) \\ &+ \frac{\partial f_2}{\partial x_2}(x_{1,0}, x_{2,0}, u_0) \cdot (x_2 - x_{2,0}) \\ &+ \frac{\partial f_2}{\partial u}(x_{1,0}, x_{2,0}, u_0) \cdot (u - u_0) \end{aligned}$$

8.3 Define $x_1 = y$, $x_2 = \theta$, and $x_3 = z$. Use block diagram algebra to find expressions for $s \cdot X_i(s)$, then use the inverse Laplace transform.

8.4 Use canonical forms.

8.5 Take the Laplace transform of $g(t)$.

8.6 $G(s) = C(sI - A)^{-1}B$.

8.7 $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$

8.8 a) Insert the control signals and take Laplace transforms. Use the final value theorem.

b) Examine the difference $u_1(t) - u_2(t)$ for arbitrarily small constant ϵ .

8.9 Check controllability.

8.10 The controllable subspace is spanned by the linearly independent columns of \mathcal{S} . The unobservable subspace is spanned by the null space of \mathcal{O} .

8.11 b) Compare what happens to the states as $t \rightarrow \infty$, to the transfer function poles.

c) Check if $\det \mathcal{S}$ and $\det \mathcal{O}$ are nonzero.

8.12 The system is minimal, compute the transfer function.

8.13 a) For small deviations around 0, $\sin(\phi) \approx \phi$, $\cos(\phi) \approx 1$. Take \ddot{z} as input.

b) $\det \mathcal{S} = \frac{1}{\alpha}(1 - \frac{1}{\alpha})^2$

8.15 a) Combine mass balance with the given equation for reaction speed.

8.17 a) Use $\dot{x} = 0$ and $u \equiv 1$.

b) $G(s) = C(sI - A)^{-1}B$

9 State Feedback

- 9.1 a) The closed loop system $\dot{x} = Ax + Bu$, $y = Cx$, $u = -Lx + y_{\text{ref}}$ has characteristic polynomial $\det(sI - A + BL) = 0$.
- b) The observer poles are given by $\det(sI - A + KC) = 0$ and should be placed to the left of the closed loop poles.
- 9.2 a) Write the system in state space form by introducing three state variables corresponding to the outputs of the three left-most integrators in the figure ($\dot{z} = \text{output}$). Design a state feedback controller $u = -Lx + y_{\text{ref}}$ and place the poles in -0.5 .
- c) Design an observer with poles to the left of the closed loop poles.
- 9.3 a) The constant l_0 can be found by using that $\dot{\theta} = \dot{\omega} = 0$ at steady state.
- b) Introduce the integrated control error as an auxillary state.
- 9.4 Decompose the system into two subsystems, one controlled by u_1 and one by u_2 , and check the controllability.
- 9.5 Is the system observable?
- 9.6 Is the system observable?
- 9.7
- Is the system controllable?
 - $\tilde{x}(t)$ converges to zero as $p(t)e^{-\alpha t}$ if $(A - KC)$ has a double eigenvalue in $-\alpha$.
 - Check the observability of the system.
- 9.9 Study the phase margin of the open loop system.
- 9.11 Use the initial value theorem.
- 9.12 Compute the transfer function from $u(t)$ to $z(t) = L\hat{x}(t)$, that is, the loop gain, and check the stability margin given a certain time delay T .
- 9.13
- The closed loop poles are given by $\det(sI - A + BL) = 0$
- 9.17 a) The closed loop system $\dot{x} = Ax + Bu$, $u = -Lx$ has the characteristic polynomial $\det(sI - A + BL) = 0$.
- b) Check observability.
- c) Introduce a new state.

11 Implementation

Answers

This version: January 2011

1 Mathematics

1.1 a) A step has Laplace transform $\frac{A}{s}$.

b) A ramp has Laplace transform $\frac{A}{s^2}$.

c) $\frac{1}{s+2}$

d) $\frac{s}{s^2+25}$

e) $sU(s) - u(0)$

f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)

g) $s^2U(s) - su(0) - \dot{u}(0)$

h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)

i) A time delayed signal has Laplace transform, $e^{-sT}U(s)$.

1.2 a) $\lim_{t \rightarrow \infty} y(t) = 5/2$

b) $Y(s) = \frac{1}{s+2}U(s)$

1.3 The general solution is given by

$$y(t) = C_1 e^{-2t} + (C_2 + C_3 t) e^{-t} - \frac{3}{100} (\cos(2t) + 7 \sin(2t))$$

1.4 a) $y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$, $t \geq 0$

b) $y(t) = 1 - 0.5e^{-t} + 0.5 \sin t - 0.5 \cos t$

1.5 a) $\sqrt{2}e^{i\frac{\pi}{4}}$

b) $\frac{\sqrt{2}}{10} e^{-i\frac{105}{180}\pi}$

c) $1 + \sqrt{3}i$

d) -5

1.6

decibel (dB ₂₀)	Definition	Amplification F
20	$20 \log F = 20 \Rightarrow$	$F = 10^1 = 10$
-3	$20 \log F = -3 \Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20 \log F = 0 \Rightarrow$	$F = 10^0 = 1$
10	$20 \log F = 10 \Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20 \log F = -10 \Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

1.7 Multiplication of the two matrices gives the unit matrix.

1.8

$$\lambda_1 = 3 \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda_3 = 4 \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

1.9

$$T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

1.10 A basis for the null space is for example

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

A basis for the range space is

$$\begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

The rank of the matrix is hence 3.

1.11 a) $f(t) = 1 - e^{-t}$; 1.

b) $f(t) = -0.5e^{-t} + 0.5e^t$; ∞ .

c) $f(t) = e^{-t} \cdot t$; 0.

1.12 $y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$

2 Dynamic Systems

2.1 a) Differential equation

$$\ddot{\theta} + \frac{1}{\tau} \cdot \dot{\theta} = k_0 \cdot u$$

where

$$\frac{1}{\tau} = \frac{R_a f + k_a k_v}{J R_a} \quad k_0 = \frac{k_a}{J R_a}$$

b) Transfer function

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

c) Step response

$$\theta(t) = k_0 \tau t - k_0 \tau^2 (1 - e^{-t/\tau})$$

2.2 (1) $K = 0.1$

(2) $K = 2.5$

(3) $K = 3$

(4) $K = 0.5$

2.3 $G(s) = \frac{10e^{-(L/V)s}}{1+3s}$

2.4 a) $a < 1$

b) $b = 2$

2.5 A-B, B-F, C-A, D-C, E-E, F-D.

2.6

System	T_r	T_s	M	poles
G_A	3.3	4.7	0%	-1, -1
G_B	1.2	13.6	52%	$-0.2 \pm i0.98$
G_C	10.6	14.6	0%	-4.8, -0.2
G_D	1.7	5.4	16%	$-0.5 \pm i0.87$
G_E	0.8	2.6	16%	$-1 \pm i1.73$

2.7 $\alpha > 0$ gives overshoot, $\alpha < 0$ gives undershoot.

2.8 $y(t) = \mathcal{L}^{-1}(G(s)\frac{1}{s})$

2.9 a) 1.5

b) $M \approx 26\%$

c) $T_r \approx 1.5$

d) $T_s \approx 7.8$

2.10 G_1 -C, G_3 -B, G_4 -A, G_5 -D.

2.11 a) The signals can be classified as

- ◇ Disturbances signal: Acid process flow (unknown pH and flow)
- ◇ Control signal: NaOH solution
- ◇ Measured and controlled signal: The pH of the outflow

b) A block diagram where the control strategy is based on feedback could look like Figure 2.11a

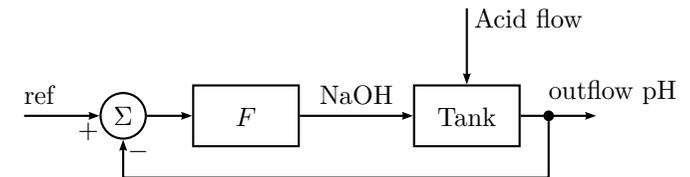


Figure 2.11a

2.12 a) $q^* = 1.5 \text{ m}^3/\text{min}$ and $c_A^* = 2.0 \text{ kmol/m}^3$.

b) The model is nonlinear since the model is described by the following non-linear equations

$$\frac{d(\rho V)}{dt} = \rho(q_{in} - q_{out})$$

$$\frac{d(Vc_A)}{dt} = q_1 c_{A,1} + q_2 c_{A,2} - qc_A$$

c) $k_0 = 2.0 \text{ kmol/m}^3$, $k_1 = 0.13 \text{ kmol/m}^3$, and $\tau = \frac{1}{1.5} = 0.67 \text{ min}$.

2.13 a) The model is given by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

$$\frac{dM_i}{dt} = L_{i-1} + V_{i+1} - L_i - V_i$$

$$M_i \frac{dx_i}{dt} = L_{i-1}(x_{i-1} - x_i) + V_{i+1}(y_{i+1} - x_i) + V_i(x_i - y_i)$$

b) The linearized model is given by

$$M_i^* \frac{dx_{i\Delta}}{dt} = L_{i-1}^* x_{i-1,\Delta} + V_{i+1}^* y_{i+1,\Delta} - L_i^* x_{i\Delta} - V_i^* y_{i\Delta}$$

$$+ x_{i-1}^* L_{i-1,\Delta} + y_{i+1}^* V_{i+1,\Delta} - x_i^* L_{i\Delta} - y_i^* V_{i\Delta}$$

$$y_{i\Delta} = \frac{\alpha}{(1 + (\alpha - 1)x_i^*)^2} x_{i\Delta}$$

3 Feedback Systems

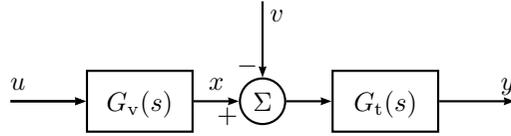


Figure 3.1a

- 3.1 a) Transfer function of the tank is $G_t(s) = \frac{1}{s}$.
 b) $k_v = 2, T = 5$
 c) A block diagram is given in Figure 3.1a.

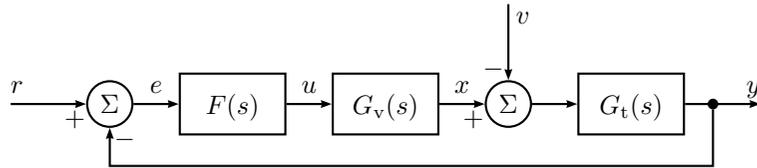


Figure 3.2a

- 3.2 b) A block diagram is given in Figure 3.2a.
 c)

$$Y(s) = \frac{2(K_D s^2 + K_P s + K_I)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} R(s) - \frac{s(1 + 5s)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I} V(s)$$

- 3.3 a) The poles are located at $-0.1 \pm \sqrt{0.01 - 0.4K_P}$, and are stable for all $K_P > 0$.
 b) Steady state level is $5 - \frac{1}{K_P}$, but the system will be oscillatory if K_P is large.
 c) As long as the system is stable, the steady state level will equal the reference.

- d) The derivative part in the feedback can be used to improve the damping in the closed loop system, that is, to make it less oscillatory.

3.4 a) $-a/K$

b) 0

- 3.5 a) For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases, that is, the closed loop system becomes unstable.

- b) The integrator in the regulator eliminates the steady state error. A too small value of K_I gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

- c) Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_D too much, however, gives that an oscillation with higher frequency appears in the step response and finally (approximately when $K_D > 65$) the closed loop system becomes unstable.

- 3.6 Root loci are shown in Figure 3.6a.

- b) Intersection with the imaginary axis for $K = 4, \omega = \pm\sqrt{2}$.

- c) Intersection with the imaginary axis for $K = 7.5, \omega = \pm\sqrt{1.5}$.

Conclusions about the step response of the corresponding systems:

- a) Asymptotically stable all $K > 0$.
 Small K : No oscillations, larger K gives faster system.
 Larger K : Oscillations. Larger K gives more oscillations.

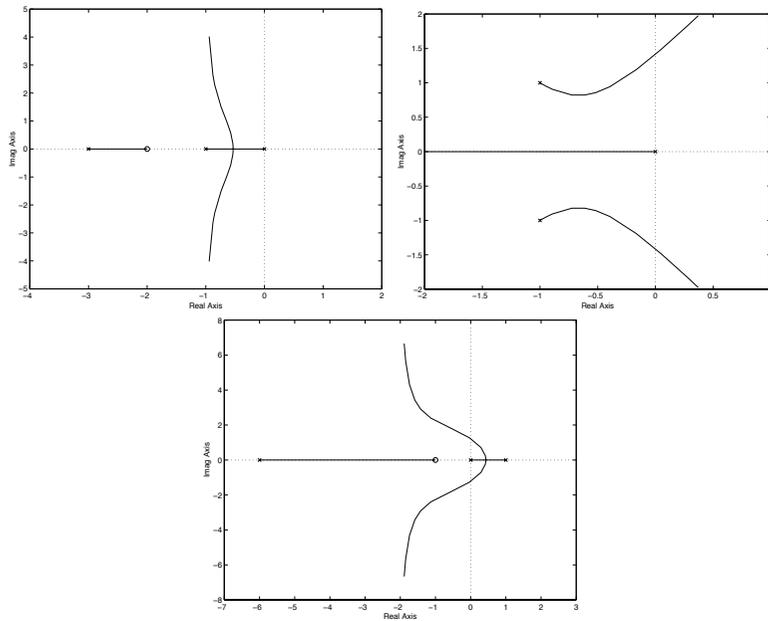


Figure 3.6a

- b) Asymptotically stable for $0 < K < 4$. Oscillating all $K > 0$.
 Small K : larger K gives faster system.
 Larger K : larger K gives more oscillating system. Unstable for large K (> 4).
- c) Asymptotically stable for $K > 7.5$. Unstable for $K < 7.5$. Stable and oscillating for $K > 7.5$. Larger K gives faster system, until the real pole becomes dominating, then larger K gives a slower system.

3.7 General characteristic equation:

$$s(s + 2) + 4K(1 + \alpha s) = 0$$

The root loci are shown in Figure 3.7a.

- a) Asymptotically stable for all $K > 0$, oscillatory for large K .

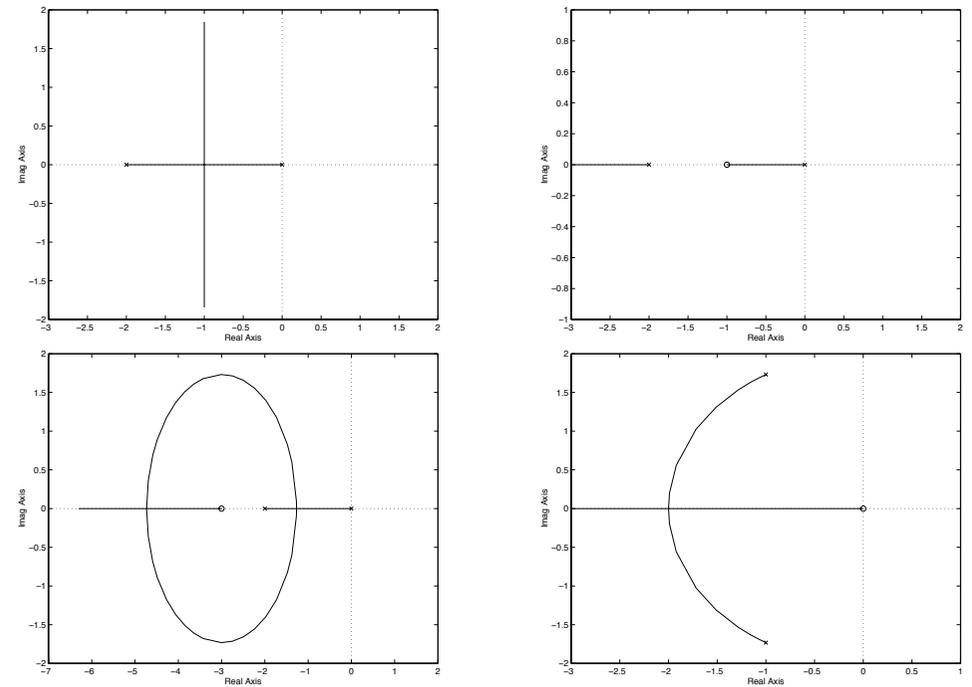


Figure 3.7a

- b) Asymptotically stable for all $K > 0$, not oscillatory for any K .
- c) Asymptotically stable for all $K > 0$, no oscillations for small and large K , faster for large K .
- d) Asymptotically stable for all $\alpha > 0$. Oscillatory for small α . Larger α gives more damped system.

With the tachometer feedback we can make the system both fast and well damped. The tachometer feedback is equivalent to the D-part in a PID controller.

- 3.8 a) The system is unstable so ω will grow to infinity.

b) The root locus is shown in Figure 3.8a. The system is asymptotically stable for $K > 12$.

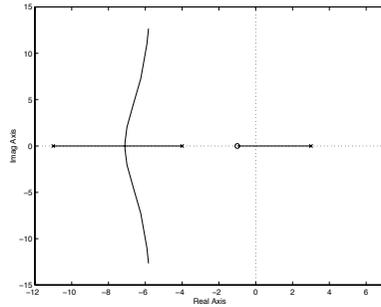


Figure 3.8a

c) No. When $K = 12$, $s = 0$ is one pole but the other two are complex.

3.9 a) Starting points: $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$. The starting points are all real for $K_1 \leq 20.25$, while we have complex starting points for $K_1 > 20.25$. The two principal root loci are shown in Figure 3.9a. The system is asymptotically stable for $0 < K_2 < 11K_1 + 110$.

b) A larger K_1 gives stability for larger K_2 .

3.10 Root loci in Figure 3.10a.

a) The system is unstable for all K .

b) Asymptotically stable for $K > 5$.

3.11 a) The root locus is shown in Figure 3.11a. The system is asymptotically stable for $a > 1$.

b) The smallest amplitude is 0.1.

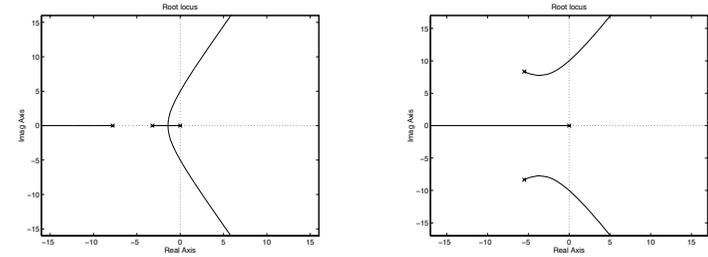


Figure 3.9a

3.12	K	Step
	4	C
	10	D
	18	B
	50	A

3.13 The poles of the system all tends to points in the LHP or to $-\infty$ for large K .

3.14 The system is stable for $K/A < \pi$.

3.15 a) The closed loop system is stable in (i), (ii), and (iv).

b) Stable when: (i) $K < 2.5$, (ii) $K > 0$, (iii) $K < 1/2$, and (iv) $K < 1/4$ or $K > 1/2$.

3.16 The Nyquist curves are shown in Figure 3.16a.

3.17 a) $K < 2/3$

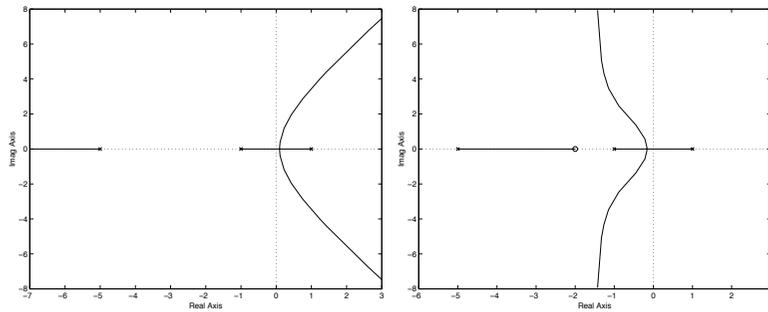


Figure 3.10a

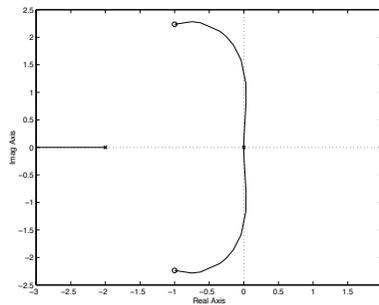


Figure 3.11a

b) $\frac{1}{1+2K}$ when $K < 2/3$

c) $K < 2/3$

3.18 $\tau = 1.69$

$T = \frac{\pi}{2} - \arctan \tau = 0.53$

$T_1 = \pi - 2 \arctan \frac{\tau}{2} = 1.74$

3.19 $K < 2$

3.20 Root locus 1.

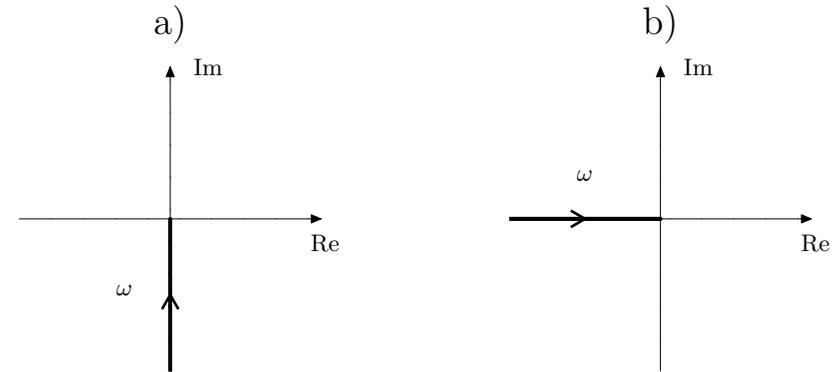


Figure 3.16a

3.21 P $\Rightarrow b_0 = b_2 = 0$
 I $\Rightarrow b_0 = b_1 = 0$
 D $\Rightarrow b_1 = b_2 = 0$

3.22 a) The root locus with respect to K_P is shown in Figure 3.22a. When K_P increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_P \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem 3.5. For small values of K_P the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the steady state error changes with the parameter.

b) The root locus with respect to K_I is shown in Figure 3.22b. For small K_I the response of the closed loop system is dominated by the poles on the real axis close to the origin. When K_I increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_I \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem 3.5 a small value of K_I , that is, a pole close to

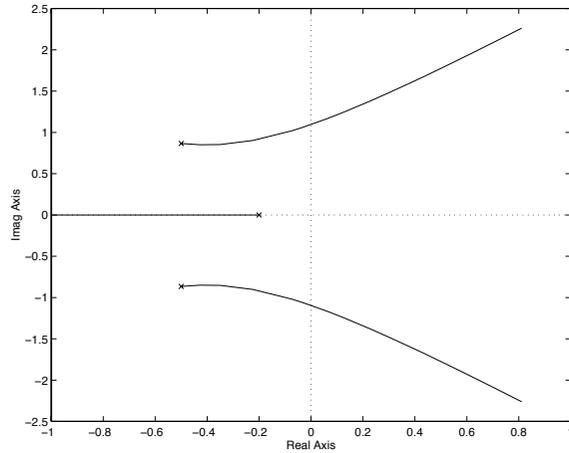


Figure 3.22a

the origin, gives a slow step response. When K_I increases the dominating poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or has poor damping. Here, the large settling time for small K_I is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

- c) The root locus with respect to K_D is shown in Figure 3.22c. When K_D increases the complex poles closest to the origin move towards the origin and at the same time the damping of the poles is increased. When K_D increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

- 3.23 a) The Nyquist curve is “far away” from the point -1 for all frequencies and the step response of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_P = 6.2$ the Nyquist curve reaches -1 and thus is the limit of stability.
- b) For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(i\omega)|$ large for low frequencies. The

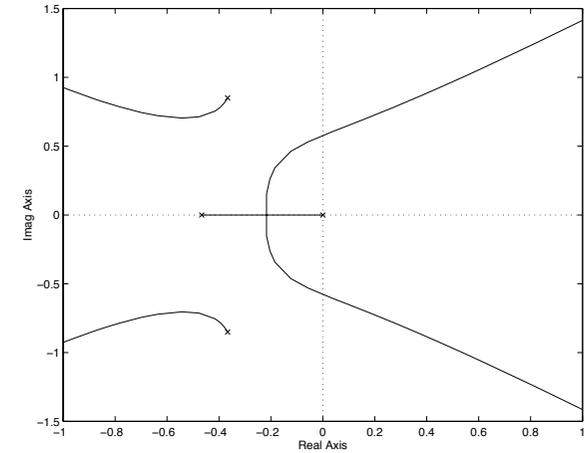


Figure 3.22b

Nyquist curve now passes closer to -1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_I = 1.44$.

- c) The Nyquist curve is now further away from -1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_D = 66$.

3.24 a) $\omega_c = 0.38$, $\omega_p = 1.1$, $\varphi_m = 94^\circ$ and $A_m = 3.1$.

- b) The closed loop system is now much more oscillatory due to the reduced phase and gain margins.

c) $K_P = 3.1$.

3.25 A-iii, B-i, C-iv, D-ii.

- 3.26 a) K_P small \Rightarrow Both poles on the real axis, but one pole very close to the origin \Rightarrow Slow but not oscillatory system.

$K_P = 1/(4\tau^2 k_0) \Rightarrow$ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.

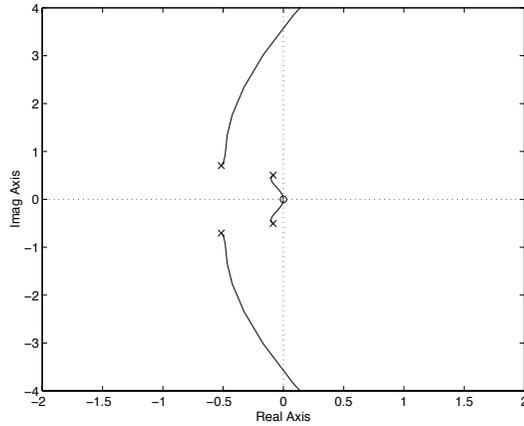


Figure 3.22c

K_P large \Rightarrow Complex poles with large imaginary part relative to the real part, that is, oscillatory system.

b) If the reference is a step,

$$\lim_{t \rightarrow \infty} e(t) = 0$$

If the reference is a ramp,

$$\lim_{t \rightarrow \infty} e(t) = \frac{A}{K_P k_0 \tau}$$

c) $\lim_{t \rightarrow \infty} e(t) = 0$

3.27 $G_c = \frac{G_o}{1+G_o}$

3.28 a) $G_o = FG$

b) $G_c = \frac{FG}{1+FG}$

c) $G_{ny} = -\frac{FG}{1+FG}$

d) $G_{re} = \frac{1}{1+FG}$

3.29 a) $\frac{3A}{3+K}$

b) $F(s) = \frac{1}{s}$ (for example)

c) Poles in $-2, -2$. No zeros.

3.30 A-4, B-2, C-3, D-1.

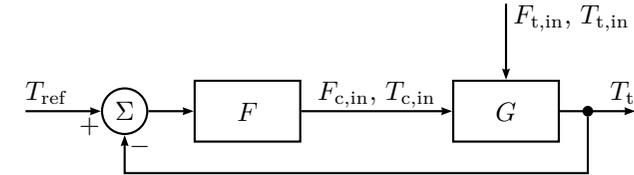


Figure 3.31a

3.31 a) See the block diagram in Figure 3.31a. There, the signals are classified as:

- ◇ Input $F_{c,in}$ and $T_{c,in}$
- ◇ Output T_t
- ◇ Disturbance $F_{t,in}$ and $T_{t,in}$

b) The model is given by

$$V_t \frac{dT_t}{dt} = F_t(T_{t,in} - T_t) + \frac{U}{c_{t,in}^p \rho_t} (T_c - T_t)$$

$$V_c \frac{dT_c}{dt} = F_c(T_{c,in} - T_c) - \frac{U}{c_c^p \rho_c} (T_c - T_t)$$

c) $T_{t\Delta}(s) = \frac{32}{(s+3.675)(s+0.185)} F_{c\Delta}(s)$

d,e) The root locus for a P controller is shown in Figure 3.31b.

3.32 a) The system has a pole in -3 .

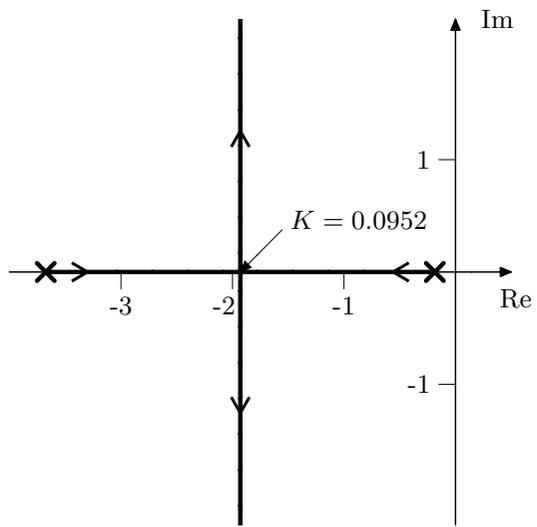


Figure 3.31b

b) The system is stable for $K \leq -3$.

3.33 $K > \mu$

3.34 $K_2 < 1$ and $K_1 = 200/K_2^2$.

4 Frequency Description

4.1 $G(s) = \frac{0.16}{s+0.16}$

4.2 a) See figure in the solution. $\omega_c = 0.025$, $\varphi_m = 31^\circ$, $A_m = 2.5$.

b) The period time will be 108 seconds, $K = 1.25$.

c) $B = 8^\circ$, $\beta = 0.02 \text{ rad/s}$ and $\varphi = -42^\circ$.

4.3 a) Figure 4.3a in Solutions.

b) Figure 4.3b in Solutions.

4.4 A-B, B-C, C-D, D-A.

4.5 a)

System	$G(0)$	ω_B	ω_r	M_p
G_A	1	0.64		
G_B	1	1.5	1	2.5
G_C	1	0.21		
G_D	1	1.27	0.7	1.15
G_E	1	2.54	1.4	1.15

b) The bandwidth of a system is (approximately) inversely proportional to the rise time. The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large overshoot.

4.6 $y(t) = \frac{1}{\sqrt{5}} \sin(2t - 1/2 - 4 - \frac{\pi}{2} - \arctan 2)$.

4.7 a) $0.45 \sin(2t - 1.1)$

b) Unstable system.

c) $0.11 \sin(2t - 2.4)$

d) $0.45 \sin(2t - 2.1)$

4.8 a,b)

ω	$ G(i\omega) $	$\arg G(i\omega)$
1	1 = 0 dB ₂₀	-0.2 rad = -11°
5	0.8 = -1.9 dB ₂₀	-0.9 rad = -52°
10	0.5 = -6 dB ₂₀	-1.6 rad = -92°
20	0.2 = -14 dB ₂₀	-2.2 rad = -126°

c) See Figure 4.8a in Solutions.

4.9 G_1 -B, G_2 -D, G_3 -A, G_4 -C, G_5 -E.

4.10 Bode gain-step response pairs: A-D, B-C, C-A, D-B.

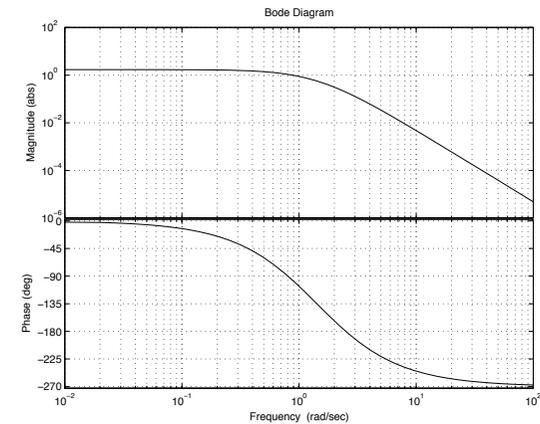


Figure 4.11a

4.11 a) The bode digram of the system is shown in Figure 4.11a.

b) $K = \frac{1}{0.1946} = 5.14$

4.12 a) $K \leq 5.04$

b) $F(s) = 1.58 \frac{100s+1}{100s}$

5 Compensation

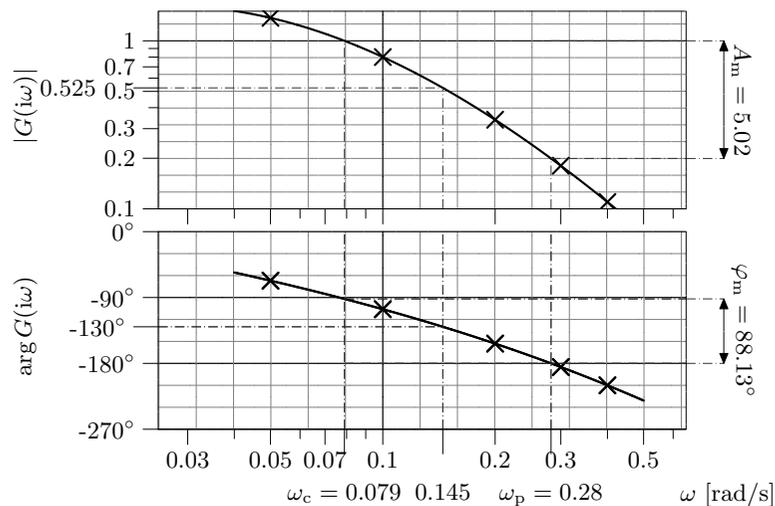


Figure 5.2a

5.1 For example, the following controller fulfills the requirements:

$$F(s) = 3.33 \cdot \frac{s + 0.185}{s + 0.555} \cdot \frac{s + 0.032}{s + 0.0036}$$

5.2 a) See Figure 5.2a.

b) Largest crossover frequency: 0.14 rad/s.

c) One controller that fulfills the requirements is the lead compensator (with gain adjustment)

$$F(s) = 1.9 \cdot 7 \frac{s + 0.106}{s + 0.106 \cdot 7}$$

5.3 a) See Figure 5.3a.

b) Smallest value of ramp error 0.067 and crossover frequency 150 rad/s.

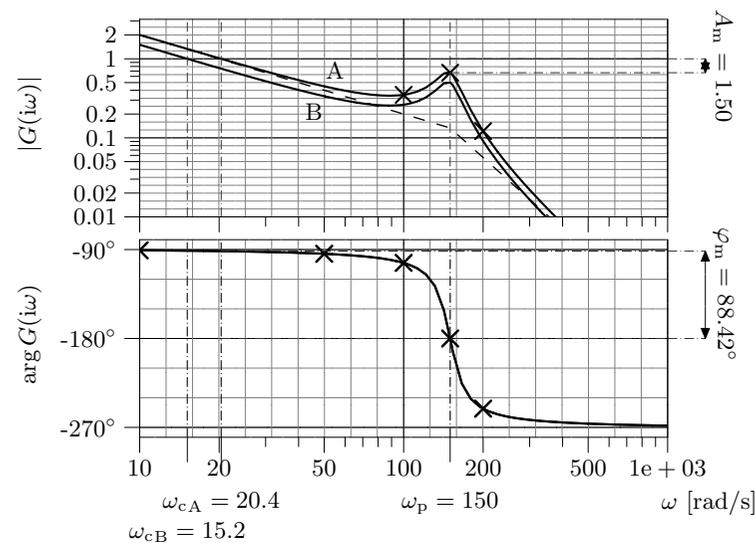


Figure 5.3a

c) One controller that fulfills the constraints is

$$F(s) = 0.75 \frac{s + 1.4}{s + 0.1}$$

5.4 One controller which fulfills the requirements is

$$F(s) = 1.12 \cdot 4 \cdot \frac{s + 6.7}{s + 27} \cdot \frac{s + 1.33}{s + 0.15}$$

5.5 A-E-C, B-C-E, C-A-B, D-D-D, E-B-A.

5.6 One controller which satisfies the demands is

$$F(s) = 1.2 \cdot 5 \frac{s + 8.0}{s + 5 \cdot 8.0} \cdot \frac{s + 1.8}{s + 1.8/84}$$

5.7 The system is stable when $0 < K < 0.2$ or $1.67 < K < 5$.

5.8 a) $T < 0.698$ s

b) 0.1 s $< T < 0.279$ s

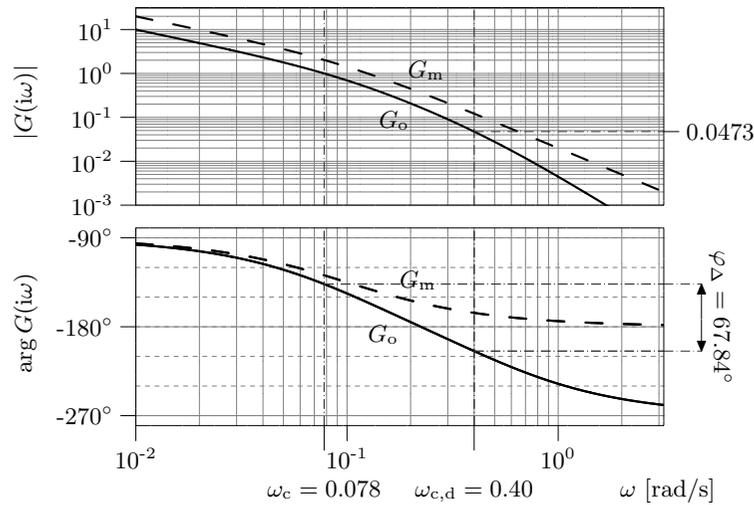


Figure 5.9a

5.9 a) $k_A = 0.25$ and $a = 0.5$. The Bode plot is given in Figure 5.9a.

b) One controller that does the job is

$$F(s) = 10.6 \cdot \left(4 \frac{(s+0.2)}{(s+0.2 \cdot 4)} \right)^2$$

5.10 The following compensator fulfills the requirements:

$$F(s) = 4.4 \cdot \left(4 \frac{s+0.53}{s+0.53 \cdot 4} \right)^2 \frac{s+0.105}{s+0.105/195}$$

5.11 a) See Figure 5.11a.

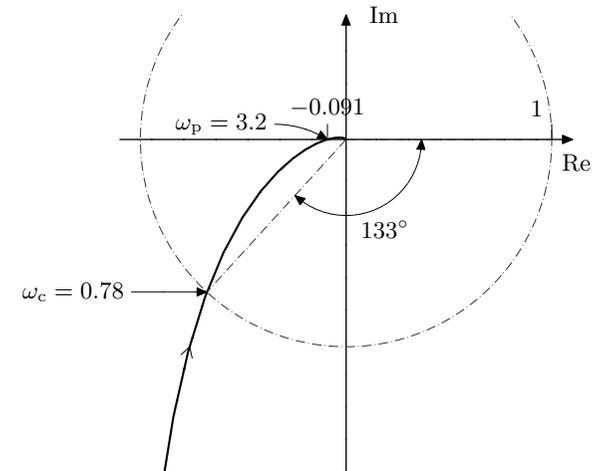


Figure 5.11a

b) Asymptotically stable for $0 < K < 10$.

c) $\lim_{t \rightarrow \infty} e(t) = 5$

d) $T < 0.4$

5.12 a) Impossible to determine.

b) It is stable.

5.13 a) $\omega_c = 5$ rad/s, $\omega_p = 9.5$ rad/s, $A_m = 3.5$ and $\varphi_m = 27^\circ$.

b)-d) See solution.

5.14 The following controller will do:

$$F(s) = 35.7 \cdot 3 \frac{s+0.12}{s+3 \cdot 0.12} \frac{s+0.02}{s}$$

5.15 a) $e_0 = 0$, $e_1 = \frac{4}{K}$, provided $K < 4000$. Larger K results in an unstable system.

b) The following controller will do:

$$F(s) = 756 \cdot 7 \frac{s + 37.8}{s + 264.6} \cdot \frac{s + 10}{s + 1.9}$$

6 Sensitivity and Robustness

6.1 The gain of the sensitivity is:

$$|S(i\omega)| = \frac{\sqrt{2}}{\sqrt{(K-1)^2 + 1}}$$

and the requirement on K becomes $K > 2$.

6.2 The maximum bandwidth is $\omega_B = 1$.

6.3 See the solution, Figure 6.3a.

6.4 Yes.

6.5 a) See the solution, Figure 6.5a.

b)

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{2}$$

6.6 a) No, stability cannot be guaranteed when $\tilde{G}(s) = 1$.

b) $\alpha > -2/25$. This is not contradictory since the robustness criterion is a *sufficient* but not *necessary* condition.

6.7 a) Asymptotically stable for $\alpha > 3$. See the solution, Figure 6.7a.

b) $\alpha > 4$

c) The robustness criterion gives a sufficient but not necessary condition.

6.8 $0 \leq \gamma < \frac{1}{35}$

$$6.9 \quad y(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right) - \sin(t)$$

$$6.10 \quad \text{a) } \frac{1}{G_\Delta(s)} = -\frac{s+1}{s}$$

b) Stability cannot be guaranteed for $F(s) = 1$, while it can be guaranteed for the regulator from Problem 5.13.

6.11 The amplitude of the steady state error will be 0.2.

6.12 The controller also stabilizes the system for the stirring speed 400 r/min.

$$6.13 \quad K \geq \sqrt{396} \approx 19.9$$

7 Special Controller Structures

- 7.1 a) See Figure 7.1a in the solution. ω_c and φ_m are undefined and $A_m = 43.5$.
The stability requirement gives $K_1 = 21.75$ which implies

$$\lim_{t \rightarrow \infty} e(t) = 0.0487 \cdot a$$

where a is the size of the step.

- b) See Figure 7.1b in the solution. ω_c and φ_m are undefined, and $A_m = 16$.
The requirement gives $K_1 = 8$ which implies

$$\lim_{t \rightarrow \infty} e(t) = 0.111 \cdot a$$

- 7.2 a) $F_f(s) = 1$, and $h(t) = -\frac{0.1}{A \cdot 2}(1 - e^{-2t})$.

- b) Zero steady state error.

- 7.3 a)

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

- b) The amplitude of the control signal is 3.

c) $\lim_{t \rightarrow \infty} y(t) = \frac{9(1-b/2)}{12+4Kb}$

- 7.4 a) $F_f(s) = -\frac{4(s+1)}{3(s+2)(s+5)}$

b) $\lim_{t \rightarrow \infty} y(t) = -0.012$

c) $\lim_{t \rightarrow \infty} y(t) = -\frac{0.012}{3K+1}$

- d) $y(t)$ doesn't have a final value.

8 State Space Description

8.1

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= (1 \quad 0) x\end{aligned}$$

8.2

$$\begin{aligned}\dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta}\end{aligned}$$

8.3

$$\begin{aligned}\dot{x}_1(t) &= K_2 x_2(t) + M_1(t) \\ \dot{x}_2(t) &= -x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t)\end{aligned}$$

8.4 a)

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= -6x_1(t) - 11x_2(t) - 6x_3(t) + 6u(t) \\ y(t) &= x_1(t)\end{aligned}$$

b)

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + x_3(t) + 4u(t) \\ \dot{x}_2(t) &= -3x_1(t) + 2u(t) \\ \dot{x}_3(t) &= -5x_1(t) + x_2(t) + u(t) \\ y(t) &= x_1(t)\end{aligned}$$

c)

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) - u(t) \\ \dot{x}_2(t) &= -3x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

8.5

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t)\end{aligned}$$

8.6 $G(s) = \frac{s}{(s+2)(s+3)}$

8.7

$$x(t_0 + T) = e^{AT} x(t_0) + \left(\int_{t_0}^{t_0+T} e^{A(t_0+T-s)} ds \right) Bu_0$$

8.8 a) The state space description of the closed loop system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} h_{\text{ref}}(t) \\ h(t) &= (1 \quad 0 \quad 0) x(t)\end{aligned}$$

b,c) The state space description of the closed loop system with noise

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} n(t) \\ h(t) &= (1 \quad 0 \quad 0) x(t)\end{aligned}$$

8.9 Yes, since the system is controllable.

8.10 a) Dimensions: 2 and 1. Subspaces: $\left\{ \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}^T, \begin{pmatrix} -2 & 3 & -6 \end{pmatrix}^T \right\}$ and $\left\{ \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^T \right\}$.

b) Dimensions: 2 and 1. Subspaces: $\left\{ \begin{pmatrix} 0 & 4 & -2 \end{pmatrix}^T, \begin{pmatrix} 0 & -8 & 8 \end{pmatrix}^T \right\}$ and $\left\{ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T \right\}$.

8.11 a) $x_1 = 1 - e^{-t}$, $x_2 = 0.5(e^{2t} - 1)$

b) No. Yes.

c) Controllable, not observable.

d) Unobservable growing state \Rightarrow simulation collapses.

8.12 Poles: $1 \pm i\sqrt{2}$. Zeros: -1 .

8.13 a)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\alpha}x_1 - \frac{u}{\alpha} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - u\end{aligned}$$

b) $\det \mathcal{S} = \frac{1}{\alpha^2}(1 - \frac{1}{\alpha})^2$. Thus, the system is controllable except for the case $\alpha = 1$, that is, when the two pendulums have the same lengths.

8.14 a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (1 \quad 1) x\end{aligned}$$

b) $u = -5x_1 + x_2 + 3.2r$

c) $Y(s) = \frac{3.2(2s+5)}{(s+4)^2}R(s)$

8.15 a) The model is given by

$$\begin{aligned}V \frac{dc_A}{dt} &= -Vk_1c_A^3 + qc_{A,in} - qc_A \\ V \frac{dc_B}{dt} &= \frac{Vk_1c_A^3}{3} - qc_B\end{aligned}$$

b) The linearized model is given by

$$\begin{aligned}\frac{d}{dt} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} &= \begin{pmatrix} \frac{-q-3k_1c_A^*V}{V} & 0 \\ k_1c_A^* & \frac{-q}{V} \end{pmatrix} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} u \\ y &= (0 \quad 1) \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix}\end{aligned}$$

8.16 a)

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= (0 \quad 1) x\end{aligned}$$

with

$$x = (x_1 \quad x_2)^T$$

b) The poles are $s = 0$ and $s = -(k_{12} + k_{21})$.

c) The system is observable when $k_{21} \neq 0$. If $k_{21} = 0$ we are not able to observe state x_1 by measuring x_2 .

d) The nonlinear state space is given by

$$\begin{aligned}\dot{x}_1 &= -k_{21}x_1 + \frac{V_{\max}x_2}{K_M + x_2} \\ \dot{x}_2 &= -\frac{V_{\max}x_2}{K_M + x_2} + k_{21}x_1 + Ku\end{aligned}$$

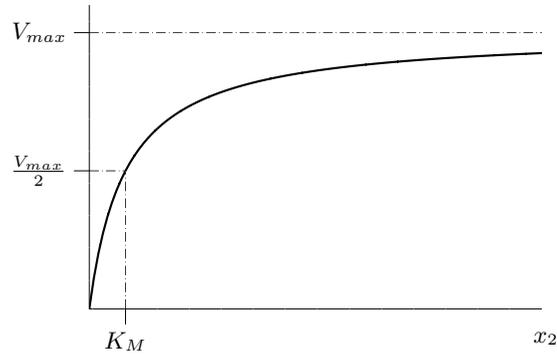


Figure 8.16a

e) The relation is shown in Figure 8.16a.

f) Linearization gives

$$A = J_f(x_0, u_0) = \begin{pmatrix} -k_{21} & \frac{V_{\max}}{4K_M} \\ k_{21} & -\frac{V_{\max}}{4K_M} \end{pmatrix}.$$

and

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ K \end{pmatrix}.$$

With $x_\Delta = x - x_0$ and $u_\Delta = u - u_0$, the linear approximation of the system around x_0, u_0 is given by

$$\dot{x}_\Delta = Ax_\Delta + Bu_\Delta.$$

g)

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ 0 & k_{32} & -k_{23} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \\ 0 \end{pmatrix} u$$

$$y = (0 \quad 1 \quad 0) x$$

h) The determinant of \mathcal{O} can be expanded along the first row to obtain

$$\det \mathcal{O} = k_{21}k_{23}(k_{23} - k_{21}).$$

I.e. the system is observable unless $k_{21} = 0$ or $k_{23} = 0$ or $k_{23} = k_{21}$. The first two cases mean that the corresponding state, x_1 and x_3 respectively, do not influence x_2 and would then not be visible in the output. In the third case, $k_{23} = k_{21}$, x_1 and x_3 do indeed influence x_2 but they do it in the exact same way and we can therefore not distinguish between them.

8.17 a) The stationary point is

$$x_0 = \begin{pmatrix} \frac{k_1(k_{12}+k_2)}{k_{21}k_2} & \frac{k_1}{k_2} \end{pmatrix}^T.$$

b)

$$G(s) = \frac{k_1k_{21}}{(s+k_{21})(s+k_{12}+k_2) - k_{12}k_{21}}.$$

c)

$$\lim_{t \rightarrow \infty} y(t) = \frac{k_1}{k_2}.$$

Since the output is defined as $y = x_2$, it approaches the stationary point $x_{0,2}$.

9 State Feedback

- 9.1 a) State feedback. Poles in $\{-3, -5\}$ gives the state feedback

$$u = -6x_1 - 14x_2 + y_{\text{ref}}$$

Poles in $\{-10, -15\}$ gives the state feedback

$$u = -23x_1 - 149x_2 + y_{\text{ref}}$$

- b) Observer poles in -20 gives the observer gain

$$K = \begin{pmatrix} 38 \\ -399 \end{pmatrix}$$

- 9.2 a)

$$\dot{x} = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u$$

b) $u = -\frac{1}{8K_1K_2}x_1 - \frac{3}{4K_1}x_2 - \frac{3}{2K_1}x_3$

c) Observer gain $K^T = (6 \quad 12/K_2 \quad 8/K_2)$

9.3 a) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}}$

b) $u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}} - \frac{c_2}{c_1}\hat{x}_3$

9.4 State feedback gain $L = (6 \quad -2)$. Observer gain $K^T = (16 \quad 9)$.

9.5 The system is observable and the poles of the observer may be placed arbitrarily.

- 9.6 a) Yes, since the system is controllable.

- b) Closed loop poles in -3 gives

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

- c) The system is observable with the sensor at x_1 or x_3 . The sensor at x_1 and observer poles in -4 give $K^T = (6 \quad 14 \quad 14)$.

9.7 $\hat{X}_3(s) = \frac{K_1}{s+K}U(s) + \frac{K^2s}{s+K}X_2(s)$

9.8 a) $L = (1 \quad 2)$

- b) In steady state: $h = -0.1$.

- c) $F_f(s) = 2$ gives, in steady state, $h = 0$.

d) $h = \frac{2(k_1-1)}{k_1}v$

- e) Introduce the integral of the height as a new state

$$z(t) = \int_0^t h(s) ds \Rightarrow \dot{z} = h$$

9.9 $T < \frac{\arctan 2\omega_c}{\omega_c} = 0.65s$

9.10 a) $K^T = (-13 \quad 38)$

- b) The transfer function from v to \tilde{x}_1 is

$$-C_1(sI - A + KC)^{-1}K = \frac{13s - 12}{s^2 + 15s + 50}$$

where $C_1 = (1 \quad 0)$.

- 9.11 a) The initial value theorem gives

$$\dot{y}(0) = -\frac{\beta^2}{\alpha}$$

and hence $\dot{y}(0)$ decreases as α decreases.

- b) No, since the zero is not affected by the feedback.

9.12 A very fast closed loop system:

- implies that the poles are far into the LHP which implies a need for generating large input signals.
- easily becomes unstable in case of model uncertainties.
- becomes sensitive to measurement noise.
- has a sensitivity function with a large peak.

9.13 a) $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$

b) $r(t) = r_0 e^{-5t}$

9.14 a) $x_2 = y$ (motor angle) and $x_1 = \dot{y}$ (angular velocity).

b) The pole locations give similar rise and settling times. With complex poles the maximum value of the input is lower.

c) Larger weight on the motor angle gives faster response.

d) Increasing weight on the input makes the system slower.

e) Increasing weight on the velocity makes the system slower.

9.15 a) Yes the system is controllable.

b) Poles in -0.1 gives the state feedback

$$u = -0.13x_1 - 0.128x_2$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large amplification of the measurement noise the poles of the observer should not be placed too far into the left hand plane.

d) Observer poles in -0.1 gives the observer gain

$$K = \begin{pmatrix} 0.45 \\ 0.33 \end{pmatrix}$$

9.16 Are the specifications 1–4 fulfilled?

1. The bandwidth requirement is not fulfilled.
2. The system is stable despite the model errors.
3. The gain is different from 1 when $\kappa \neq 1$
4. Both measurement and process noise are amplified for some frequency.

9.17 a) $u = 4x_1 + x_2$

b) Yes! It is essential that the input u is known since u is required in the observer design to get an asymptotically vanishing state estimation error.

c) Yes, by introducing a third state $x_3 = u$. This new system is observable hence a observer can be designed to estimate u .

9.18 a) The poles are pure complex and thus the system doesn't have a well defined stationary error or speed of response.

b) A linear combination of r and x_2 is given by

$$u = l_0 r - l_2 x_2$$

with this controller the poles can be placed with l_2 as

$$s = \frac{-l_2}{2} \pm \sqrt{\frac{l_2^2 - 4}{4}}$$

and by setting $l_0 = 1$ the stationary error will be zero when $w = 0$. If $w \neq 0$ and $l_0 = 1$ then there will be stationary error of size $l_2 w$.

c) Designing a observer with the following observer gains $k_1 = -11$, $k_2 = 6$, and $k_3 = -8$. Let the control law be $u = l_0 r - l_2 \hat{x}_2 - l_3 \hat{x}_3$. With $l_3 = l_2$ and $l_0 = 1$ there will be no error. Place the poles to the closed loop with l_2 .

11 Implementation

11.1 $\beta_1 = 0.905$, $\alpha_1 = 19.14$, and $\alpha_2 = -18.95$.

11.2 a) $y_{k+1} - y_k = Tu_k$

b) $0 < K < \frac{2}{T}$

11.3 a)

$$A = \frac{1}{\sqrt{1 + (\omega_2 T_1)^2}}$$
$$\omega_1 = \frac{2\pi}{T} - \omega_2$$
$$\varphi = \pi + \arctan \omega_2 T_1$$

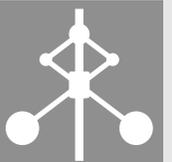
b) $T_1 = T/\pi$ gives $A = \frac{1}{\sqrt{1 + (\omega_2 T/\pi)^2}}$.

Reglerteknik: Solutions

- Solutions

This version: January 2011

AUTOMATIC CONTROL
REGLERTEKNIK
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Solutions

This version: January 2011

1 Mathematics

- 1.1 a) A step has Laplace transform $\frac{A}{s}$.
 b) A ramp has Laplace transform $\frac{A}{s^2}$.
 c) $\frac{1}{s+2}$
 d) $\frac{s}{s^2+25}$
 e) $sU(s) - u(0)$
 f) $sU(s)$. ($u(0) = 0$ is a common assumption in the course.)
 g) $s^2U(s) - su(0) - \dot{u}(0)$
 h) $s^2U(s)$. ($u(0) = \dot{u}(0) = 0$ is a common assumption in the course.)
 i) A time delayed signal has Laplace transform, $e^{-sT}U(s)$.
- 1.2 a) Insert $\dot{y}(t) = 0$ och $u(t) = 5$ directly into the differential equation $\Rightarrow y(t) = 5/2$. It is also possible to solve the differential equation and let $t \rightarrow \infty$, or to use b) and the final value theorem.
 b) Use Laplace transform on the differential equation $Y(s) = \frac{1}{s+2}U(s)$. The denominator coincides with the characteristic polynomial of the differential equation. Note that we also have assumed $y(0) = 0$.

1.3 The general solution is given by

$$y(t) = C_1 e^{-2t} + (C_2 + C_3 t) e^{-t} - \frac{3}{100} (\cos(2t) + 7 \sin(2t))$$

1.4 a)

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}, \quad t \geq 0$$

b) The Laplace transform of the input

$$u(t) = 1 + \sin t$$

yields

$$U(s) = \frac{1}{s} + \frac{1}{s^2 + 1}$$

The differential equation

$$\dot{y}(t) + y(t) = u(t)$$

may be represented by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1}$$

Hence, the Laplace transform of the system output is given by

$$Y(s) = \underbrace{\frac{1}{s} \cdot \frac{1}{s+1}}_{Y_1(s)} + \underbrace{\frac{1}{s+1} \cdot \frac{1}{s^2+1}}_{Y_2(s)}$$

Rewriting the first term using partial fractions leads to

$$Y_1(s) = \frac{1}{s} \cdot \frac{1}{s+1} = \frac{1}{s} - \frac{1}{s+1}$$

with inverse transform

$$y_1(t) = 1 - e^{-t}$$

Rewriting the second term using partial fractions leads to

$$Y_2(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{0.5}{s+1} - \frac{0.5s}{s^2+1} + \frac{0.5}{s^2+1}$$

with inverse transform

$$y_2(t) = 0.5e^{-t} - 0.5 \cos t + 0.5 \sin t$$

Hence, the system output is

$$y(t) = 1 - 0.5e^{-t} + 0.5 \sin t - 0.5 \cos t$$

- 1.5 a) The absolute value is $|1 + i| = \sqrt{2}$, and the argument is $\arctan \frac{1}{1} = \frac{\pi}{4} = 45^\circ$. Hence, the polar form is

$$\sqrt{2} e^{i\frac{\pi}{4}}$$

b) The absolute value is

$$\frac{|1+i|}{5|1+\sqrt{3}i|} = \frac{\sqrt{2}}{5 \cdot 2} \approx 0.14$$

The argument is

$$\begin{aligned} \arg\left(\frac{1+i}{5i(1+\sqrt{3}i)}\right) &= \arg(1+i) - \arg 5i - \arg(1+\sqrt{3}i) \\ &= \arctan 1 - 90^\circ - \arctan \sqrt{3} = 45^\circ - 90^\circ - 60^\circ \\ &= -105^\circ \end{aligned}$$

Hence, the polar form is

$$\frac{\sqrt{2}}{10} e^{-i\frac{105}{180}\pi}$$

c) $2e^{i\frac{\pi}{3}} = 2 \cos \frac{\pi}{3} + 2i \sin \frac{\pi}{3} = 1 + \sqrt{3}i$

d) $5e^{-i\pi} = 5 \cos(-\pi) + 5i \sin(-\pi) = -5$

1.6 The amplification in deciBel is computed as $10 \log |F|^2 = 20 \log |F|$, where F is the absolute value of the amplification. The amplification $F = 100$ hence corresponds to $20 \log 100 = 40 \text{ dB}_{20}$.

deciBel (dB ₂₀)	Definition	Amplification F
20	$20 \log F = 20 \Rightarrow$	$F = 10^1 = 10$
-3	$20 \log F = -3 \Rightarrow$	$F = 10^{-3/20} \approx 0.708 \approx \frac{1}{\sqrt{2}}$
0	$20 \log F = 0 \Rightarrow$	$F = 10^0 = 1$
10	$20 \log F = 10 \Rightarrow$	$F = 10^{0.5} = \sqrt{10} \approx 3.16$
-10	$20 \log F = -10 \Rightarrow$	$F = 10^{-0.5} = \frac{1}{\sqrt{10}} \approx 0.316$

1.7 Multiplication of the two matrices gives the unit matrix.

1.8 The eigenvalues (λ) of the matrix A are given by the equation $\det(\lambda I - A) = 0$,

and the corresponding eigenvectors (v) are given by the equation $(\lambda I - A)v = 0$.

$$\lambda_1 = 3 \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\lambda_3 = 4 \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

1.9

$$T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

1.10 A basis for the null space is for example

$$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

A basis for the range space is

$$\begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}$$

The rank of the matrix is hence 3.

1.11 a) Writing the function with partial fractions yields

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = 1 - e^{-t}$$

This means that $f(t) \rightarrow 1$ as $t \rightarrow \infty$. The same result can also be obtained by use of the final value theorem, that is, by computing $\lim_{s \rightarrow 0} sF(s)$.

b) Writing the function with partial fractions yields

$$F(s) = -\frac{0.5}{s+1} + \frac{0.5}{s-1}$$

The inverse transform is then computed by use of a Laplace transform table:

$$f(t) = -0.5e^{-t} + 0.5e^t$$

This means that $f(t)$ will grow without bound as $t \rightarrow \infty$. Here, the final value theorem cannot be used since $f(t)$ lacks a final value.

c) The inverse transform can be computed by use of the relation

$$\mathcal{L}^{-1}\{G(s+a)\} = e^{-at} \cdot g(t)$$

Here, $G(s) = \frac{1}{s^2}$ and $a = 1$. The inverse transform of G is $g(t) = t$, so

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t} \cdot t$$

which tends to 0 as $t \rightarrow \infty$. This result can also be obtained by use of the final value theorem.

1.12 The relation between inflow and water level is given by the transfer function

$$Y(s) = \frac{1}{s+1}Z(s)$$

and the relation between control signal and inflow may be written as

$$Z(s) = \frac{1}{s^2+s+1}U(s)$$

This means that the Laplace transforms of the control signal and water level are related by

$$Y(s) = \frac{1}{(s+1)} \frac{1}{(s^2+s+1)} U(s) = \frac{1}{s^3+2s^2+2s+1} U(s)$$

which corresponds to the differential equation

$$y^{(3)} + 2\ddot{y} + 2\dot{y} + y = u$$

2 Dynamic Systems

2.1 a) We start from the equations

$$J\ddot{\theta} = -f\dot{\theta} + M \quad (2.1)$$

$$M = k_a i \quad (2.2)$$

$$v = k_v \dot{\theta} \quad (2.3)$$

Voltage equilibrium gives

$$u - R_a i - L_a \frac{di}{dt} - v = 0 \quad (2.4)$$

where $L_a = 0$. Equation (2.2) in (2.1) gives

$$J\ddot{\theta} + f\dot{\theta} = k_a i \quad (2.5)$$

From (2.4) and (2.3) we get

$$i = (u - k_v \dot{\theta}) / R_a$$

which in (2.5) gives

$$J\ddot{\theta} + f\dot{\theta} = k_a(u - k_v \dot{\theta}) / R_a$$

that is

$$\ddot{\theta} + \frac{R_a f + k_a k_v}{JR_a} \dot{\theta} = \frac{k_a}{JR_a} u$$

Let

$$\frac{1}{\tau} = \frac{R_a f + k_a k_v}{JR_a} \quad k_0 = \frac{k_a}{JR_a}$$

which gives

$$\ddot{\theta} + \frac{1}{\tau} \cdot \dot{\theta} = k_0 u \quad (2.6)$$

b) Laplace transformation of (2.6) gives

$$(s^2 + \frac{1}{\tau} \cdot s)\theta(s) = k_0 U(s)$$

and this gives the transfer function

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{k_0}{s(s + 1/\tau)}$$

c) Suppose that u is a unit step, that is,

$$u = \begin{cases} 0, & t < 0 \\ 1 & t \geq 0 \end{cases}$$

that is

$$U(s) = \frac{1}{s}$$

This gives

$$\theta(s) = G(s)U(s) = \frac{k_0}{s(s + 1/\tau)} \cdot \frac{1}{s} = \left(\frac{k_0\tau}{s} - \frac{k_0\tau}{s + 1/\tau} \right) \cdot \frac{1}{s}$$

Inverse Laplace transformation gives

$$\theta(t) = k_0\tau t - k_0\tau^2(1 - e^{-t/\tau})$$

that is, θ will grow unlimited when t increases.

- 2.2 (1) Asymptotically stable system. Monotonic step response, that is, real poles: $K = 0.1$.
- (2) Very oscillative system. Poles close to the imaginary axis: $K = 2.5$.
- (3) Unstable system. Poles in the right half plane: $K = 3$.
- (4) Asymptotically stable system. Oscillative step response, that is, complex poles in the left half plane: $K = 0.5$.

2.3 The inverse Laplace transform gives the step response

$$d_1(t) = \mathcal{L}^{-1} \left\{ \frac{\beta}{1 + sT} \cdot \frac{1}{s} \right\} = \beta(1 - e^{-t/T})$$

For the final value, we have

$$d_1(t) \rightarrow \beta, \quad t \rightarrow \infty$$

The figure gives $\beta = 10$. At the time $t = T$, the system time constant, the step response has reached 63% of the final value, that is,

$$d_1(T) = 0.63 \cdot 10$$

The figure gives $T = 3$, which gives the total transfer function

$$G(s) = \frac{10}{1 + 3s}$$

If we measure the signal $d_2(t)$ we introduce an additional time delay of $\frac{L}{V}$ time units. The total transfer function then becomes

$$G(s) = \frac{10e^{-\frac{L}{V}s}}{1 + 3s}$$

Answer:

$$G(s) = \frac{10e^{-\frac{L}{V}s}}{1 + 3s}$$

2.4 Use the system description

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}$$

In the first figure $\omega_0 = 1$ and $\zeta = 0.5$.

a) For the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

we have $\omega_0 = 1$ and $\zeta = 0.5a$. The step response is more oscillative than in the case $\zeta = 0.5$, that is, $\zeta < 0.5$. This gives $a < 1$.

b) For the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

we have $\omega_0 = b$ and $\zeta = 0.5$. The step response is in this case pure time scaling compared to the case $\omega_0 = 1$. The figure shows that the step response is twice as fast as in the case $\omega_0 = 1$. This gives $b = \omega_0 = 2$.

2.5 The pairs of plots that belong to the same system will be written in the form *pole-zero-letter-step-response-letter*.

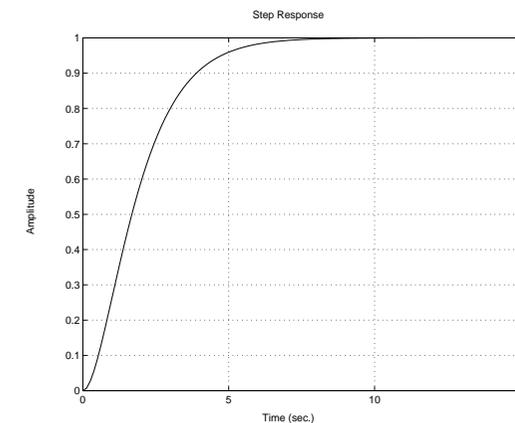
Pole-zero diagram B has a single pole in the origin which gives a ramp as step response, that is, B-F. Pole-zero diagram D also has a pole in the origin which gives an infinitely growing step response, D-C. Pole-zero diagram F has complex poles which gives an oscillative step response, F-D. Pole-zero diagram A has a zero in the origin which gives final value zero, A-B. Pole-zero diagram C cannot be step response E, since two real poles and no zeros give no overshoot. Hence C-A, and step response E is the only alternative left for pole-zero diagram E.

Answer: A-B, B-F, C-A, D-C, E-E, F-D.

2.6 a) Enter the systems.

```
>> s = tf( 's' );
>> GA = 1 / ( s^2 + 2*s + 1 );
>> GB = 1 / ( s^2 + 0.4*s + 1 );
>> GC = 1 / ( s^2 + 5*s + 1 );
>> GD = 1 / ( s^2 + s + 1 );
>> GE = 4 / ( s^2 + 2*s + 4 );
>> step( GA ); grid
```

Compute and plot the step response.



The systems $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ can be simulated in a similar way. The values of T_r , T_s , and M for the different step responses can be found by a right click in the figure and selecting “Characteristics” and then selecting the desired property. Use the “Properties...” menu item

(of the right click menu) to change the interval for the settling time. (The default interval is 2%, while we use 5% in the course.)

b) Compute the poles.

```
>> pole( GA )
ans =
    -1
    -1
```

The other systems are handled in the same way.

c) The results from a) and b) can be summarized in the following table.

System	T_r	T_s	M	poles
G_A	3.37	4.74	0%	-1, -1
G_B	1.21	13.7	52.7%	$-0.2 \pm i0.98$
G_C	10.5	14.6	0%	-4.8, -0.2
G_D	1.65	5.29	16.3%	$-0.5 \pm i0.87$
G_E	0.824	2.64	16.3%	$-1 \pm i1.73$

Using this table we can draw the following conclusions. (i): The speed of the step response (mainly) depends on the distance between the poles and the origin. Poles further away from the origin give a faster step response and shorter rise time. (ii): The damping of the system depends on the relationship between the imaginary part and the real part of the poles. Poles with large imaginary part relative to the real part give a poorly damped (oscillatory) step response.

Remark: We see that even though the distance to the origin is nearly the same in system G_A and G_B the rise time is almost 3 times faster in system B. Note that speed is not only rise time, also the settling time should be considered! Look at the following system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2}$$

The poles of this system are given by $s = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2}) = \omega_0(-\cos\phi \pm i\sin\phi)$ where $\cos\phi = \zeta$. The parameter ζ is called relative damping and $0 \leq \zeta \leq 1$. We see that ω_0 is the distance from the origin to the poles and in Figure 2.6a the step responses for different ζ are shown when ω_0 is constant. We see clearly that the rise time is faster when ζ is small but when ζ is small the settling time is big!

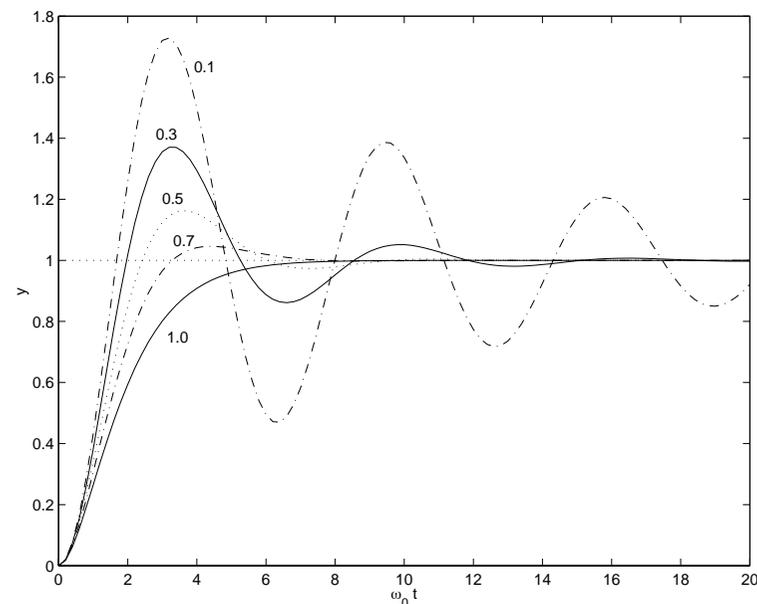


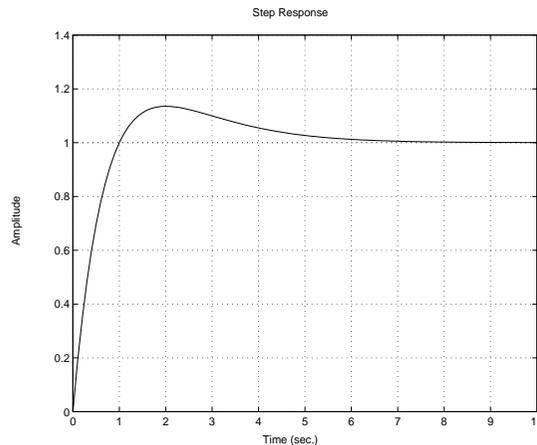
Figure 2.6a

2.7 Enter the system. Here we consider the case $\alpha = 2$, that is the system has a zero in -0.5 .

```
>> s = tf( 's' );
>> G1 = ( 2*s + 1 ) / ( s^2 + 2*s + 1 );
```

Plot the step response.

```
>> step( G1, 10 ); grid
```



A zero located close to the origin on the negative real axis causes an overshoot in the step response. A zero on the positive real axis causes the step response to initially move in the negative direction. This means that in some cases the zeros of the system can have significant influence in the system properties. Systems with zeros in the right half plane normally imply extra difficulties for the design of control systems.

2.8 The Laplace transform of a step is $U(s) = \frac{1}{s}$. The step response is hence given by

$$y(t) = \mathcal{L}^{-1}\left(G(s)\frac{1}{s}\right).$$

If $G(s)$ is a rational function the inverse Laplace transform can be computed by first doing a partial fraction expansion and then using a transform table. When the system is available one can let the input $u(t)$ be a step and measure $y(t)$.

2.9 a) The steady state value is 1.5.

b) The output signal almost reaches 1.9, which is slightly less than 0.4 over the final value. The overshoot is hence $\frac{0.4}{1.5} \approx 26\%$.

c) Find the time points where the output is 10% (0.15) and 90% (1.35) of the steady state value. The rise time is the difference between these values, here approximately $T_r \approx 1.5$ s.

d) Find the earliest time such that the output then lies within $\pm 5\%$ of the steady state value. Here, the interval is $[1.425, 1.575]$, and the settling time is $T_s \approx 7.8$.

2.10 G_1 -C: G_1 is poorly damped, which gives an oscillatory behavior.

G_2 : Can be excluded since it is the only system having static gain $\frac{1}{2}$, and among the step responses there is always more than one match for each of the present final values.

G_3 -B: This case has the shortest rise time, and some overshoot due to the pair of complex poles. The static gain is 2.

G_4 -A: The pole in -2 dominates, which gives slower step response than systems G_3 and G_5 . The static gain is 1.

G_5 -D: The dominating pole is in -3 , which is slower than for G_3 but faster than for G_4 . The static gain is 2.

G_6 : Can be excluded due to instability.

2.11 a) The signals can be classified as

- ◇ Disturbances signal: Acid process flow (unknown pH and flow)
- ◇ Control signal: NaOH solution
- ◇ Measured and controlled signal: The pH of the outflow

b) A block diagram where the control strategy is based on feedback could look like Figure 2.11a

2.12 a) At steady state the inflow is equal to the outflow (constant volume). From mass balance

$$\rho^* q^* = \rho_1^* q_1^* + \rho_2^* q_2^*$$

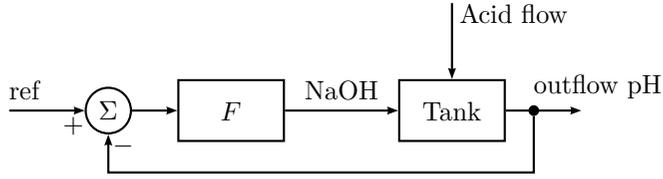


Figure 2.11a

Assuming the densities are equal ($\rho = \rho_1 = \rho_2$) gives $q^* = q_1^* + q_2^* = 1 + 0.5 = 1.5 \text{ m}^3/\text{min}$. From component balance for component A

$$q^* c_A^* = q_1^* c_{A,1}^* + q_2^* c_{A,2}^*$$

which gives $c_A^* = 2.0 \text{ kmol/m}^3$.

- b) The amount of mass in the tank is given by ρV (assuming ρ is constant). The change in mass is given by the mass coming in subtracted by the mass going out of the tank

$$\frac{d(\rho V)}{dt} = \rho(q_{in} - q_{out}) \quad (2.1)$$

where $q_{in} = q_1 + q_2$ and $q_{out} = q$. Assuming that the volume is constant gives $\frac{d(\rho V)}{dt}$ which means

$$q_1 + q_2 = q \quad (2.2)$$

The amount of component A contained in the tank is given by $V c_A$. The change is then given by

$$\frac{d(V c_A)}{dt} = q_1 c_{A,1} + q_2 c_{A,2} - q c_A \quad (2.3)$$

Constant V and (2.2) gives

$$V \frac{dc_A}{dt} = q_1 (c_{A,1} - c_A) + q_2 (c_{A,2} - c_A) \quad (2.4)$$

The model (2.1), (2.3) is nonlinear since it contains products between variables. Assuming volumes and flows to be constant gives a linear model.

- c) Assume that all the other independent variables ($q_1, q_2, c_{A,2}$) are constant. Take their values from a). Equation (2.4) then gives

$$V \frac{dc_A}{dt} = q_1^* (c_{A,1}(t) - c_A(t)) + q_2^* (c_{A,2}^* - c_A(t)) = -1.5 c_A(t) + c_{A,1}(t) + 2$$

The equation can be written

$$\frac{dc_A}{dt} = -1.5 c_A(t) + 3.2$$

for $t \geq 0$. The corresponding Laplace transform equation is

$$s(\mathcal{L}c_A)(s) - (\mathcal{L}c_A)(0) = -1.5(\mathcal{L}c_A)(s) + 3.2 \frac{1}{s}$$

or

$$(\mathcal{L}c_A)(s) = \frac{1}{s + 1.5} \left(2 + \frac{3.2}{s} \right) = 2 \frac{1}{s + 1.5} + 3.2 \frac{1}{s + 1.5} \frac{1}{s}$$

which transforms back to

$$c_A(t) = 2e^{-1.5t} + \frac{3.2}{1.5} (1 - e^{-1.5t})$$

Rearranging yields

$$\frac{3.2}{1.5} + \left(2 - \frac{3.2}{1.5} \right) e^{-1.5t} = 2 - \left(2 - \frac{3.2}{1.5} \right) \left(1 - e^{-\frac{t}{0.67}} \right)$$

where the sought constants can be identified: $k_0 = 2.0 \text{ kmol/m}^3$, $k_1 = 0.13 \text{ kmol/m}^3$, and $\tau = \frac{1}{1.5} = 0.67 \text{ min}$.

- 2.13 a) The equilibrium equation is

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i} \quad (2.1)$$

Mass balance gives

$$\frac{dM_i}{dt} = L_{i-1} + V_{i+1} - L_i - V_i \quad (2.2)$$

Component balance gives

$$\frac{d(M_i x_i)}{dt} = M_i \frac{dx_i}{dt} + x_i \frac{dM_i}{dt} = L_{i-1} x_{i-1} + V_{i+1} y_{i+1} - L_i x_i - V_i y_i \quad (2.3)$$

Combining (2.2)–(2.3) gives

$$\begin{aligned} M_i \frac{dx_i}{dt} &= -x_i (L_{i-1} + V_{i+1} - L_i - V_i) + L_{i-1} x_{i-1} + V_{i+1} y_{i+1} - L_i x_i - V_i y_i \\ &= L_{i-1} (x_{i-1} - x_i) + V_{i+1} (y_{i+1} - x_i) + V_i (x_i - y_i) \end{aligned} \quad (2.4)$$

The dynamic model for $M_i(t)$ and $x_i(t)$ is described by (2.1), (2.2), and (2.4).

- b) The stationary point for (2.2) gives $L_{i-1}^* + V_{i+1}^* - L_i^* - V_i^* = 0$. Introduce the difference variables

$$\begin{aligned} x_{i\Delta} &= x_i - x_i^* & x_{i+1,\Delta} &= x_{i+1} - x_{i+1}^* & y_{i\Delta} &= y_i - y_i^* \\ y_{i-1,\Delta} &= y_{i-1} - y_{i-1}^* & L_{i+1,\Delta} &= L_{i+1} - L_{i+1}^* & V_{i-1,\Delta} &= V_{i-1} - V_{i-1}^* \\ L_{i\Delta} &= L_i - L_i^* & V_{i\Delta} &= V_i - V_i^* \end{aligned}$$

The assumption that the change of mass on the plate is zero gives

$$\frac{dM_{i\Delta}}{dt} = 0$$

which means that

$$L_{i-1} + V_{i+1} - L_i - V_i = 0$$

this will simplify (2.4) to

$$M_i \frac{dx_i}{dt} = L_{i-1}x_{i-1} + V_{i+1}y_{i+1} - L_i x_i - V_i y_i \quad (2.5)$$

Linearization of (2.5) gives

$$\begin{aligned} M_i^* \frac{dx_{i\Delta}}{dt} &= L_{i-1}^* x_{i-1,\Delta} + V_{i+1}^* y_{i+1,\Delta} - L_i^* x_{i\Delta} - V_i^* y_{i\Delta} \\ &+ x_{i-1}^* L_{i-1,\Delta} + y_{i+1}^* V_{i+1,\Delta} - x_i^* L_{i\Delta} - y_i^* V_{i\Delta} \end{aligned} \quad (2.6)$$

Linearization of (2.1) gives

$$y_{i\Delta} = \frac{\alpha}{(1 + (\alpha - 1)x_i^*)^2} x_{i\Delta} \quad (2.7)$$

The linearized model is described by (2.6)–(2.7).

3 Feedback Systems

3.1 a) The balance equation is, with the bottom area of the tank being 1 m^2 ,

$$\dot{y}(t) = x(t) - v(t)$$

that is (note that all initial conditions are zero when deriving transfer functions)

$$sY(s) = X(s) - V(s)$$

Hence

$$Y(s) = G_t(s)(X(s) - V(s))$$

where

$$G_t(s) = \frac{1}{s}$$

b) The transfer function for the valve is

$$G_v(s) = \frac{k_v}{1 + Ts}$$

With the input taken as a unit step signal, that is,

$$U(s) = \frac{1}{s}$$

it follows that

$$X(s) = \frac{k_v}{1 + Ts} \cdot \frac{1}{s}$$

The final value theorem gives

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = k_v$$

The transfer function for the valve corresponds to the differential equation

$$T\dot{x}(t) + x(t) = k_v u(t)$$

Assuming the initial value $x(0) = 0$ and that $u(t)$ is a step with amplitude one gives the solution

$$x(t) = k_v(1 - e^{-t/T})$$

The time constant T is the time it takes for the step response to reach 63% of its final value. (This comes from the observation $(1 - e^{-1}) \approx 0.63$.) From the plot it follows that $T = 5$ and $k_v = 2$, that is

$$G_v(s) = \frac{2}{1 + 5s}$$

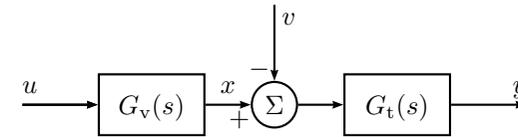


Figure 3.1a

c) The outflow caused by the pump enters in the block diagram as an additive disturbance on the input of the tank. The resulting structure of the open loop system is shown in Figure 3.1a.

3.2 a) Taking the Laplace transform of both sides of (3.1) gives

$$U(s) = K_P E(s) + K_I \frac{1}{s} E(s) + K_D s E(s)$$

which can be written

$$U(s) = F(s)(R(s) - Y(s))$$

if

$$F(s) = K_P + K_I \frac{1}{s} + K_D s$$

and

$$E(s) = R(s) - Y(s)$$

b) By using the controller $F(s)$, the closed loop system shown in Figure 3.2a is obtained.

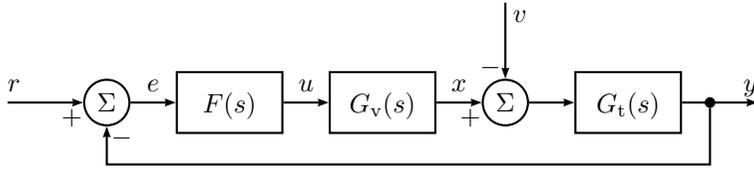


Figure 3.2a

c) From the block diagram, the following equations are obtained:

$$\begin{aligned} Y(s) &= G_t(s)(-V(s) + F(s)G_v(s)(R(s) - Y(s))) \\ &= -G_t(s)V(s) + F(s)G_v(s)G_t(s)R(s) - F(s)G_v(s)G_t(s)Y(s) \end{aligned}$$

Collecting all terms involving $Y(s)$ on the left hand side gives

$$Y(s)(1 + F(s)G_v(s)G_t(s)) = -G_t(s)V(s) + F(s)G_v(s)G_t(s)R(s)$$

and hence

$$\begin{aligned} Y(s) &= \frac{F(s)G_v(s)G_t(s)}{1 + F(s)G_v(s)G_t(s)}R(s) \\ &\quad - \frac{G_t(s)}{1 + F(s)G_v(s)G_t(s)}V(s) \end{aligned}$$

Inserting

$$F(s) = K_P + K_I \frac{1}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

and

$$G_t(s) = \frac{1}{s} \quad G_v(s) = \frac{2}{1 + 5s}$$

gives

$$\begin{aligned} Y(s) &= \frac{2(K_D s^2 + K_P s + K_I)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I}R(s) \\ &\quad - \frac{s(1 + 5s)}{5s^3 + (1 + 2K_D)s^2 + 2K_P s + 2K_I}V(s) \end{aligned}$$

Note that both transfer functions have the same denominator, that is, stability analysis can be carried out using either of the two transfer functions.

3.3 (a) The needed transfer functions were derived in Problem 3.2 and give, putting $K_D = K_I = 0$,

$$Y(s) = \frac{2K_P}{5s^2 + s + 2K_P}R(s) - \frac{(1 + 5s)}{5s^2 + s + 2K_P}V(s)$$

with characteristic equation

$$5s^2 + s + 2K_P = 0$$

or, equivalently,

$$s^2 + 0.2s + 0.4K_P = 0$$

The equation has the roots

$$s = -0.1 \pm \sqrt{0.01 - 0.4K_P}$$

$K_P = 0.02$ gives the poles $\{-0.14, -0.06\}$. Since both poles are real this implies that the level will change without oscillations when a step is applied to the reference or disturbance signal.

$K_P = 1$ gives the poles $-0.1 \pm 0.62i$. In this case the poles are complex and since the magnitude of the imaginary part is large compared to the real part, the closed loop system will not be well damped.

For $K_P \geq 0.025$ the poles are given by

$$s = -0.1 \pm i\sqrt{0.4K_P - 0.01}$$

When K_P increases, the imaginary part of the poles increases while the real part is constant.

(b) The Laplace transforms of $r(t) = 5$ and $v(t) = 2$ are

$$R(s) = \frac{5}{s} \quad V(s) = \frac{2}{s}$$

The Laplace transform of the level is then given by

$$Y(s) = \frac{2K_P}{(5s^2 + s + 2K_P)} \cdot \frac{5}{s} - \frac{(1 + 5s)}{(5s^2 + s + 2K_P)} \cdot \frac{2}{s}$$

Since the closed loop system is stable for all $K_P > 0$ the final value theorem can be applied. This gives

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = 5 - \frac{1}{K_P}$$

Hence, if K_P is chosen very large the steady state level will be close to the desired level, but the system will be very oscillatory due to the location of the poles.

- c) The transfer functions from Problem 3.2 give, putting $K_D = 0$,

$$Y(s) = \frac{2(K_P s + K_I)}{5s^3 + s^2 + 2K_P s + 2K_I} R(s) - \frac{s(1 + 5s)}{5s^3 + s^2 + 2K_P s + 2K_I} V(s)$$

The Laplace transform of the level, using $r(t) = 5$ and $v(t) = 2$, becomes

$$Y(s) = \frac{2(K_P s + K_I)}{(5s^3 + s^2 + 2K_P s + 2K_I)} \cdot \frac{5}{s} - \frac{s(1 + 5s)}{(5s^3 + s^2 + 2K_P s + 2K_I)} \cdot \frac{2}{s}$$

Provided that the coefficients K_P and K_I are chosen such that the closed loop system is stable, the steady state level can be determined using the final value theorem. This gives

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = 5$$

which means that the desired level is reached. Hence, if the regulator coefficients are chosen such that the closed loop system is stable, the integrating part eliminates the steady state error.

- (d) The transfer functions from Problem 3.2 give, putting $K_I = 0$, $K_P = 1$,

$$Y(s) = \frac{2(K_D s + 1)}{5s^2 + (1 + 2K_D)s + 2} R(s) - \frac{(1 + 5s)}{5s^2 + (1 + 2K_D)s + 2} V(s)$$

with characteristic equation

$$5s^2 + (1 + 2K_D)s + 2 = 0$$

or, equivalently,

$$s^2 + (0.2 + 0.4K_D)s + 0.4 = 0$$

This equation can be compared to the general characteristic equation for the case of complex roots

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

where ω_0 denotes the absolute value of the roots and ζ is the relative damping. Here one gets $\omega_0 = \sqrt{0.4}$, which implies

$$\zeta = \frac{0.2 + 0.4K_D}{2\sqrt{0.4}}$$

The condition $\zeta > 1/\sqrt{2}$ implies $K_D > 1.7$. Poles with such damping ratio correspond to an overshoot of less than 5% (the overshoot may also be affected by the zeros). Hence, the derivative part in the feedback can be used to improve the damping in the closed loop system, that is, to make it less oscillatory.

- 3.4 We shall determine how the control error $e(t) = y_{\text{ref}}(t) - y(t)$ depends on the disturbance signal f_c . We can assume that $y_{\text{ref}}(t) = 0$, since the size of the error as a function of f_c is sought for.

$$E(s) = Y_{\text{ref}}(s) - G(s) \cdot (F_c(s) + F(s)E(s))$$

where

$$G(s) = \frac{1}{ms^2 + ds}$$

gives

$$E(s) = -\frac{G(s)}{1 + G(s)F(s)} \cdot F_c(s)$$

$f_c(t)$ is a step disturbance, that is

$$F_c(s) = \frac{a}{s}$$

- a) Proportional control, $F(s) = K$, gives

$$E(s) = -\frac{1}{ms^2 + ds + K} \cdot \frac{a}{s}$$

Using the final value theorem it follows that (provided that K is chosen such that the closed loop is asymptotically stable)

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = -a/K$$

- b) Proportional-Integral control

$$F(s) = \frac{K_1 s + K_2}{s}$$

gives

$$E(s) = -\frac{s}{ms^3 + ds^2 + K_1 s + K_2} \cdot \frac{a}{s}$$

The final value theorem in this case gives (provided that K_1 and K_2 are chosen such that the closed loop is asymptotically stable)

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0$$

3.5 a) Enter the system.

```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
```

Generate a proportional regulator.

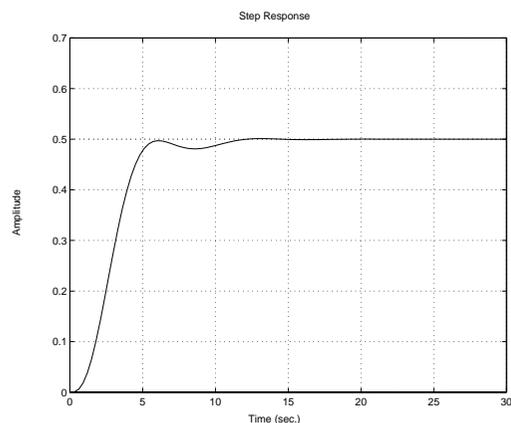
```
>> F = 1;
```

Generate the closed loop system.

```
>> Gc = feedback( F * G, 1 );
```

Compute and plot the step response.

```
>> step( Gc, 30 ); grid
```



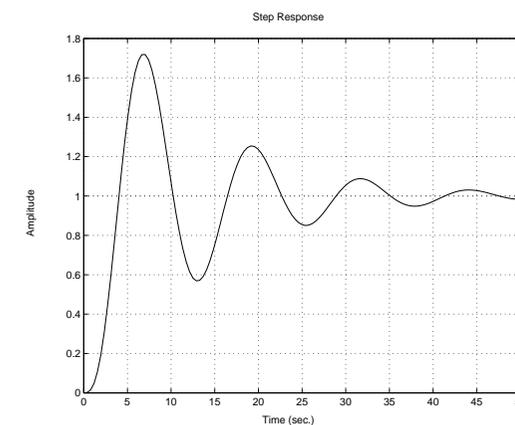
By trying some different values of K_P the following behavior can be seen: For small values of K_P the step response is slow, well damped and the steady state error is large. For increasing K_P the step response becomes faster but more oscillatory, while the error is reduced. For large K_P the amplitude of the oscillations increases over time, that is, the closed loop system becomes unstable.

b) Generate a PI controller with $K_P = 1$ and $K_I = 1$.

```
>> KP = 1; KI = 1;
>> F = KP + KI / s;
```

Plot the result.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



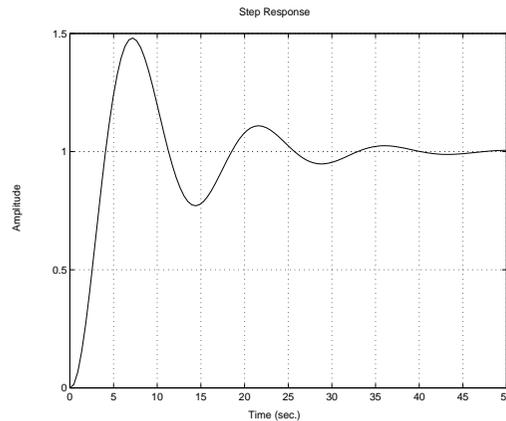
The following effects of the integrator can be found by trying some different values of K_I . (i): The integrator in the regulator eliminates the steady state error. (ii): A too small value of K_I gives a large settling time while a too large value gives an oscillatory (finally unstable) closed loop system.

c) Generate a PID controller with $K_P = 1$, $K_I = 1$, $K_D = 2$ and $T = 0.1$.

```
>> KP = 1; KI = 1; T = 0.1; KD = 1;
>> FP = KP;
>> FI = KI / s;
>> FD = KD * s / ( s*T + 1 );
>> F = FP + FI + FD;
```

Plot the result.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 ); grid
```



Using the (approximate) derivative of the error in the regulator increases the damping of the closed loop system. Increasing K_D too much, however, gives that an oscillation with higher frequency appears in the step response and finally (approximately when $K_D > 65$) the closed loop system becomes unstable.

3.6 a) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K(s+2)}{s(s+1)(s+3) + K(s+2)}$$

The characteristic equation is

$$s(s+1)(s+3) + K(s+2) = P(s) + KQ(s) = 0$$

that is

$$P(s) = s(s+1)(s+3) \quad Q(s) = s+2$$

- ◇ Starting points: \Leftrightarrow zeros of $P(s)$: 0, -1, -3
End points: \Leftrightarrow zeros of $Q(s)$: -2
- ◇ Number of asymptotes: 2
Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
Intersection with the real axis: $\frac{1}{2}[0 + (-1) + (-3) - (-2)] = -1$

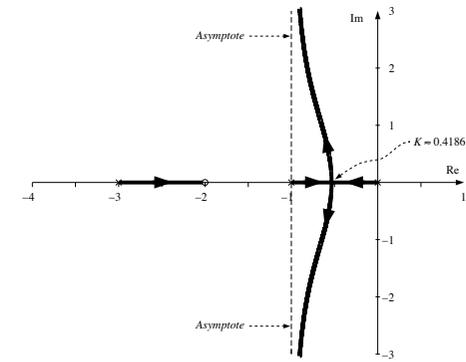


Figure 3.6a

- ◇ Real axis: $[-3, -2]$ and $[-1, 0]$ belongs to the root locus
- ◇ Intersection with the imaginary axis: Set $s = i\omega$ and solve the characteristic equation

$$\begin{aligned} i\omega(i\omega+1)(i\omega+3) + K(i\omega+2) &= -i\omega^3 - 4\omega^2 + (3+K)i\omega + 2K = 0 \\ \Rightarrow \left. \begin{aligned} (-\omega^2 + 3 + K)\omega &= 0 \\ -4\omega^2 + 2K &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} \omega &= K = 0 \\ & \text{(starting point)} \end{aligned} \end{aligned}$$

This gives the root locus in Figure 3.6a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$. For small values of K there are no oscillations and the speed is increasing with increasing K . For a certain value of K the system becomes oscillating. The damping is decreasing with increasing K .

b) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K}{s(s^2 + 2s + 2) + K}$$

The characteristic equation reads

$$s(s^2 + 2s + 2) + K = 0$$

that is

$$P(s) = s(s^2 + 2s + 2) \quad Q(s) = 1$$

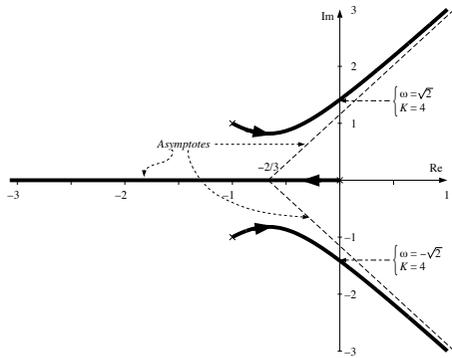


Figure 3.6b

- ◇ Starting points: \Leftrightarrow zeros of $P(s)$: $0, -1 \pm i$
End points: \Leftrightarrow There are no zeros of $Q(s)$
- ◇ Number of asymptotes: 3
Directions: $\frac{1}{3}[\pi + 2k\pi] = \pi, \pm\pi/3$
Intersection of asymptotes: $\frac{1}{3}[0 + (-1 + i) + (-1 - i)] = -2/3$
- ◇ Part of the real axis that belongs to the root locus: $(-\infty, 0]$
- ◇ Intersection with the imaginary axis: Set $s = i\omega$ and solve the characteristic equation

$$i\omega((i\omega)^2 + 2i\omega + 2) + K = -i\omega^3 - 2\omega^2 + 2i\omega + K = 0$$

$$\Rightarrow \left. \begin{array}{l} (-\omega^2 + 2)\omega = 0 \\ -2\omega^2 + K = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \omega = K = 0 \\ \text{(start point)} \end{array} \right\} \text{ or } \left. \begin{array}{l} \omega = \pm\sqrt{2} \\ K = 4 \end{array} \right\}$$

This gives the root locus in Figure 3.6b.

Answer: All poles are in the left half plane. That is, the system is asymptotically stable for $0 < K < 4$. The step response is oscillating for all K . To begin with the system will be faster with increasing K . However, for K sufficiently large the oscillating part is dominating. The damping will decrease with increasing K and for $(K \geq 4)$ the closed loop system is unstable.

c) The transfer function for the closed loop system is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)} = \frac{K(s+1)}{s(s-1)(s+6) + K(s+1)}$$

The characteristic equation is

$$s(s-1)(s+6) + K(s+1) = P(s) + KQ(s) = 0$$

$$P(s) = s(s-1)(s+6) \quad Q(s) = s+1$$

- ◇ Starting points: \Leftrightarrow zeros of $P(s)$: $0, 1, -6$
End points: \Leftrightarrow zeros of $Q(s)$: -1
- ◇ Number of asymptotes: $3 - 1 = 2$
Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$
Intersection of the asymptotes: $\frac{1}{2}[0 + 1 + (-6) - (-1)] = -2$
- ◇ Part of the real axis that belongs to the root locus: $[-6, -1]$ and $[0, 1]$
- ◇ Intersection with the imaginary axis: Set $s = i\omega$ and solve the characteristic equation:

$$i\omega(i\omega-1)(i\omega+6) + K(i\omega+1) = -i\omega^3 - 5\omega^2 + (K-6)i\omega + K = 0$$

$$\Rightarrow \left. \begin{array}{l} (-\omega^2 + K - 6)\omega = 0 \\ -5\omega^2 + K = 0 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \omega = K = 0 \\ \text{(start point)} \end{array} \right\} \text{ or } \left. \begin{array}{l} \omega = \sqrt{\frac{3}{2}} \\ K = 7.5 \end{array} \right\}$$

This gives the root locus in Figure 3.6c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for $K > 7.5$. For small values on K the closed loop system is (as the open loop system) unstable. For $K > 7.5$ the closed loop system is stable and oscillating. As K is increasing from the critical value both the damping and the response speed are increasing (the time constant is always $\geq 1/2s$), until they both are beginning to decrease. The damping is decreasing with increasing K .

3.7 The transfer function for the closed loop system is obtained from

$$\theta(s) = \frac{1}{s} \dot{\theta}(s) = \frac{1}{s} \cdot \frac{k}{1+s\tau} \cdot K \cdot (\theta_{\text{ref}}(s) - \alpha s\theta(s) - \theta(s))$$

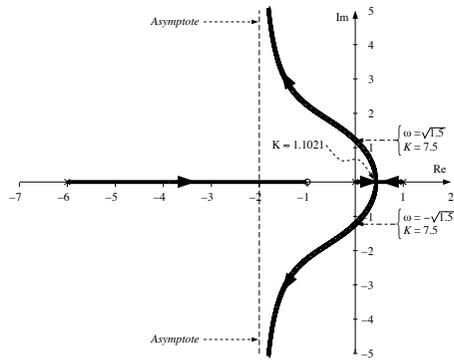


Figure 3.6c

⇒

$$G(s) = \frac{\theta(s)}{\theta_{\text{ref}}(s)} = \frac{k \cdot K}{s(1 + s\tau) + k \cdot K(1 + \alpha s)} = \frac{4K}{s(s + 2) + 4K(1 + \alpha s)}$$

The characteristic equation is:

$$s(s + 2) + 4K(1 + \alpha s) = 0$$

a) $\alpha = 0$. The characteristic equation is then

$$s(s + 2) + 4K = s^2 + 2s + 4K = 0$$

with the solution

$$s = -1 \pm \sqrt{1 - 4K}$$

This gives the root locus in Figure 3.7a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

b) $\alpha = 1$. The characteristic equation is then

$$s(s + 2) + 4K(1 + s) = 0$$

that is

$$P(s) = s(s + 2) \quad Q(s) = 4(1 + s)$$

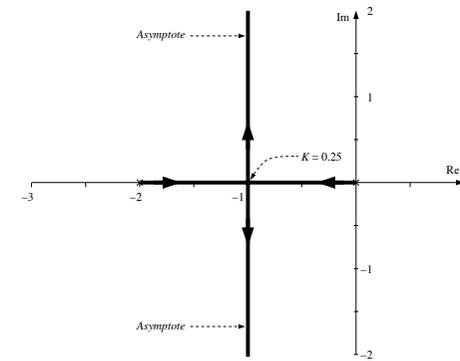


Figure 3.7a

◇ Starting points: ⇔ zeros of $P(s)$: 0, -2
End points: ⇔ zeros of $Q(s)$: -1

◇ Number of asymptotes: $2 - 1 = 1$.
Direction of asymptotes: $\frac{1}{1} \cdot \pi$, that is, the negative real axis.

◇ Part of the real axis that belongs to the root locus: $(-\infty, -2]$ and $(-1, 0]$

◇ Intersection with the imaginary axis: Set $s = i\omega$ and solve the characteristic equation:

$$i\omega(i\omega + 2) + 4K(1 + i\omega) = -\omega^2 + (2 + 4K)i\omega + 4K = 0$$

$$\Rightarrow \left. \begin{array}{l} (2 + 4K)\omega = 0 \\ -\omega^2 + 4K = 0 \end{array} \right\} \Rightarrow \begin{array}{l} \omega = K = 0 \\ \text{(start point)} \end{array}$$

This gives the root locus in Figure 3.7b.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

c) $\alpha = 1/3$. The characteristic equation is then

$$s(s + 2) + 4K(1 + s/3) = P(s) + KQ(s) = 0$$

which gives

$$P(s) = s(s + 2) \quad Q(s) = 4(1 + s/3)$$

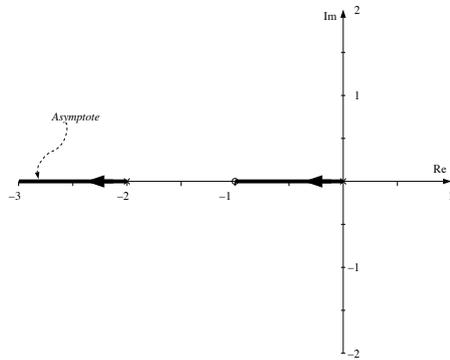


Figure 3.7b

- ◇ Starting points \Leftrightarrow zeros of $P(s)$: 0, -2
End points \Leftrightarrow zeros of $Q(s)$: -3
- ◇ Number of asymptotes: $2 - 1 = 1$ Direction: $\frac{1}{1} \cdot \pi$, that is, the negative real axis
- ◇ Part of the real axis that belongs to the real axis $(-\infty, 3)$ and $[-2, 0]$
- ◇ Intersection with the imaginary axis. Set $s = i\omega$ and solve the characteristic equation:

$$i\omega(i\omega + 2) + 4K(1 + i\omega/3) = -\omega^2 + (2 + \frac{4}{3}K)i\omega + 4K = 0$$

$$\Rightarrow \left. \begin{array}{l} (2 + \frac{4}{3}K)\omega = 0 \\ -\omega^2 + 4K = 0 \end{array} \right\} \Rightarrow \omega = K = 0 \quad (\text{start point})$$

This gives the root locus in Figure 3.7c.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 0$.

- d) $K = 1$. The characteristic equation becomes

$$s(s + 2) + 4(1 + \alpha s) = s^2 + 2s + 4 + 4\alpha s = 0$$

that is

$$P(s) = s^2 + 2s + 4 \quad Q(s) = 4s$$

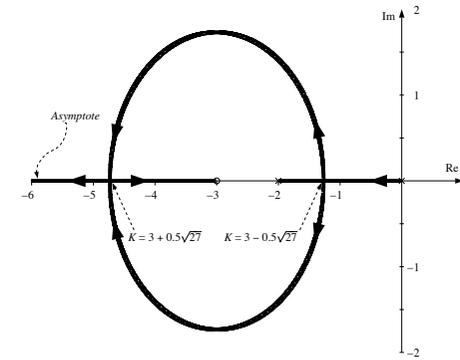


Figure 3.7c

- ◇ Starting points \Leftrightarrow zeros of $P(s)$: $-1 \pm i\sqrt{3}$
End points \Leftrightarrow zeros of $Q(s)$: 0
- ◇ Number of asymptotes: $2 - 1 = 1$
Direction: $\frac{1}{1} \cdot \pi$, that is, the negative real axis
- ◇ Part of the real axis that belongs to the root locus: $(-\infty, 0)$
- ◇ Intersection with the imaginary axis: $s = i\omega$ solves the characteristic equation

$$-\omega^2 + 2i\omega + 4 + 4i\omega\alpha = 0$$

$$\Rightarrow \left. \begin{array}{l} \omega(2 + 4\alpha) = 0 \\ -\omega^2 + 4 = 0 \end{array} \right\} \begin{array}{l} \text{has no solution} \\ (\alpha < 0 \text{ is of no interest}) \end{array}$$

To get further insights into the behavior of the closed loop system the intersection with the real axis is determined. That is, a real valued double root to the characteristic equation has to be determined

$$s^2 + 2s + 4 + 4\alpha s = (s + a)^2 = s^2 + 2as + a^2$$

$$\Rightarrow \left. \begin{array}{l} 2a = 2 + 4\alpha \\ a^2 = 4 \end{array} \right\} \Rightarrow \left. \begin{array}{l} a = 2 \\ \alpha = 1/2 \end{array} \right\}$$

This gives the root locus in Figure 3.7d.

Answer: All poles are in the left half plane, that is, the closed loop system is stable for all $\alpha \geq 0$. From d) it follows that the system will be

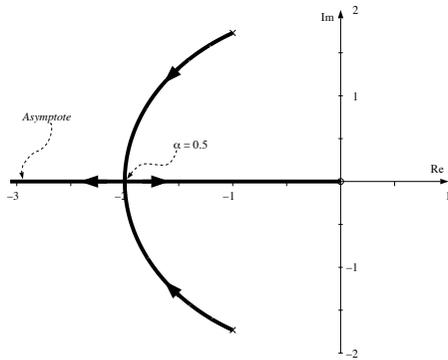


Figure 3.7d

more damped for larger values on α (compare b, c: in b) the system is not oscillating for any value on K). For α sufficiently large, the time constant can be arbitrary large. This is natural since the term $-\alpha\dot{\theta} \cdot K$ (D-term) that appears in the input voltage of the motor reduces the velocity of the axis. The effect is as if the motor has been drained with thick oil. With a suitable viscosity α the system can be made fast and stable as in c). With $\alpha = 0$ as in a) and K large enough, the system is not becoming faster* just less damped.

3.8 a)

$$\frac{\omega(s)}{\delta_{\text{ref}}(s)} = G_1(s) \cdot G_2(s) = \frac{10(s+1)}{(s+10)(s+4)(s-3)}$$

The open loop system is unstable (a pole in $s = 3$). Hence $\omega(t)$ is increasing when $\delta_{\text{ref}}(t)$ is a step signal. Observe that the model is valid for small changes with respect to a large reference input θ_0 for the pitch, and for predetermined values on the static and the dynamic pressure.

b)

$$\omega(s) = G_1(s) \cdot G_2(s) \cdot K \cdot (\omega_{\text{ref}}(s) - \omega(s))$$

*Note that although the system *does* get faster with respect to risetime, this is not a very useful notion of speed in highly oscillative systems. Rather, the settling time should be used, and this property of a system is related to the real part of the poles, not their distance to the origin.

gives

$$\begin{aligned} \omega(s) &= \frac{K \cdot G_1(s) \cdot G_2(s)}{1 + K \cdot G_1(s) \cdot G_2(s)} \omega_{\text{ref}}(s) \\ &= \frac{10K(s+1)}{(s+10)(s+4)(s-3) + 10K(s+1)} \omega_{\text{ref}}(s) \end{aligned}$$

The characteristic equation is

$$(s+10)(s+4)(s-3) + 10K(s+1) = 0$$

which gives

$$P(s) = (s+10)(s+4)(s-3) \quad Q(s) = 10(s+1)$$

◇ Starting points: \Leftrightarrow zeros of $P(s)$: $-10, -4, 3$
End points: \Leftrightarrow zeros of $Q(s)$: -1

◇ Number of asymptotes: $3 - 1 = 2$

Directions: $\frac{1}{2}(\pi + 2k\pi) = \pm\pi/2$

Intersection with the real axis: $\frac{1}{3-1}[(-10) + (-4) + 3 - (-1)] = -5$

◇ Part of the real axis that belong to the root locus: $[-10, -4]$ and $(-1, 3]$

◇ Intersection with the imaginary axis: $s = i\omega$ solves the characteristic equation

$$\begin{aligned} (i\omega + 10)(i\omega + 4)(i\omega - 3) + 10K(i\omega + 1) &= \\ = -i\omega^3 - 11\omega^2 + (10K - 2)i\omega + 10K - 120 &= 0 \\ \Rightarrow \left. \begin{aligned} (-\omega^2 + 10K - 2)\omega &= 0 \\ -11\omega^2 + 10K - 120 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \omega &= 0 \\ K &= 12 \end{aligned} \right\} \end{aligned}$$

This gives the root locus in Figure 3.8a.

Answer: All poles are in the left half plane, that is, the closed loop system is asymptotically stable for all $K > 12$.

c) The question is: Is there any $K > 12$ for which all poles are real valued? For $K = 12$ it is known that $s = 0$ is a solution to the characteristic equation. The other roots are given by

$$(s+10)(s+4)(s-3) + 10 \cdot 12(s+1) = s^2 + 11s + 118$$

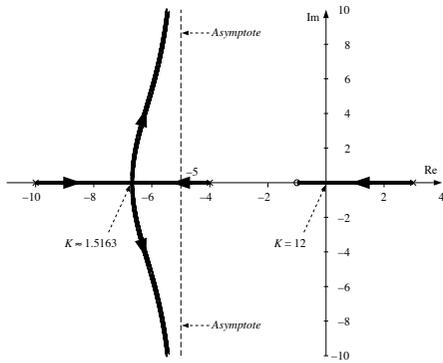


Figure 3.8a

That is, the two remaining poles are

$$-\frac{11}{2} \pm \sqrt{\left(\frac{11}{2}\right)^2 - 118} = -5.5 \pm 9.4i$$

which shows that they are complex for $K > 12$. The answer is hence no.

3.9 This text serves as a workaround for an obscure bug in L^AT_EX.

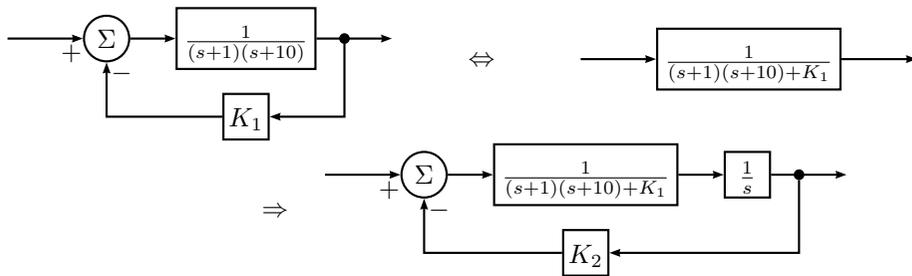


Figure 3.9a

- a) The block diagram is given in Figure 3.9a. Hence, the characteristic equation is $s \underbrace{((s+1)(s+10)+K_1)}_{P(s)} + K_2 = 0$

Start points: 0 and the roots to $s^2 + 11s + 10 + K_1 = 0$, that is, $s = -5.5 \pm \sqrt{5.5^2 - 10 - K_1}$

The roots are real when $K_1 \leq 20.25$. (I)

The roots are complex when $K_1 > 20.25$. (II)

- (I) \diamond start points: 0, $-5.5 \pm \alpha$
 \diamond end points: missing
 \diamond asymptotes: $3: \frac{\pi}{3}, \pi, \frac{5\pi}{3}$
 intersection of the asymptotes: $\frac{1}{3}(-11) = -\frac{11}{3}$
 \diamond parts of real axis: $(-\infty, -5.5 - \alpha], [-5.5 + \alpha, 0]$

The root locus is given in Figure 3.9b.

- (II) \diamond start points: 0, $-5.5 \pm i\beta$
 \diamond end points: missing
 \diamond asymptotes: as in (I)
 \diamond parts of real axis: $(-\infty, 0]$

The root locus is given in Figure 3.9c

Imaginary axis crossings: put $s = i\omega$ in the characteristic equation \Rightarrow

$$\begin{aligned} i\omega(-\omega^2 + 11i\omega + 10 + K_1) + K_2 &= 0 \quad \Leftrightarrow \\ -i\omega^3 - 11\omega^2 + 10i\omega + K_1i\omega + K_2 &= 0 \quad \Rightarrow \\ \begin{cases} -\omega^3 + 10\omega + K_1\omega = 0 & (1) \\ -11\omega^2 + K_2 = 0 & (2) \end{cases} \end{aligned}$$

$\omega = 0$ solution to (1) \Rightarrow in (2) $K_2 = 0$.

$\omega^2 = 10 + K_1$ solution to (1) $\Rightarrow -110 - 11K_1 + K_2 = 0 \Rightarrow K_2 = 11K_1 + 110$

Answer: The closed loop system is asymptotically stable when $0 < K_2 < 11K_1 + 110$

- b) By using the inner feedback ($K_1 > 0$) a larger value of K_2 is allowed.

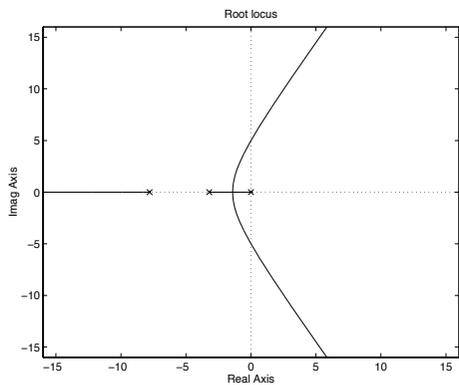


Figure 3.9b

3.10 Set

$$G(s) = \frac{1}{(s+1)(s-1)(s+5)}$$

With $U(s) = F(s)E(s)$, the transfer function of the closed loop system becomes

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

a) Here, $F(s) = K$, so

$$G_c(s) = \frac{K}{(s+1)(s-1)(s+5) + K}$$

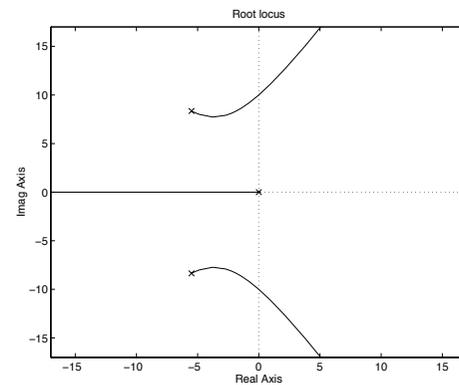


Figure 3.9c

The characteristic equation is

$$(s+1)(s-1)(s+5) + K = 0$$

which gives

$$P(s) = (s+1)(s-1)(s+5) \quad Q(s) = 1$$

- ◇ Starting points: \Leftrightarrow Zeros of $P(s)$: $-1, 1, -5$
End points: \Leftrightarrow Zeros of $Q(s)$: none
- ◇ Number of asymptotes: $3 - 0 = 3$
Directions: $\frac{1}{3}[\pi + 2k\pi] = \pi, \pm\pi/3$
Intersection point: $\frac{1}{3}[-1 + 1 + (-5)] = -5/3$

◇ Real axis: $(-\infty, -5]$ and $[-1, 1]$ belongs to the root locus

◇ Intersection with the imaginary axis, set $s = i\omega$:

$$(i\omega + 1)(i\omega - 1)(i\omega + 5) + K = -i\omega^3 - 5\omega^2 - i\omega + K - 5 = 0$$

$$\iff \begin{cases} (\omega^2 + 1)\omega = 0 \\ -5\omega^2 + K - 5 = 0 \end{cases} \iff \begin{cases} \omega = 0 \\ K = 5 \end{cases}$$

(A simple root!)

This gives the root locus in Figure 3.10a.

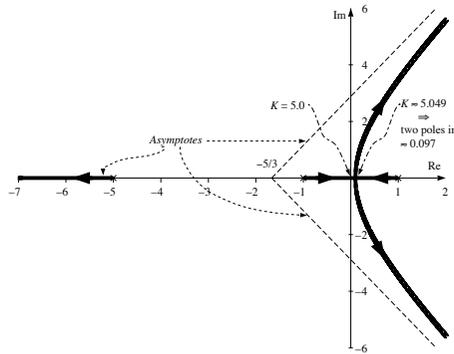


Figure 3.10a

Answer: There exists at least one pole in the RHP. Hence, the system is not asymptotically stable for any value of K .

b) Here, $F(s) = K(1 + 0.5s)$. Hence

$$G_c(s) = \frac{K(1 + 0.5s)}{(s + 1)(s - 1)(s + 5) + K(1 + 0.5s)}$$

The characteristic equation is

$$(s + 1)(s - 1)(s + 5) + K(1 + 0.5s) = 0$$

which gives

$$P(s) = (s + 1)(s - 1)(s + 5) \quad Q(s) = 1 + 0.5s$$

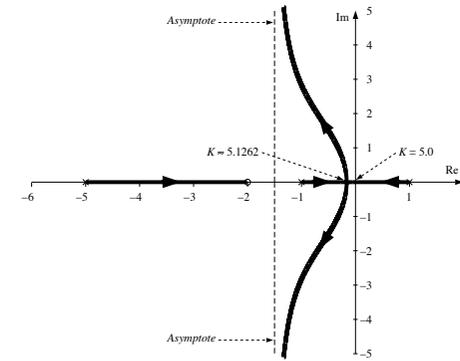


Figure 3.10b

◇ Starting points \iff Zeros of $P(s)$: $-1, 1, -5$

End points \iff Zeros of $Q(s)$: -2

◇ Number of asymptotes: $3 - 1 = 2$

Directions: $\frac{1}{2}[\pi + 2k\pi] = \pm\pi/2$

Intersection point: $\frac{1}{2}[-1 + 1 - 5 - (-2)] = -\frac{3}{2}$

◇ Real axis: $[-5, -2)$ and $[-1, 1]$ belongs to the root locus

◇ Intersection with the imaginary axis, set $s = i\omega$:

$$(i\omega + 1)(i\omega - 1)(i\omega + 5) + K(1 + 0.5i\omega) = 0$$

$$\iff \begin{cases} -\omega^3 + \omega(0.5K - 1) = 0 \\ -5\omega^2 - 5 + K = 0 \end{cases}$$

$$\iff \begin{cases} \omega = 0 \\ K = 5 \end{cases} \text{ or } \begin{cases} \omega^2 = -1, \text{ not real!} \\ K = 0 \end{cases}$$

This gives the root locus in Figure 3.10b.

Answer: The system is asymptotically stable (all poles in the LHP) if $K > 5$.

3.11 a) The closed loop system

$$G_c(s) = \frac{\frac{k}{s(s+2)}}{1 + \frac{ka}{s(s+2)(s+a)}} = \frac{k(s+a)}{s(s+2)(s+a) + ka}$$

has the characteristic equation

$$s(s+2)(s+a) + ka = 0$$

Choose $k = 6$, and draw a root locus with respect to a . The characteristic equation can be written

$$s^3 + 2s^2 + a(s^2 + 2s + 6) = 0$$

that is,

$$P(s) = s^2(s+2) \quad Q(s) = s^2 + 2s + 6$$

- ◇ Starting points: $0, 0, -2$
End points: $-1 \pm i\sqrt{5}$
- ◇ Number of asymptotes: $3 - 2 = 1$, direction: π .
- ◇ Parts of the real axis: $(-\infty, -2]$
- ◇ Intersection with the imaginary axis: $s = i\omega$

$$6a - \omega^2(2+a) + i\omega(2a - \omega^2) = 0$$

$$\text{Im: } \omega(2a - \omega^2) = 0 \quad \Leftarrow \quad \omega = 0 \text{ or } \omega^2 = 2a$$

$$\text{Re: } 6a - \omega^2(2+a) = 0$$

$$\omega = 0 \quad \Leftarrow \quad a = 0$$

$$\omega^2 = 2a \quad \Leftarrow \quad 2a - 2a^2 = 0 \quad \Leftarrow \quad a = 0 \text{ or } a = 1$$

$$\text{Intersection points: } s = 0, s = \pm\sqrt{2}$$

This gives the root locus in Figure 3.11a.

Answer: The system is asymptotically stable for $a > 1$

- b) For y to have a stationary value of 1 the system must first of all be stable. When the system is stable, the stationary value will be 1 when r is a unit step since the system contains an integrator.

Next, consider

$$y_m(t) = \sin(10t) \Rightarrow y_f(t) = \left| \frac{a}{a+10i} \right| \sin(10t + \varphi)$$

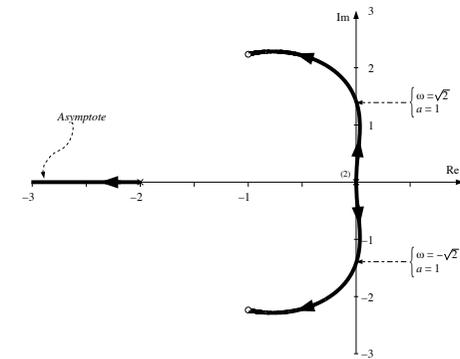


Figure 3.11a

(the expression for $y_f(t)$ is valid after a long time, that is, when the transient has vanished). The amplitude is given by

$$A = \left| \frac{a}{a+10i} \right| = \frac{1}{\sqrt{1 + \frac{100}{a^2}}}$$

Now, choose a as small as possible but $a > 1$ to maintain stability. The lowest amplitude is $A \approx 0.1$.

Answer: $A = 0.1$

- 3.12 When K is small the system has a real unstable pole, that is, the magnitude of the step response grows without bound and the step response has no oscillations $\Rightarrow K = 4$ corresponds to step response C.

When K is larger we have an unstable complex-conjugated pole pair, that is, the magnitude of the step response grows without bound and the step response is oscillative. $\Rightarrow K = 10$ corresponds to step response D.

For even larger values of K all poles end up in the LHP. As K grows the step response becomes faster since the dominating poles move away from the origin. $K = 18$ corresponds to step response B and $K = 50$ to step response A.

Answer:	K	Step
	4	C
	10	D
	18	B
	50	A

3.13

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{T_{n-1}(s)}{N_n(s)}$$

With a proportional feedback the closed loop system becomes

$$G_c(s) = \frac{G(s)}{1 + KG(s)} = \frac{T_{n-1}(s)}{N_n(s) + KT_{n-1}(s)}$$

with the characteristic equation

$$N_n(s) + KT_{n-1}(s) = 0$$

that is,

$$P(s) = N_n(s) \quad Q(s) = T_{n-1}(s)$$

- Starting points: The zeros of $N_n(s)$
End points: The zeros of $T_{n-1}(s)$
- Number of asymptotes: 1 since $\deg N_n(s) - \deg T_{n-1}(s) = 1$
Direction: π

When K tends to infinity, one root approaches $-\infty$, the remaining roots approaches the zeros of $T_{n-1}(s)$. The zeros of $T_{n-1}(s)$ are in the LHP according to the problem formulation. Hence, if K is large enough, the system is asymptotically stable.

3.14 Since $q_{\text{out},\Delta}(t) = 0$ we get

$$\begin{aligned} \frac{d}{dt} h_{\Delta}(t) &= \frac{1}{A} (q_{\text{in},\Delta}(t) - q_{\text{out},\Delta}(t)) = \frac{1}{A} q_{\text{dam},\Delta}(t - T) \\ &= \frac{K}{A} (h_{\text{ref},\Delta}(t - T) - h_{\Delta}(t - T)) \end{aligned}$$

which gives

$$sH_{\Delta}(s) = \frac{K}{A} e^{-sT} (H_{\text{ref},\Delta}(s) - H_{\Delta}(s))$$

The transfer function of the open loop system is hence

$$G_o(s) = \frac{K}{A} \cdot \frac{e^{-sT}}{s}$$

Now draw the Nyquist curve:

- Big semi-circle in the RHP:

$$s = Re^{i\theta} \quad -\pi/2 < \theta < \pi/2$$

Since $\text{Re } s > 0$ we have $|e^{-sT}| < 1$, that is,

$$|G_o(s)| < \frac{K}{A} \cdot \frac{1}{R}$$

The large half circle is hence mapped onto the origin.

- Imaginary axis:

$$|G_o(i\omega)| = \frac{K}{A} \cdot \frac{1}{\omega} \quad \arg G_o(i\omega) = -\frac{\pi}{2} - \omega T$$

As ω goes from r to R , the gain monotonically decreases towards zero and the argument goes from $-\pi/2$ to $-\infty$. The resulting Nyquist curve makes a spiral motion towards the origin. The first time the curve crosses the real axis is for $\omega T = \pi/2$, that is, $\omega = \pi$. The absolute value is then $\frac{K/A}{\pi}$.

- Small semi-circle to the right of the origin:

$$G_o(re^{i\omega}) \approx \frac{K}{A} \cdot \frac{1}{r} \cdot e^{-i\omega}$$

The small half circle is hence mapped into a large half circle in the RHP.

This gives the Nyquist path in Figure 3.14a. The system $G_o(s)$ has no poles in the RHP. According to the Nyquist criterion, the closed loop system is asymptotically stable if the Nyquist curve does not enclose the point -1 . In this case the condition reads

$$\frac{K/A}{\pi} < 1$$

Answer: $K/A < \pi$

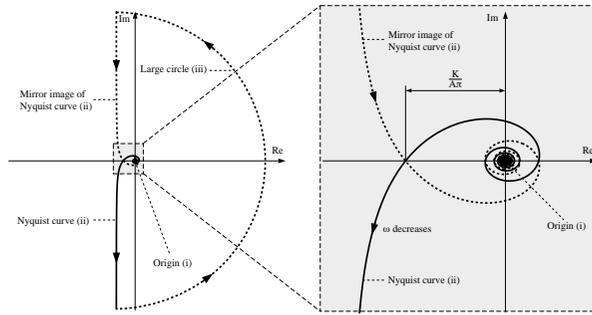


Figure 3.14a

3.15 The system $G(s)$ has no poles in the RHP. The closed loop system is asymptotically stable if the Nyquist curve of $KG_o(s)$ does not enclose the point -1 . In the problem, Nyquist diagrams for $G(s)$ are given. The axes must hence be rescaled with a factor K .

- a) (i) Yes. (ii) Yes. (iii) No. (iv) Yes.
- b) (i) Stable if $0.4K < 1$, that is, $K < 2.5$.
- (ii) Stable for $K > 0$.
- (iii) Stable if $2K < 1$, that is, $K < 1/2$.
- (iv) Stable if $4K < 1$ or $2K > 1$, that is, $K < 1/4$ or $K > 1/2$.

3.16 a) $G(i\omega) = \frac{1}{i\omega}$ gives

$$|G(i\omega)| = \frac{1}{\omega} \quad \arg G(i\omega) = -90^\circ$$

b) $G(i\omega) = \frac{1}{-\omega^2}$ gives

$$\arg G(i\omega) = \frac{1}{\omega^2} \quad \arg G(i\omega) = -180^\circ$$

This gives the Nyquist curves in Figure 3.16a.

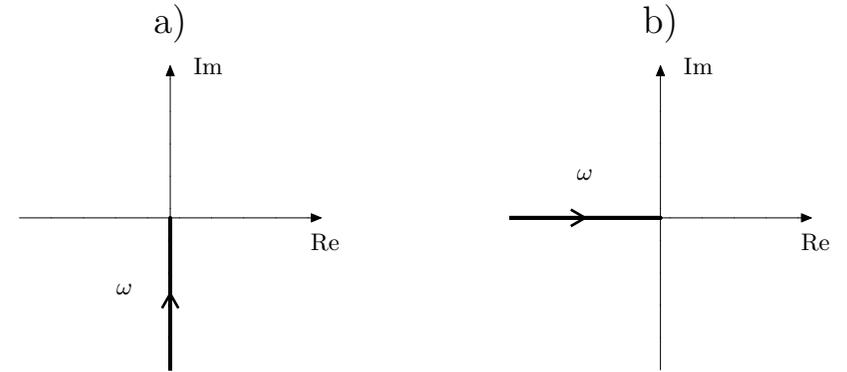


Figure 3.16a

3.17 a) Since $G(i\omega) \rightarrow 0$, $\omega \rightarrow \infty$, we assume that the large half circle is mapped onto the origin. The small half circle is mapped onto the point 2. The point -1 must not be encircled by the curve. This means that the closed loop system is stable if $1.5 \cdot K < 1$. Hence $K < 2/3$.

b)

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \cdot \frac{1}{1 + KG(s)} \cdot \frac{1}{s} = \frac{1}{1 + 2K}$$

for $K < 2/3$ according to a.

c) The Nyquist criterion can also be applied to

$$\frac{K}{s} \cdot G(s)$$

as the open loop system. On the large half circle $\frac{1}{s} \approx 0$ which means that it is mapped onto the origin even for $\frac{1}{s} \cdot G(s)$. On the small half circle

$$s = r \cdot e^{i\theta} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

we have $G(s) \approx 2$ and

$$\frac{1}{s} = \frac{1}{r} e^{-i\theta}$$

Hence, it is transformed by $\frac{1}{s} \cdot G(s)$ to a large half circle in the RHP. Setting $s = i\omega$ in $\frac{1}{s}$ gives the absolute value $\frac{1}{\omega}$ and the argument $-\pi/2$.

The Nyquist curve is turned 90° and “increased” by a factor $\frac{1}{\omega}$. This gives the Nyquist path in Figure 3.17a.

Answer: The closed loop system is asymptotically stable if $\frac{3}{2}K < 1$. This means that also in this case we have $K < 2/3$.

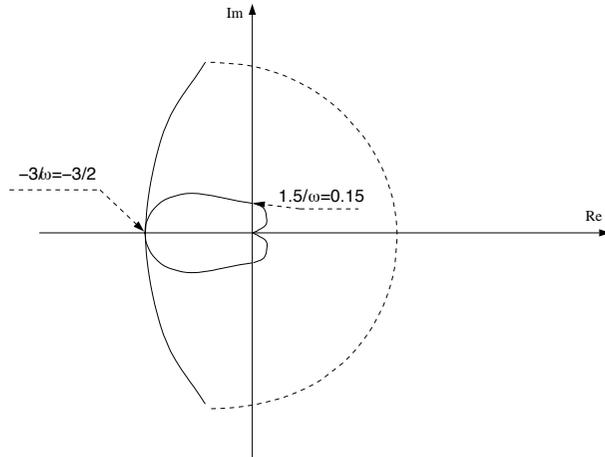


Figure 3.17a

3.18 The system

$$G(s) = \frac{1}{s(1 + \tau s)}$$

is controlled by

$$u(t) = -Ky(t - T)$$

which gives the open loop system

$$G_o(s) = Ke^{-sT}G(s)$$

During self oscillations the open-loop gain is equal to -1 :

$$Ke^{-i\omega T}G(i\omega) = -1$$

that is,

$$Ke^{-i\omega T} \cdot \frac{1}{\omega} e^{-i\frac{\pi}{2}} \frac{1}{\sqrt{\omega^2\tau^2 + 1}} e^{-i\arctan \omega\tau} = e^{-i\pi}$$

$\omega = 1$ gives

$$\begin{cases} -T - \frac{\pi}{2} - \arctan \tau = -\pi & (1) \\ \frac{K}{\sqrt{\tau^2 + 1}} = 1 & (2) \end{cases}$$

$K_1 = \frac{1}{3}K$ gives a self oscillations with $\omega = 0.5$. This gives

$$\begin{cases} -\frac{T_1}{2} - \frac{\pi}{2} - \arctan \frac{\tau}{2} = -\pi & (3) \\ \frac{K_1}{0.5 \cdot \sqrt{\frac{\tau^2}{4} + 1}} = 1 & (4) \end{cases}$$

The equations (1) - (4) give $\tau = 1.69$ and hence

$$T = \frac{\pi}{2} \cdot \arctan \tau = 0.53$$

$$T_1 = \pi - 2 \arctan \frac{\tau}{2} = 1.74$$

3.19 From the Nyquist curve it is seen that for $\omega = 1$

$$\arg G(1i) = -135^\circ \quad |G(1i)| = 1/\sqrt{2}$$

and

$$\arg F(1i) = -45^\circ \quad |F(1i)| = K/\sqrt{2}$$

This gives $\arg F(1i)G(1i) = -180^\circ$. According to the Nyquist criterion, asymptotic stability is achieved if

$$|F(1i)G(1i)| = K/2 < 1 \quad \Rightarrow \quad K < 2$$

3.20 Since $|G(i\omega)|$ does not tend to ∞ as $\omega \rightarrow 0$ the system does not have integrating factor for $K = 0$. Thus reject root locus no 2. Furthermore, since the gain can be increased arbitrarily without causing the Nyquist curve to encircle -1 , that is, without making the closed loop system unstable, we reject root loci 3 and 4.

Answer: Root locus no 1.

$$\begin{array}{lll} 3.21 & \text{P} & \Rightarrow \quad b_0 = b_2 = 0 \\ & \text{I} & \Rightarrow \quad b_0 = b_1 = 0 \\ & \text{D} & \Rightarrow \quad b_1 = b_2 = 0 \end{array}$$

3.22 a) The characteristic equation of the closed loop system is given by

$$(s^2 + s + 1)(s + 0.2) + K_P \cdot 0.2 = 0$$

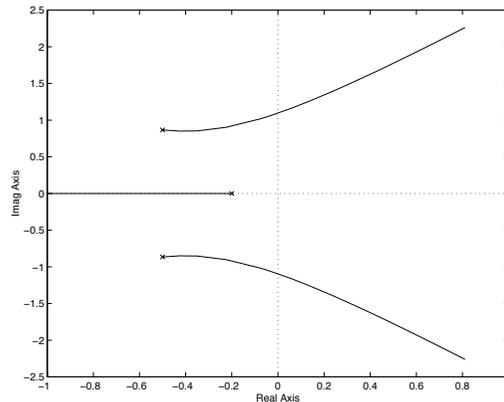
that is,

$$P(s) = (s^2 + s + 1)(s + 0.2) \quad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> s = tf( 's' );
>> P = ( s^2 + s + 1 ) * ( s + 0.2 );
>> Q = 0.2;
>> rlocus( Q / P )
```

Draw the root locus. Click in the figure to determine the imaginary axis crossings.



When K_P increases the two complex poles move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_P \approx 6.2$, the poles cross the imaginary axis and the closed loop system becomes unstable. This result is in accordance with Problem 3.5. For small values of K_P the properties of the step response are mainly determined by the real pole close to the origin. For larger values the complex poles start to dominate and when the complex poles cross the imaginary axis the amplitude of the oscillations in the step response increases and the system becomes unstable.

Note, however, that the root locus alone does not give sufficient information to tell how the stationary error changes with the parameter.

b) The characteristic equation of the closed loop system using the PI controller with $K_P = 1$ is given by

$$s((s^2 + s + 1)(s + 0.2) + 0.2) + K_I \cdot 0.2 = 0$$

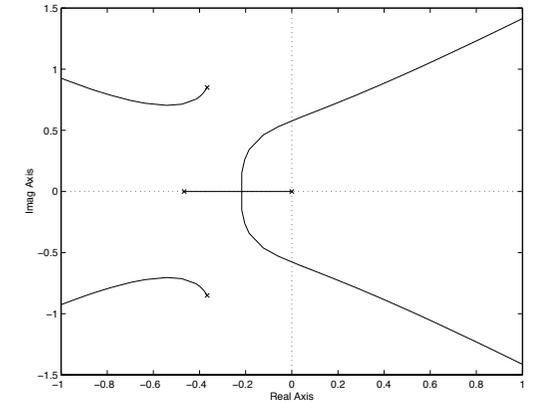
that is,

$$P(s) = s(s^3 + 1.2s^2 + 1.2s + 0.4) \quad Q(s) = 0.2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = s * ( s^3 + 1.2*s^2 + 1.2*s + 0.4 );
>> Q = 0.2;
>> rlocus( Q / P )
```

Draw the root locus. Click in the figure to determine the imaginary axis crossings.



For small K_I the response of the closed loop system is dominated by the poles on the real axis close to the origin. When K_I increases the poles become complex and move towards the imaginary axis, that is, the closed loop system becomes more oscillatory. Finally, for $K_I \approx 1.5$, the poles cross the imaginary axis, that is, the closed loop system becomes unstable. As can be seen in Problem 3.5 a small value of K_I , that is, a pole close to the origin, gives a slow step response. When K_I increases the dominating poles become complex and the step response becomes oscillatory.

A large settling time will typically follow if the system is slow or have poor damping. Here, the large settling time for small K_I is due to the system being slow. That the steady state error is eliminated cannot easily be seen in the root locus.

c) Using PID control with $K_P = 1$, $K_I = 1$ and $T = 0.1$ the characteristic equation of the closed loop system is given by

$$(0.1s + 1)(s(s^2 + s + 1)(s + 0.2) + 0.2(s + 1)) + K_D \cdot 0.2s^2 = 0$$

that is,

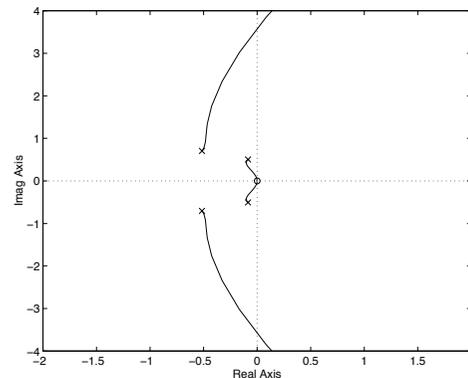
$$P(s) = (0.1s + 1)(s^4 + 1.2s^3 + 1.2s^2 + 0.4s + 0.2) \quad Q(s) = 0.2s^2$$

Enter $P(s)$ and $Q(s)$.

```
>> P = ( 0.1*s + 1 ) * ...
      ( s^4 + 1.2*s^3 + 1.2*s^2 + 0.4*s + 0.2 );
>> Q = 0.2*s^2;
```

Draw the root locus. By changing the axes or using the function `zoom` the region of interest can be seen more clearly (there is also a fifth pole which is of less interest since it is located on the negative real axis, far away from the origin).

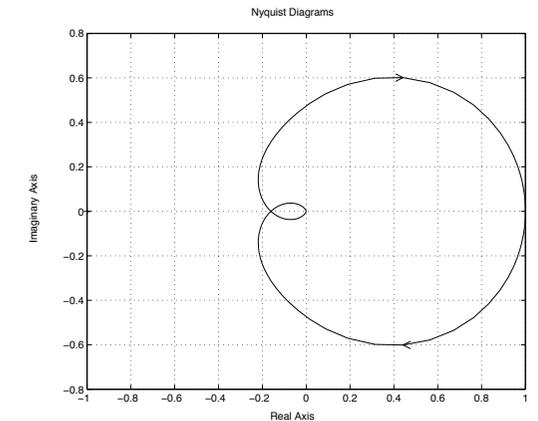
```
>> rlocus( Q / P )
>> axis([ -2 2 -4 4 ])
```



When K_D increases the complex poles closest to the origin move towards the origin and at the same time the damping of the system is increased. When K_D increases even more the second pair of complex poles moves towards the imaginary axis giving a high frequency oscillation which finally gives instability.

3.23 a) Enter the system and the regulator. Plot the Nyquist curve of the open loop system.

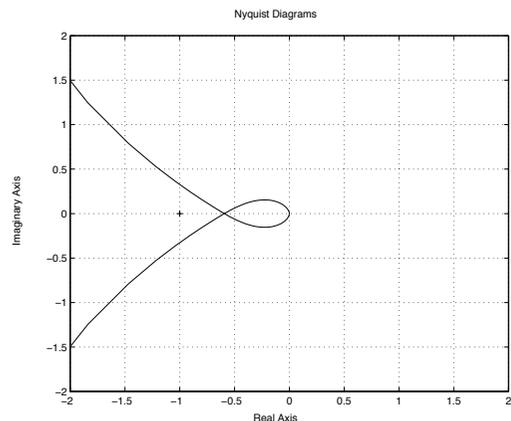
```
>> s = tf( 's' );
>> G = 0.2 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
>> F = 1;
>> nyquist( F * G )
```



The Nyquist curve is “far away” from the point -1 for all frequencies and the step response of the closed loop system is well damped. As K_P increases the Nyquist curve grows in size and for $K_P = 6.2$ the Nyquist curve reaches -1 and thus is the limit of stability.

b) Generate a PI controller.
Plot the Nyquist curve of the open loop system.

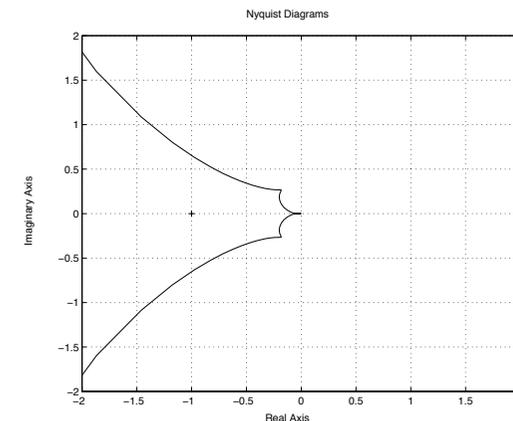
```
>> F = 1 + 1/s;
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```



For low frequencies the Nyquist curve is now far away from the origin since the integrating part makes $|G(i\omega)|$ large for low frequencies. The Nyquist curve now passes closer to -1 which results in a more oscillatory closed loop system. The system becomes unstable around $K_I = 1.44$.

c) Generate a PID controller.
Plot the Nyquist curve of the open loop system. Here with the parameters $K_P = 1$, $K_I = 1$, $K_D = 2$, and $T = 0.1$

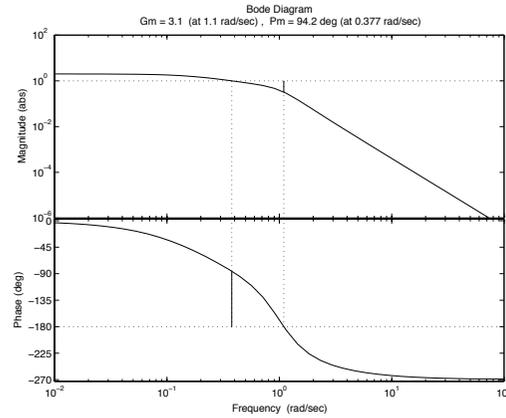
```
>> F = 1 + 1/s + 2*s / ( 0.1*s + 1 );
>> nyquist( F * G )
>> axis([ -2 2 -2 2 ])
```



The Nyquist curve is now further away from -1 which corresponds to an improved damping of the closed loop system. The system becomes unstable around $K_D = 66$.

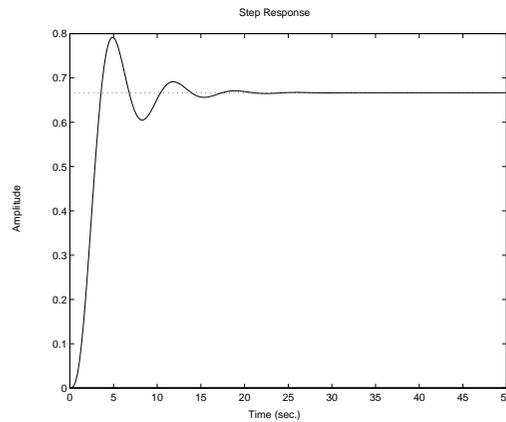
3.24 a) Enter the systems and the regulator. Make a Bode plot of the open loop system when the regulator and the system are put in series. This gives $\omega_c = 0.38$, $\omega_p = 1.1$, $\varphi_m = 94^\circ$ and $A_m = 3.1$.

```
>> s = tf( 's' );
>> G = 0.4 / ( ( s^2 + s + 1 ) * ( s + 0.2 ) );
>> F = 1;
>> margin( F * G )
```



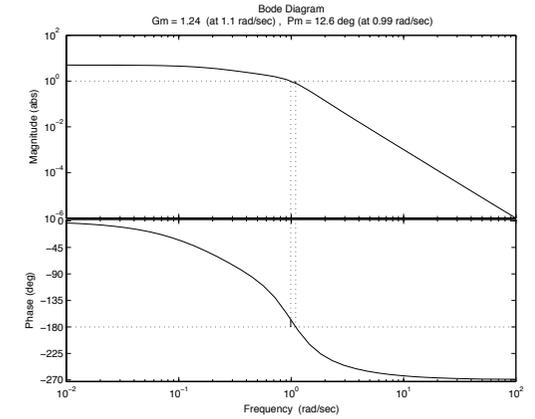
Plot the step response.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



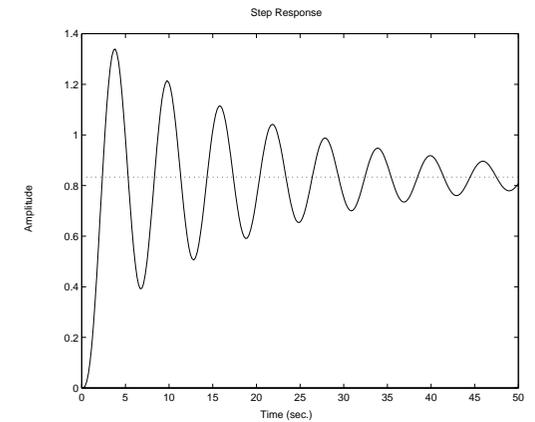
b) Increase the gain in the regulator. Make a Bode plot. The crossover frequency ω_c has increased while ω_p is the same, since only the amplitude curve is changed when the gain is changed. Both the gain and phase margins have decreased.

```
>> F = 2.5;
>> margin( F * G )
```



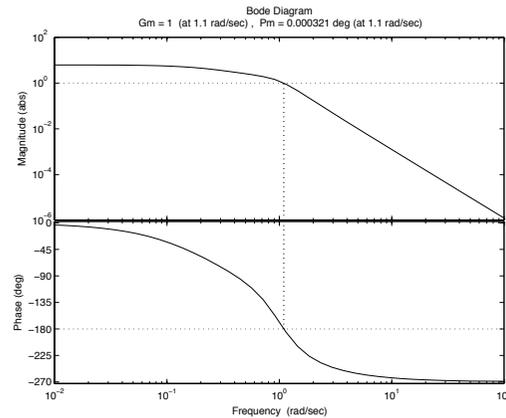
Plot the step response. The closed loop system is now much more oscillatory due to the reduced phase and gain margins.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



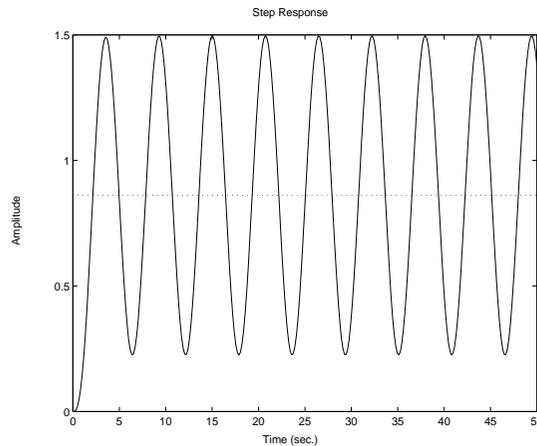
c) Increase the gain to 3.1, that is, the value of A_m in a). Both the gain and phase margin are at the limit between what would give an stable or unstable closed loop system. Any further increase of the gain will give an unstable closed loop system.

```
>> F = 3.1;
>> margin( F * G )
```



Plot the step response. The output now oscillates with constant amplitude.

```
>> Gc = feedback( F * G, 1 );
>> step( Gc, 50 )
```



3.25 The top row gives a steady state error $\Rightarrow K_I = 0$. Left column less oscillative than the right one $\Rightarrow K_D \neq 0$.

Answer: A-iii, B-i, C-iv, D-ii.

3.26 a) The motor transfer function is (from Solution 2.1)

$$\frac{\theta(s)}{U(s)} = G(s) = \frac{k_0}{s(s + 1/\tau)}$$

Feedback control

$$U(s) = F(s)(\theta_{\text{ref}}(s) - \theta(s))$$

where $F(s)$ is the control law transfer function and θ_{ref} is the reference signal. The closed loop transfer function is given by

$$G_c(s) = \frac{\theta(s)}{\theta_{\text{ref}}(s)} = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

Proportional feedback $F(s) = K_P$ and $G(s)$ according to above give

$$G_c(s) = \frac{K_P k_0}{s^2 + s/\tau + K_P k_0}$$

The poles of the closed loop system are given by

$$s^2 + s/\tau + K_P k_0 = 0$$

that is,

$$s = \frac{-1 \pm \sqrt{1 - 4\tau^2 K_P k_0}}{2\tau}$$

- (1) K_P small \Rightarrow Both poles on the real axis, but one pole very close to the origin \Rightarrow Slow but not oscillatory system.
- (2) $K_P = 1/(4\tau^2 k_0) \Rightarrow$ Both poles in $-1/(2\tau)$, that is, faster than in (1) but still no oscillations.
- (3) K_P large \Rightarrow Complex poles with large imaginary part relative to the real part, that is oscillative system.

b) The transfer function from the reference signal to the tracking error $e = \theta_{\text{ref}} - \theta$ is given by

$$E(s) = \frac{1}{1 + F(s)G(s)} \theta_{\text{ref}}(s) = \frac{s(s + 1/\tau)}{s(s + 1/\tau) + K_P k_0} \theta_{\text{ref}}(s)$$

The reference signal is a step

$$\theta_{\text{ref}}(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\text{ref}}(s) = \frac{A}{s}$$

The final value theorem gives

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot \frac{s(s + 1/\tau)}{s(s + 1/\tau) + K_P k_0} \cdot \frac{A}{s} = 0$$

The reference signal is a ramp

$$\theta_{\text{ref}}(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$$

which gives

$$\theta_{\text{ref}}(s) = \frac{A}{s^2}$$

The final value theorem gives (the closed loop is asymptotically stable for all K_P according to a))

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s \cdot \frac{s(s + 1/\tau)}{s(s + 1/\tau) + K_P k_0} \cdot \frac{A}{s^2} = \frac{A}{K_P k_0 \tau}$$

The error can be decreased by selecting K_P large, but according to a) the system becomes very oscillative for large K_P .

c) PI controller

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau$$

that is

$$F(s) = K_P + K_I \frac{1}{s}$$

gives

$$E(s) = \frac{1}{1 + F(s)G(s)} \theta_{\text{ref}}(s) = \frac{s^2(s + 1/\tau)}{s^2(s + 1/\tau) + k_0(K_P s + K_I)} \theta_{\text{ref}}(s)$$

When θ_{ref} is a ramp according to b) we get

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0$$

Comment: The final value theorem can only be used when the denominator of $G(s)U(s)$ has all zeros in the left half plane or at the origin. $G(s)$ is the system transfer function and $U(s)$ is the input signal.

3.27 The transfer function for the loop gain is G_o .

The transfer function from the reference signal R to the output Y is obtained by using the block diagram and observing that

$$Y = G_o(R - Y)$$

Solving this equation for Y gives

$$Y = \frac{G_o}{1 + G_o} R$$

that is, the transfer function for the closed loop system is $G_c = \frac{G_o}{1 + G_o}$.

3.28 a) The loop gain, G_o , is FG .

b) The influence of the disturbance ($N = 0$) can be neglected. Use the solution to problem Solution 3.27. The transfer function from R to Y is $G_c = \frac{FG}{1 + FG}$, that is, $Y = G_c R$.

c) The influence of the reference signal can be neglected. ($R = 0$). The block diagram gives

$$Y = FGE = -FG(Y + N)$$

which implies that the transfer function from N to Y is $G_{ny} = -\frac{FG}{1 + FG}$.

- d) The influence of the disturbance can be neglected ($N = 0$). The block diagram gives

$$E = R - Y = R - FGE$$

Solving for E gives

$$E = \frac{1}{1 + FG}R$$

that is, the transfer function from R to E is $G_{re} = \frac{1}{1+FG}$.

- 3.29 a) The transfer function from reference signal to error signal is (see Solution 3.28d)

$$E = \frac{1}{1 + FG}R(s) = \frac{1}{1 + \frac{K}{(s+1)(s+3)}}R(s) = \frac{(s+1)(s+3)}{(s+1)(s+3) + K}R(s)$$

$r(t)$ step $\Rightarrow R(s) = \frac{A}{s}$. The steady state value of the error is given by the final value theorem

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{(s+1)(s+3)}{(s+1)(s+3) + K}A = \frac{3A}{3+K}$$

- b) In order to make the steady state error equal to zero the regulator has to contain an integrator. Using, for example, $F(s) = \frac{1}{s}$ one gets

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} \frac{A}{1 + F(s)G(s)} = \lim_{s \rightarrow 0} \frac{A}{1 + \frac{1}{s} \frac{1}{(s+1)(s+3)}} = 0$$

Notice though, that the integrating feedback normally has to be combined with proportional feedback.

- c) The transfer function from R to Y using $F(s) = 1$ is

$$G_c(s) = \frac{FG}{1 + FG} = \frac{1}{(s+1)(s+3) + 1} = \frac{1}{s^2 + 4s + 4} = \frac{1}{(s+2)^2}$$

The system has two poles in -2 and no zeros.

- 3.30 • The four step responses are characterized by, for example, that A and D have a steady state error, while C and B do not. Further, A shows better damping than D, and C shows better damping than B. It can also be noticed (although it is not as apparent as the other characteristics) that the error decays more slowly in C than in B.

- The four regulators are characterized by, for example, that regulators 1 and 4 don't have any integral action. Regulator 2 has more integral action than 3, and regulator 4 gives better damping than 1.
- The derivative part in the regulator improves the damping, while integral action eliminates the steady state error and reduces the damping. Besides, for small values of K_I , the error will decay slowly to zero.

Answer: A-4, B-2, C-3, D-1.

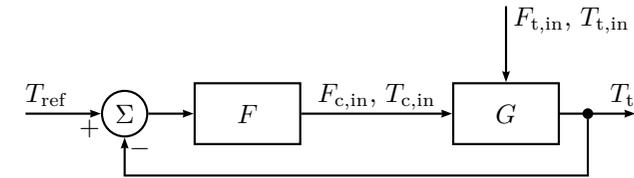


Figure 3.31a

- 3.31 a) See the block diagram in Figure 3.31a. There, the signals are classified as:

- ◇ Input $F_{c,in}$ and $T_{c,in}$
- ◇ Output T_t
- ◇ Disturbance $F_{t,in}$ and $T_{t,in}$

- b) Assume perfect mixing in the tank. Mass balance for the tank

$$\frac{d(\rho_t V_t)}{dt} = \rho_{t,in} F_{t,in} - \rho_t F_t$$

Assume $\rho_{t,in} = \rho_t$ and that ρ_t is constant which gives

$$\frac{d(\rho_t V_t)}{dt} = 0 = F_{t,in} - F_t \Rightarrow F_t = F_{t,in}$$

Assume that there are no heat losses to the surroundings. The energy balance for the tank is

$$\begin{aligned} \frac{d(\rho_t V_t c_{t,in}^p (T_t - T_{ref}))}{dt} \\ = \rho_t F_{t,in} c_{t,in}^p (T_{t,in} - T_{ref}) - \rho_t F_t c_t^p (T_t - T_{ref}) + U(T_c - T_t) \end{aligned} \quad (3.1)$$

where U is a heat transfer constant. Assume that $c_{t,\text{in}}^{\text{p}} = c_t^{\text{p}}$ is constant and that T_{ref} is constant. This means that (3.1) can be simplified to

$$V_t \frac{dT_t}{dt} = F_t(T_{t,\text{in}} - T_t) + \frac{U}{c_t^{\text{p}} \rho_t} (T_c - T_t) \quad (3.2)$$

Mass balance for the heating system

$$\frac{d(\rho_c V_c)}{dt} = \rho_{c,\text{in}} F_{c,\text{in}} - \rho_c F_c$$

Assume $\rho_{c,\text{in}} = \rho_c$ and that ρ_c is constant which gives $F_c = F_{c,\text{in}}$. Assume that there are no heat losses to the surroundings. The energy balance for the heating system is

$$\begin{aligned} \frac{d(\rho_c V_c c_c^{\text{p}} (T_c - T_{\text{ref}}))}{dt} \\ = \rho_c F_{c,\text{in}} c_{t,\text{in}}^{\text{p}} (T_{c,\text{in}} - T_{\text{ref}}) - \rho_c F_c c_c^{\text{p}} (T_c - T_{\text{ref}}) - U (T_c - T_t) \end{aligned} \quad (3.3)$$

Assume that $c_c^{\text{p}} = c_{t,\text{in}}^{\text{p}}$ is constant. This means that (3.3) can be simplified to

$$V_c \frac{dT_c}{dt} = F_c (T_{c,\text{in}} - T_c) - \frac{U}{c_c^{\text{p}} \rho_c} (T_c - T_t) \quad (3.4)$$

The dynamical model is described by (3.2) and (3.4).

c) Linearization of (3.2) and (3.4) (assuming $\rho_c = \rho_t$ and $c_c^{\text{p}} = c_t^{\text{p}}$) gives

$$\begin{aligned} V_t^* \frac{dT_{t\Delta}}{dt} &= - \left(F_t^* + \frac{U}{c_t^{\text{p}} \rho_t} \right) T_{t\Delta} + F_t^* T_{t,\text{in}\Delta} \\ &\quad + \frac{U}{c_t^{\text{p}} \rho_t} T_{c\Delta} + (T_{t,\text{in}}^* - T_t^*) F_{t\Delta} \\ V_c^* \frac{dT_{c\Delta}}{dt} &= - \left(F_c^* + \frac{U}{c_t^{\text{p}} \rho_t} \right) T_{c\Delta} + F_c^* T_{c,\text{in}\Delta} \\ &\quad + \frac{U}{c_t^{\text{p}} \rho_t} T_{c\Delta} + (T_{c,\text{in}}^* - T_c^*) F_{c\Delta} \end{aligned}$$

d) With numerical values for the stationary points and assuming that F_t , $T_{t,\text{in}}$, and $T_{c,\text{in}}$ is constant, the linearized model is

$$\frac{dT_{t\Delta}}{dt} = -0.26T_{t\Delta} + 0.16T_{c\Delta} \quad (3.5)$$

$$\frac{dT_{c\Delta}}{dt} = -3.6T_{c\Delta} + 1.6T_{t\Delta} + 200F_{c\Delta} \quad (3.6)$$

Taking the Laplace transform of (3.5) and (3.6) gives

$$sT_{t\Delta}(s) = -0.26T_{t\Delta}(s) + 0.16T_{c\Delta}(s) \quad (3.7)$$

$$sT_{c\Delta}(s) = -3.6T_{c\Delta}(s) + 1.6T_{t\Delta}(s) + 200F_{c\Delta}(s)$$

$$\Rightarrow T_{c\Delta}(s) = \frac{1.6}{s + 3.6} T_{t\Delta}(s) + \frac{200}{s + 3.6} F_{c\Delta}(s) \quad (3.8)$$

Combining (3.7) and (3.8) gives

$$\begin{aligned} sT_{t\Delta}(s) &= -0.26T_{t\Delta}(s) + \frac{0.256}{s + 3.6} T_{t\Delta}(s) + \frac{32}{s + 3.6} F_{c\Delta}(s) \\ \Rightarrow T_{t\Delta}(s) &= \frac{32}{(s + 3.675)(s + 0.185)} F_{c\Delta}(s) \end{aligned}$$

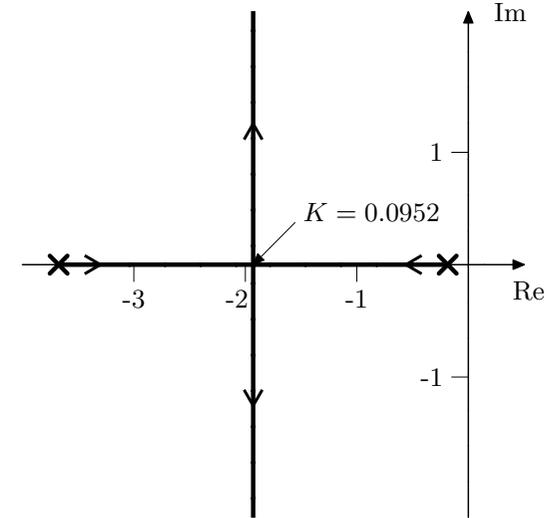


Figure 3.31b

e) The transfer function for the closed loop is

$$G_c(s) = \frac{32K}{s^2 + 3.86s + 0.68 + 32K}$$

The characteristic equation is

$$s^2 + 3.86s + 0.68 + 32K = 0$$

with the solution

$$s = -1.93 \pm \sqrt{1.93^2 - 0.68 - 32K}$$

This gives the root locus in Figure 3.31b.

3.32 a) The system $G_o(s) = \frac{-1}{s^2+2s-3}$ has one pole in -3 and one pole in 1 , hence the system is unstable.

b) The closed loop is given by

$$G_c(s) = \frac{-K}{s^2 + 2s - 3 - K}$$

The poles of the closed loop are given by

$$s = -1 \pm \sqrt{1 + 3 + K}$$

For $K \leq -3$ the closed loop will have all its pole in the LHP.

3.33 Given $\dot{y} = \mu y + u$ and $u = K(r - y)$ we have $\dot{y} = (\mu - K)y + Kr$. This system converges when the pole $(\mu - K)$ is in the LHP, that is, when $K > \mu$.

3.34 The dynamics of the astronaut is given by

$$F = ma$$

where $m = 100$, F is the control signal u and $a = \ddot{y}$. This gives the model

$$100\ddot{y} = u$$

and

$$Y(s) = \frac{1}{100s^2}U(s)$$

The control law is given by

$$u = K_1(r - y) - K_1K_2\dot{y} = K_1((r - y) - K_2\dot{y})$$

or

$$U(s) = K_1(R(s) - Y(s) - K_2sY(s))$$

The transfer function from r to e is given by

$$E(s) = \frac{s^2 + 0.01K_1K_2s}{s^2 + 0.01K_1K_2s + 0.01K_1}R(s)$$

When $r(t) = t$ we have

$$R(s) = \frac{1}{s^2}$$

The final value theorem then gives (provided that K_1 and K_2 are chosen such that the closed loop is asymptotically stable) (also note that the transfer function from r to e must have at least one zero at the origin for the final value to exist, but this is satisfied regardless of the choice of K_1 and K_2)

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = K_2 < 1$$

The transfer function from r to y is given by

$$G_c(s) = \frac{Y(s)}{R(s)} = \frac{0.01K_1}{s^2 + 0.01K_1K_2s + 0.01K_1}$$

The standard form for the characteristic equation

$$s^2 + 2\zeta\omega_0s + \omega_0^2 = 0$$

gives with $\zeta = 1/\sqrt{2} \approx 0.7$

$$s^2 + \sqrt{2}\omega_0s + \omega_0^2 = 0$$

A comparison with

$$s^2 + 0.01K_1K_2s + 0.01K_1 = 0$$

gives $\omega_0 = 0.1\sqrt{K_1}$. We hence obtain

$$K_1 = \frac{200}{K_2^2}$$

Answer: Choose $K_2 < 1$ and $K_1 = 200/K_2^2$.

4 Frequency Description

4.1 If we let $\bar{u}(t)$ and $\bar{y}(t)$ denote the actual temperature and the measured temperature, respectively, we can divide the temperatures into their mean values and variations as follows:

$$\bar{u}(t) = u_0 + u(t)$$

and

$$\bar{y}(t) = y_0 + y(t)$$

where $u_0 = y_0 = 30^\circ\text{C}$.

The thermometer is modeled as the following first order linear time invariant dynamic system with

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s + b}$$

Based on the assumption that the thermometer has been calibrated so that y_0/u_0 reflects the static gain of the system, it would be possible to conclude that $|G(0)| = 1$, that is, $a = b$. However, we will be able to find enough equations to determine both a and b without using $a = b$, allowing us to avoid this additional assumption. (Using this assumption leads to slightly different estimates for a and b , depending on which additional equation we use to determine the common value. That the estimates are not the same no matter which equations we use is explained by the uncertainty in the equations.)

Since

$$u(t) = A \sin(\omega t)$$

it follows that after the transients have vanished (that is, in steady state)

$$y(t) = |G(i\omega)| A \sin(\omega t + \phi)$$

where

$$\phi = \arg(G(i\omega)) = -\arctan(\omega/b)$$

From the relationship $\omega = 2\pi/T$ and from the figure the following is obtained:

1. $\omega = \frac{2\pi}{0.314 \cdot 60} \text{ rad/s} = 0.33 \text{ rad/s}$
2. $\phi = \frac{-0.056}{0.314} \cdot 2\pi \text{ rad} = -1.12 \text{ rad}$
3. $|G(i\omega)| = \frac{0.9}{2.0} = 0.45$

Hence

$$\tan(\phi) = -\frac{\omega}{b} \Rightarrow b = \frac{0.33}{2.066} = 0.16$$

and

$$|G(i\omega)| = \frac{a}{\sqrt{\omega^2 + b^2}} \Rightarrow a = 0.16$$

Answer:

$$G(s) = \frac{0.16}{s + 0.16}$$

4.2 The equation

$$\omega = \dot{\psi}$$

and

$$T_1 \cdot \dot{\omega} = -\omega + K_1 \cdot \delta$$

give the transfer function

$$G_s(s) = \frac{K_1}{s(1 + T_1 s)} = \frac{0.1}{s(1 + s/0.01)}$$

The transfer function of the rudder machine is

$$G_r(s) = \frac{1}{1 + sT_2} = \frac{1}{1 + s/0.1}$$

and the controller has the transfer function

$$F(s) = K \frac{1 + s/a}{1 + s/b} = K \frac{1 + s/0.02}{1 + s/0.05}$$

a) $K = 0.5$ gives

$$G_o(s) = F(s)G_r(s)G_s(s) = \frac{0.05(1 + s/0.02)}{s(1 + s/0.01)(1 + s/0.05)(1 + s/0.1)}$$

It thus follows that

$$|G_o(i\omega)| = \frac{0.05\sqrt{1 + (\frac{\omega}{0.02})^2}}{\omega\sqrt{1 + (\frac{\omega}{0.01})^2}\sqrt{1 + (\frac{\omega}{0.05})^2}\sqrt{1 + (\frac{\omega}{0.1})^2}}$$

with low frequency asymptote

$$|G_o(i\omega)| \rightarrow \frac{0.05}{\omega}, \quad \omega \rightarrow 0$$

and

$$\arg G_o(i\omega) = \arctan \frac{\omega}{0.02} - 90^\circ - \arctan \frac{\omega}{0.01} - \arctan \frac{\omega}{0.05} - \arctan \frac{\omega}{0.1}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, $|\frac{0.05}{0.005}| = 10$, and the breakpoints and slopes of the asymptotes:

Frequency [rad/s]		0.01		0.02		0.05		0.1	
Slope	-1		-2		-1		-2		-3

The phase shift is drawn based on a couple of samples:

Frequency [rad/s]	0.005	0.01	0.02	0.04	0.08
Phase	-111°	-125°	-142°	-163°	-194°

The Bode plot in Figure 4.2a gives: $\omega_c = 0.026$ rad/s, $\varphi_m = 32^\circ$, $A_m = 4.2$.

- b) The system starts to oscillate if K is chosen so that $\arg(G_o(i\omega_c)) = -180^\circ$. This gives the crossover frequency $\omega_c = \omega_p = 0.06$ rad/s. This implies that the gain should be amplified 4.2 times. Therefore, choose $K = 0.5 \cdot 4.2 = 2.1$.

$$\omega = 2\pi/T \quad \Rightarrow \quad T = \frac{2\pi}{\omega_c} = \frac{2\pi}{0.06} = 105 \text{ s}$$

Answer: The period time will be 105 seconds, and $K = 2.1$.

c)

$$\Psi_{\text{ref}}(t) = A \sin(\alpha t)$$

gives

$$\Psi(t) = B \sin(\beta t + \varphi)$$

where $A = 5^\circ$, $\alpha = 0.02$, $\beta = \alpha$, $B = A |G_c(i\alpha)|$ and $\varphi = \arg G_c(i\alpha)$. The transfer function for the closed loop system when $K = 0.5$ is

$$G_c(s) = \frac{G_o(s)}{1 + G_o(s)}$$

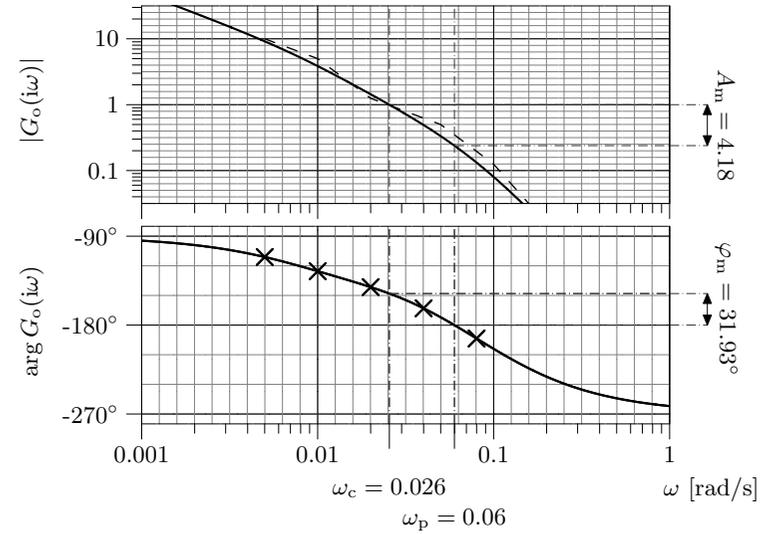


Figure 4.2a

where

$$|G_o(i0.02)| = 1.44 \quad \arg G_o(i0.02) = -142^\circ$$

That is

$$G_o(i0.02) = -1.135 - i0.886$$

which gives

$$|G_c(i0.02)| = \frac{1.44}{\sqrt{0.135^2 + 0.886^2}} = 1.61 \quad \Rightarrow \quad B = 8^\circ$$

and

$$\arg G_c(i0.02) = -142^\circ + 180^\circ - \arctan\left(\frac{0.886}{0.135}\right) = -0.76 \text{ rad}$$

Answer: $B = 8^\circ$, $\beta = 0.02$ rad/s and $\varphi = -0.76$ rad.

- 4.3 a) As $\omega \rightarrow 0$, $|G(i\omega)| \rightarrow \infty$ and $\arg G(i\omega) \rightarrow -90^\circ$. The gain is first decreasing (low frequencies). It then increases, and finally decreases again

(approaching zero for high frequencies). The phase shift is increasing at low frequencies. As the frequency becomes higher the phase shift is positive in an interval until it decreases towards -90° . This gives the plot in Figure 4.3a.

- b) A system with a Bode plot as the one shown above must have one pole in the origin since $\arg G(i\omega) \rightarrow -90^\circ$ as $\omega \rightarrow 0$. Then two break points appear (up), since there is a positive phase shift. After that, there must be two break points (down), since the phase shift should approach -90° . Hence, the plot in Figure 4.3b is possible.

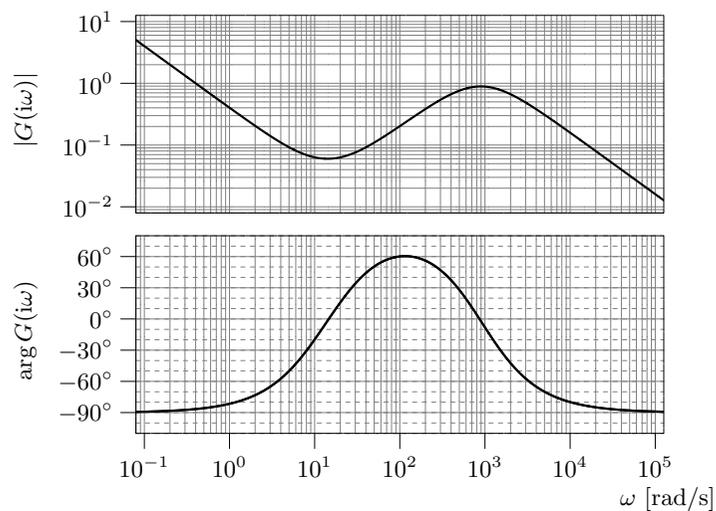


Figure 4.3a

- 4.4 From the final value of step response B (the only one greater than 1) and static gain in Bode gain C (the only one greater than 1), the step response–Bode gain pair B–C follows. Step responses C and A have approximately the same overshoots but different fundamental frequencies. Bode gains B and D have equal resonance peaks but D has a lower resonance frequency. This gives the combinations C–D and A–B. The remaining combination is D–A, which is a good match with small overshoot (resonance peak) and final value (static gain) 1.

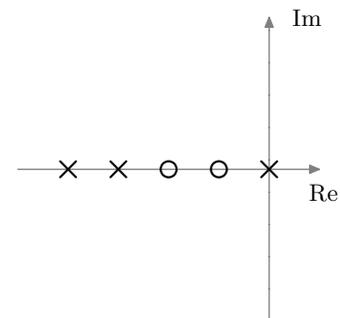
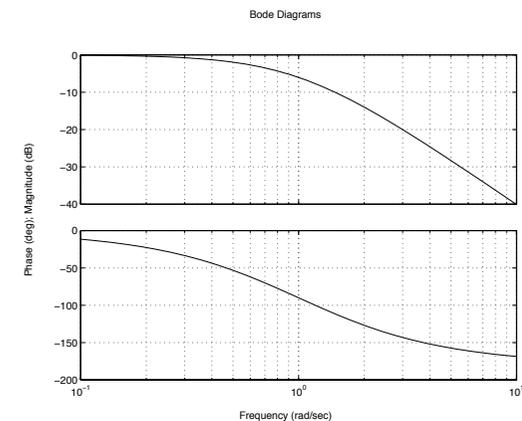


Figure 4.3b. Pole-zero diagram. Not accurate in scale; the diagram shall only be interpreted as a right to left ordering of poles and zeros, with the first pole at the origin.

- 4.5 a) Enter the system and make a Bode plot.
- ```
>> s = tf('s');
>> GA = 1 / (s^2 + 2*s + 1);
>> bode(GA)
```



Use, for example, curve handles and “Characteristics” in the right click menu to find static gain, bandwidth, resonance frequency, and resonance peak. The other systems are treated in the same way. The results can be summarized in the following table. (Note that gain values may be presented in  $\text{dB}_{20}$  in MATLAB.)

| System | $G(0)$ | $\omega_B$ | $\omega_r$ | $M_p$ |
|--------|--------|------------|------------|-------|
| $G_A$  | 1      | 0.64       |            |       |
| $G_B$  | 1      | 1.5        | 1          | 2.5   |
| $G_C$  | 1      | 0.21       |            |       |
| $G_D$  | 1      | 1.27       | 0.7        | 1.15  |
| $G_E$  | 1      | 2.54       | 1.4        | 1.15  |

- b) Using the results in a) and in Problem 2.6, the following observations can be made. (i): The bandwidth of a system is (approximately) inversely proportional to the rise time. High bandwidth implies a short rise time and hence a fast system. (ii): The damping is inversely proportional to the height of the resonance peak. A large peak implies low damping and large overshoot.

4.6 From the frequency response interpretation of the transfer function (“a sinusoid in gives a sinusoid out”) and the input being

$$u(t) = 2 \sin(2t - 1/2)$$

it follows that the output is

$$y(t) = 2 |G(i2)| \sin(2t - 1/2 + \arg G(i2))$$

Here  $G(s) = \frac{e^{-2s}}{s(s+1)}$ , and hence

$$|G(i2)| = \frac{1}{2\sqrt{2^2 + 1}} = \frac{1}{2\sqrt{5}}$$

$$\arg G(i2) = -4 - \frac{\pi}{2} - \arctan 2$$

4.7 The input is a sinusoid with amplitude 1 and angular frequency  $\omega = 2$  rad/s.

- a)  $0.45 \sin(2t - 1.1)$ .  
 (Gain:  $\left| \frac{1}{i2+1} \right| = \frac{1}{\sqrt{5}} \approx 0.45$ , phase:  $-\arg(i2+1) \approx -1.1$  rad =  $-63^\circ$ .)

- b) The system is unstable. Hence, the system output will tend to infinity, and the system will not reach a steady state. To be more precise, the

general form of the solution to the differential equation describing the system output is  $y(t) = C_0 e^t + \frac{1}{\sqrt{5}} \sin(2t - \pi + \arctan 2)$ , and any initial state  $y(0) \neq \frac{1}{\sqrt{5}} \sin(-\pi + \arctan 2)$  will lead to a solution that tends to infinity. This will almost always be the case in practice.

- c)  $0.11 \sin(2t - 2.4)$   
 (Gain:  $\left| \frac{1}{(i2+1)(i4+1)} \right| = \frac{1}{\sqrt{5}\sqrt{17}} \approx 0.11$ , phase:  $-\arg(i2+1) - \arg(i4+1) \approx -2.4$  rad =  $-139^\circ$ .)
- d)  $0.45 \sin(2(t - 0.5) - 1.1) = 0.45 \sin(2t - 2.1)$ .  
 Similar to problem a), with an extra time delay of 0.5 s.

- 4.8 a) To determine the phase difference,  $\phi$ , given a diagram with two sinusoids,  $\sin(\omega t)$  and  $K \sin(\omega t + \phi)$ , one possibility is to consider the time points when the two curves pass 0. Determine  $t_1$  and  $t_2$  such that

$$\sin(\omega t_1) = 0$$

$$K \sin(\omega t_2 + \phi) = 0$$

This gives that  $\omega t_1 = \omega t_2 + \phi$ , that is,

$$\phi = -\omega t_\Delta = -\frac{2\pi \text{ rad}}{T} t_\Delta = -\frac{t_\Delta}{T} 2\pi \text{ rad}$$

where  $t_\Delta = t_2 - t_1$  and  $T$  is the common period time. Here, the last expression may be interpreted as the delay expressed in parts ( $\frac{t_\Delta}{T}$ ) of a whole revolution ( $2\pi$ ). For example, consider the second graph where  $t_\Delta \approx 0.18$  s and  $T \approx 1.25$  s (which can either be read from the figure, or, in this problem, computed using  $\omega = 5$  rad/s). Hence,  $\phi = -\frac{0.18 \text{ s}}{1.25 \text{ s}} 2\pi \text{ rad} = -0.9$  rad. This results in the table below, where the answer to part b is also included.

| $\omega$ | $ G(i\omega) $              | $\arg G(i\omega)$ |
|----------|-----------------------------|-------------------|
| 1        | 1 = 0 dB <sub>20</sub>      | -0.2 rad = -11°   |
| 5        | 0.8 = -1.9 dB <sub>20</sub> | -0.9 rad = -52°   |
| 10       | 0.5 = -6 dB <sub>20</sub>   | -1.6 rad = -92°   |
| 20       | 0.2 = -14 dB <sub>20</sub>  | -2.2 rad = -126°  |

- b) Just evaluate the decibel formula to obtain the values in the table above.

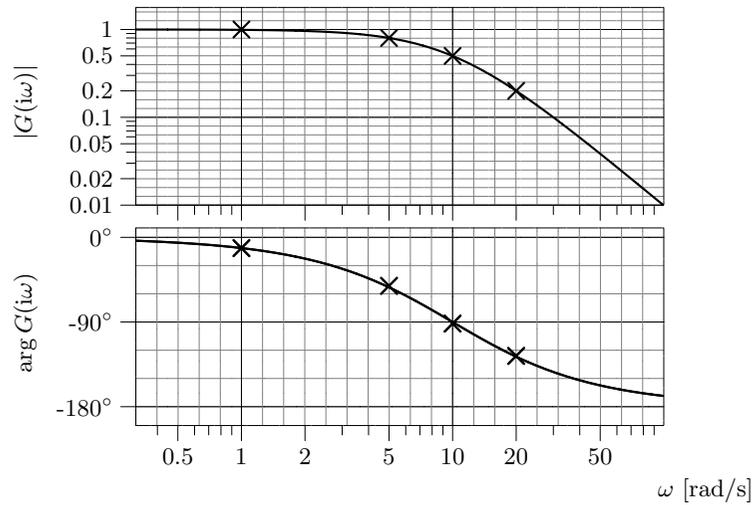


Figure 4.8a

c) A Bode plot of the system is given in Figure 4.8a

4.9 **Answer:**  $G_1$ -B,  $G_2$ -D,  $G_3$ -A,  $G_4$ -C,  $G_5$ -E.

- The Bode plot B has static gain 1 and no resonance peak, and hence  $G_1$ -B. It can also be seen that the Bode plot B decays by one decade (20 dB<sub>20</sub>) when the frequency increases by a factor of ten (“the slope is -1”) and that  $G_1$  has one pole.
- The Bode plots A and C have both infinite gain for when the frequency tends to zero, that is, they correspond to systems containing an integrator  $\Rightarrow$  systems  $G_3$  and  $G_4$ . The Bode plot C decays more rapidly for high frequencies  $\Rightarrow$  the relative degree (number of poles – number of zeros) is higher. Hence  $G_3$ -A,  $G_4$ -C.
- The Bode plots D and E have peaks  $\Rightarrow$  systems  $G_2$  and  $G_5$ . (For  $G_2$  the peak is caused by the zero where the curve “turns up” at  $\omega = 1$ .) The Bode plot E has larger slope than D for high frequencies, that is, E corresponds to a system with higher relative degree.  $G_2$  has one pole more than zeros,  $G_5$  has 2 poles, and hence  $G_2$ -D,  $G_5$ -E.

- 4.10
- In step response A and D the step responses tend to one, that is, they correspond to Bode gain A and C. Step response D has larger overshoot, that is, it corresponds to Bode gain A, and consequently step response A corresponds to Bode gain C. This gives the Bode gain–step response pairs A–D and C–A.
  - Step response B has no overshoot, which implies that it corresponds to Bode gain D, which has no peak. This gives the combination D–B.
  - The remaining combination is B–C. Step response C has an overshoot which can be related to the peak in the Bode gain plot. It can also be seen that this pair belongs to the fastest system.

4.11 a) The system can be rewritten as

$$G(s) = \frac{1.7}{(s+1)\left(\frac{s}{1.43}+1\right)\left(\frac{s}{2}+1\right)}$$

It thus follows that

$$|G(i\omega)| = \frac{1.7}{\sqrt{1+\omega^2}\sqrt{1+\left(\frac{\omega}{1.43}\right)^2}\sqrt{1+\left(\frac{\omega}{2}\right)^2}}$$

and

$$\arg G(i\omega) = -\arctan \omega - \arctan \frac{\omega}{1.43} - \arctan \frac{\omega}{2}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote,  $|G(i0)| = 1.7$ , and the breakpoints and slopes of the asymptotes:

|                   |  |   |  |      |  |    |  |    |
|-------------------|--|---|--|------|--|----|--|----|
| Frequency [rad/s] |  | 1 |  | 1.43 |  | 2  |  |    |
| Slope             |  | 0 |  | -1   |  | -2 |  | -3 |

The phase shift is drawn based on a couple of samples:

|                   |         |         |         |
|-------------------|---------|---------|---------|
| Frequency [rad/s] | 0.1     | 0.5     | 1       |
| Phase             | -12.7°  | -59.9°  | -106.6° |
| Frequency [rad/s] | 2       | 3       | 10      |
| Phase             | -162.9° | -192.4° | -244°   |

The bode plot in Figure 4.11a gives:  $\omega_c = 0.874$  rad/s,  $\varphi_m = 83.8^\circ$ ,  $A_m = 5.14$ , and  $\omega_p = 2.51$  rad/s.

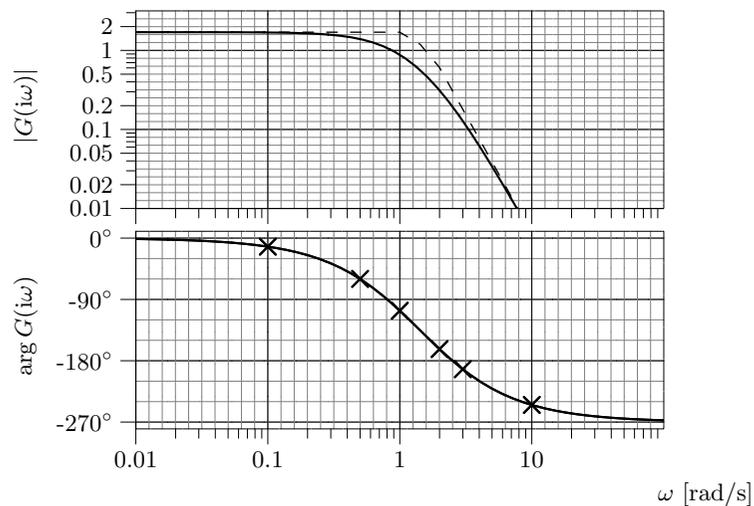


Figure 4.11a

- b) The phase is  $-180^\circ$  at  $\omega_p = 2.51$  where the amplitude is 0.1946. To make the pH oscillate with constant amplitude one has to choose  $K = \frac{1}{0.1946} = 5.14$ .
- 4.12 a) The phase is  $-180^\circ$  at  $\omega_p = 0.334$  where the amplitude is 0.1984. To keep the reactor stable one has to choose  $K \leq \frac{1}{0.1984} = 5.04$ .
- b) This is a lead-lag design task. The amplitude and phase of  $G$  at  $\omega_{c,d} = 0.1$  is 0.6325 and  $-100^\circ$ . Thus we have a phase margin of  $80^\circ$  which is sufficient, and hence no lead controller is needed. To remove the steady-state error we need a lag controller with  $\gamma = 0$ . This results in the controller structure

$$F(s) = K \underbrace{\frac{\tau_1 s + 1}{\tau_1 s}}_{F_{\text{lag}}}$$

Chose  $\tau_1 = 10/\omega_{c,d} = 100$  (a smaller value of  $\tau_1$  makes the error go to zero faster) and  $K = \frac{1}{|G(i\omega_{c,d})F_{\text{lag}}(i\omega_{c,d})|} = \frac{1}{0.6325} = 1.58$ . This gives the controller  $F(s) = 1.58 \frac{100s+1}{100s}$ .

# 5 Compensation

5.1 The compensator is constructed using lead-lag design. “Twice as fast” is interpreted as a doubling of the bandwidth, which, in turn, is approximated by a doubling of the gain crossover frequency. “Same damping” is interpreted as maintaining the old phase margin, which is accomplished using a lead compensator in the controller. The error in static reference following is controlled by adjusting the static gain of the open loop system, which is accomplished using a lag compensator in the controller. Sensitivity to measurement disturbances is given by the complementary sensitivity function,  $1 - (1 + G_o)^{-1}$ . It is small where the open loop gain is small. Thus, to make it small at high frequencies, the high frequency gain of the controller should be kept as low as possible.

First, the open loop system when  $F(s) = 1 \Rightarrow G_o = G$  is examined in order to quantify the requirements.

$$G(s) = \frac{0.4}{(s + 0.1)(s + 0.5)(s^2 + 0.4s + 4)}$$

$$= \frac{2}{(1 + s/0.1)(1 + s/0.5)(1 + 2 \cdot 0.1 \cdot s/2 + (s/2)^2)}$$

which implies that

$$|G(i\omega)| = \frac{2}{\sqrt{1 + (\frac{\omega}{0.1})^2} \sqrt{1 + (\frac{\omega}{0.5})^2} \sqrt{(1 - (\frac{\omega}{2})^2)^2 + 4 \cdot 0.01(\frac{\omega}{2})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow 2, \omega \rightarrow 0$$

and

$$\arg G(i\omega) = -\arctan \frac{\omega}{0.1} - \arctan \frac{\omega}{0.5} - \arctan \frac{2 \cdot 0.1 \frac{\omega}{2}}{1 - (\frac{\omega}{2})^2}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 2 (at any point), and the breakpoints and slopes of the asymptotes:

|                   |   |     |    |     |    |   |    |
|-------------------|---|-----|----|-----|----|---|----|
| Frequency [rad/s] |   | 0.1 |    | 0.5 |    | 2 |    |
| Slope             | 0 |     | -1 |     | -2 |   | -4 |

The system has two complex conjugated poles which implies that the amplitude curve has a resonance peak. The approximative amplitude curve must be modified at the resonance peak. An exact calculation of the gain gives

|                   |      |      |      |       |
|-------------------|------|------|------|-------|
| Frequency [rad/s] | 1    | 1.5  | 2    | 2.5   |
| Gain              | 0.12 | 0.09 | 0.12 | 0.025 |

The phase curve is drawn based on a couple of samples:

|                   |       |       |       |       |
|-------------------|-------|-------|-------|-------|
| Frequency [rad/s] | 0.01  | 0.1   | 1     | 1.5   |
| Phase             | -7°   | -57°  | -155° | -177° |
| Frequency [rad/s] | 2     | 2.5   | 10    |       |
| Phase             | -253° | -322° | -354° |       |

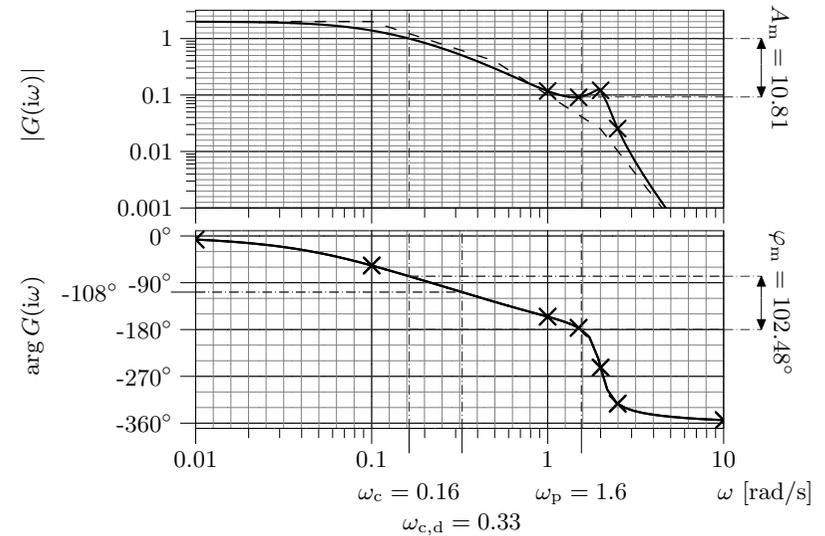


Figure 5.1a

The Bode plot in Figure 5.1a gives

$$\omega_c = 0.16 \text{ rad/s} \quad \varphi_m = 102^\circ \quad A_m = 10.6$$

and hence

$$\omega_{c,d} = 0.32 \text{ rad/s} \quad \varphi_{m,d} = 102^\circ$$

The phase of  $G$  at the  $\omega_{c,d}$  is  $-108^\circ$ . Hence, in order to obtain the desired phase margin of  $102^\circ = -78^\circ - (-180^\circ)$ , a phase advance of approximately

$(-78^\circ) - (-108^\circ) = 30^\circ$  is required. To this end, introduce a lead compensator in the controller:

$$F_{\text{lead}} = N \frac{s + b}{s + bN}$$

See the discussion of lead compensators in Glad&Ljung! To keep the high frequency gain of the controller as small as possible,  $N$  should be chosen as small as possible. The desired phase advance is obtained with  $N = 3$ . This phase lead is obtained at the desired crossover frequency if

$$b = \frac{\omega_{c,d}}{\sqrt{N}} = 0.185$$

The desired crossover frequency is obtained by adjusting the gain of the open loop system by introducing a factor,  $K$ , in the controller:

$$1 = K |F_{\text{lead}}| \cdot |G(i0.32)| = K\sqrt{N} \cdot 0.52 \Rightarrow K = 1.11$$

With

$$F(s) = 3.33 \frac{s + 0.185}{s + 0.555}$$

and  $\omega_{\text{ref}}(s) = A/s$ , where  $A$  is constant, it follows that (using the notation  $e(t) = \omega_{\text{ref}}(t) - \omega(t)$ )

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + F(s)G(s)} \frac{A}{s} \\ &= \frac{A}{1 + 1.1 \cdot 2 \cdot F_{\text{lag}}(0)} \leq 0.05A. \end{aligned}$$

This is equivalent to

$$F_{\text{lag}}(0) \geq 8.63$$

If the low frequency gain of  $F(s)$  is increased approximately 9 times the stationary error will be smaller than 5%. To this end, introduce a lag (phase-retarding) compensator

$$F_{\text{lag}} = \frac{s + a}{s + \frac{a}{M}}$$

in the controller, where  $M$  should be kept as small as possible to avoid unnecessary high gain at low frequencies. See the discussion of lag compensators in Glad&Ljung! With  $M = 9$  and  $a = 0.1\omega_{c,d} = 0.032$  the desired low frequency gain increase is obtained without altering the phase margin too much.

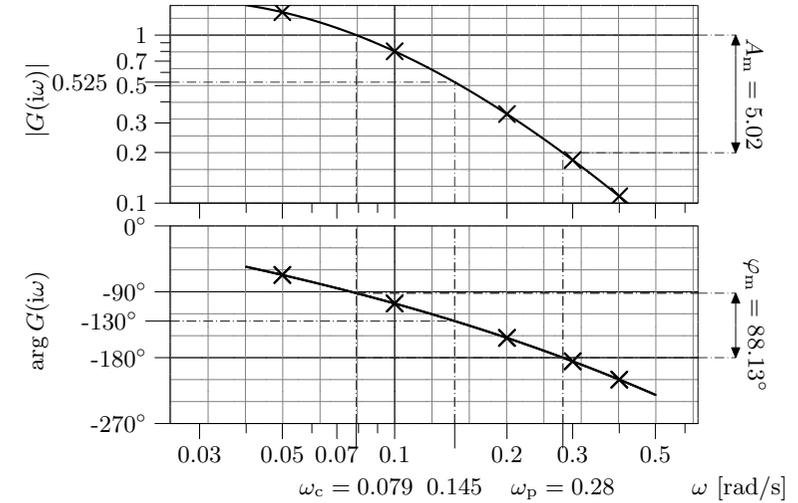


Figure 5.2a

This gives the controller

$$F(s) = K \cdot F_{\text{lead}}(s) \cdot F_{\text{lag}}(s) = 3.33 \frac{(s + 0.185)}{(s + 0.555)} \frac{(s + 0.032)}{(s + 0.0036)}$$

5.2 Let  $G$  denote the heat exchanger's transfer function.

- a) Draw the Bode plot using the given table. From the diagram in Figure 5.2a it follows that

$$\omega_c = 0.079 \text{ rad/s} \quad \varphi_m = 88^\circ \quad A_m = 5.0$$

- b) A proportional controller does not change the phase curve. According to Figure 5.2a, the phase curve crosses  $-130^\circ$  at the frequency 0.15 rad/s. A gain crossover at this frequency will yield exactly the required phase margin, and any higher crossover frequency will yield one that is too small.\*

\*The controller gain that yields the desired gain crossover frequency can be computed as

$$K = \frac{1}{|G(0.15i)|} = \frac{1}{0.525} = 1.9$$

c) Twice as large crossover frequency is desired:

$$\omega_{c,d} = 0.29 \text{ rad/s} \quad \varphi_{m,d} = 50^\circ$$

At the frequency 0.29 rad/s the phase margin is approximately  $0^\circ$  (actually a little less, since  $\omega_p = 0.28 \text{ rad/s}$ ). Hence, a phase lead of  $50^\circ$  is needed. To this end, use the lead-compensator  $KF_{\text{lead}}$ , where  $F_{\text{lead}}(s) = (\tau_D s + 1)/(\beta\tau_D s + 1)$ . Set  $\beta = 0.13$  (according to the diagram in Glad&Ljung) in order to achieve the required phase lead. To obtain the maximum phase lead at the desired crossover frequency, let

$$\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 9.47$$

Finally,  $K$  is chosen so that  $\omega_{c,d}$  is obtained:

$$1 = K |F_{\text{lead}}(i\omega_{c,d})| \cdot |G(i\omega_{c,d})| \approx K \cdot \frac{1}{\sqrt{\beta}} \cdot \frac{1}{A_m} \Leftrightarrow$$

$$K \approx \sqrt{\beta} A_m = 1.83$$

**Answer:**

$$F(s) = 1.83 \frac{(9.47s + 1)}{(0.13 \cdot 9.47s + 1)}$$

5.3 a)

$$G(s) = \frac{20}{s(1 + 2 \cdot 0.1 \cdot \frac{s}{150} + (\frac{s}{150})^2)}$$

which implies that

$$|G(i\omega)| = \frac{20}{\omega \sqrt{(1 - (\frac{\omega}{150})^2)^2 + 4 \cdot 0.01 \cdot (\frac{\omega}{150})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow \frac{20}{\omega}, \quad \omega \rightarrow 0$$

and

$$\arg G(i\omega) = -90^\circ - \arctan \frac{2 \cdot 0.1 \cdot \omega}{(1 - (\frac{\omega}{150})^2)}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote,  $|\frac{20}{20}| = 1$ , and the breakpoints and slopes of the asymptotes:

|                   |  |     |    |
|-------------------|--|-----|----|
| Frequency [rad/s] |  | 150 |    |
| Slope             |  | -1  | -3 |

The system has two complex conjugated poles which implies that the amplitude curve has a resonance peak. The approximative amplitude curve must be modified at the resonance peak. An exact calculation of the gain gives

|                   |      |      |      |
|-------------------|------|------|------|
| Frequency [rad/s] | 100  | 150  | 200  |
| Gain              | 0.35 | 0.67 | 0.12 |

The phase curve is drawn based on a couple of samples:

|                   |             |             |              |              |              |
|-------------------|-------------|-------------|--------------|--------------|--------------|
| Frequency [rad/s] | 10          | 50          | 100          | 150          | 200          |
| Phase             | $-91^\circ$ | $-94^\circ$ | $-103^\circ$ | $-180^\circ$ | $-251^\circ$ |

In addition, one can also use

$$\arg G(i\omega) \rightarrow -90^\circ, \quad \omega \rightarrow 0$$

$$\arg G(i\omega) \rightarrow -270^\circ, \quad \omega \rightarrow \infty$$

The Bode plot with the gain curve labeled "A" in Figure 5.3a gives

$$\omega_c = 20 \text{ rad/s} \quad \varphi_m = 88^\circ \quad A_m = 1.5$$

b) If  $K$  would be chosen to the gain margin,  $A_m = 1.5$ , the new gain margin would be 1. Thus, if

$$K = \frac{A_m}{2} = 0.75$$

the resulting gain margin becomes 2. With this amplification the final value theorem gives the ramp error

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{1}{sKG(s)} = \frac{1}{0.75 \cdot 20} = 0.067$$

Note that the system is stable by construction (the new gain margin is greater than 1).

c) The new gain crossover frequency obtained in part b is 15 rad/s, see the gain curve labeled "B" in Figure 5.3a. The low frequency gain of  $F(s)$

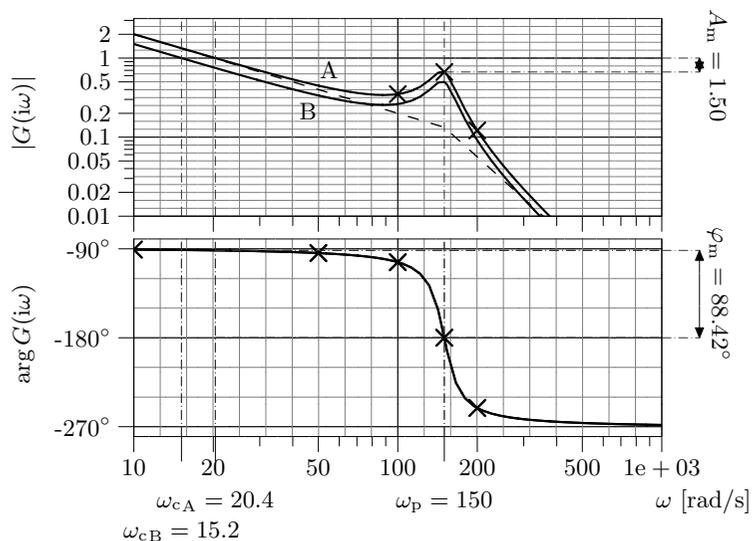


Figure 5.3a

must be increased at least 15 times. A lag-compensator with  $M = 15$  can be used. Choose, according to the rule of thumb,  $a = 0.1 \cdot \omega_{c,d}$ , where  $\omega_{c,d} = 15$ , and hence  $a = 1.5$ .

**Answer:**

$$F(s) = 0.75 \cdot \frac{(s + 1.5)}{(s + 0.1)}$$

5.4 We begin by drawing a Bode plot of the system.

$$G(s) = \frac{10}{s(1 + \frac{s}{10})(1 + \frac{s}{100})}$$

which implies that

$$|G(i\omega)| = \frac{10}{\omega \sqrt{1 + (\frac{\omega}{10})^2} \sqrt{1 + (\frac{\omega}{100})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow \frac{10}{\omega}, \omega \rightarrow 0$$

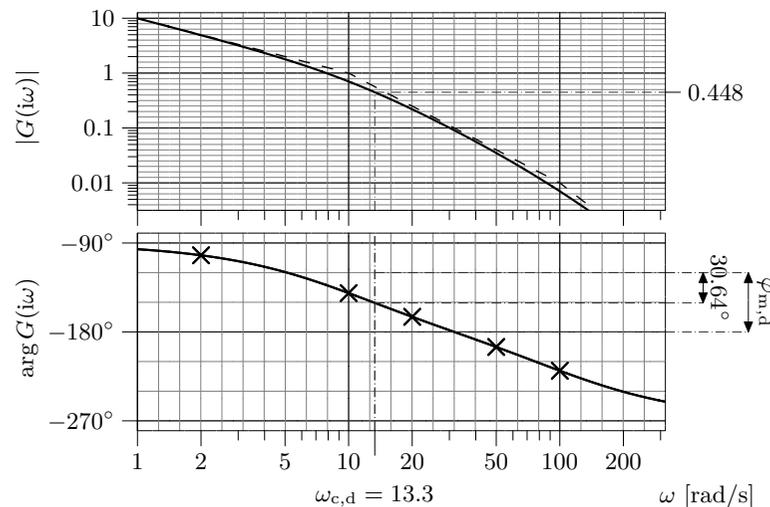


Figure 5.4a

and

$$\arg G(i\omega) = -90^\circ - \arctan \frac{\omega}{10} - \arctan \frac{\omega}{100}$$

The gain is drawn approximately based on a known gain at some point of the low frequency asymptote,  $|\frac{10}{1}| = 10$ , and the breakpoints and slopes of the asymptotes:

|                   |    |    |    |     |    |
|-------------------|----|----|----|-----|----|
| Frequency [rad/s] |    | 10 |    | 100 |    |
| Slope             | -1 |    | -2 |     | -3 |

The phase curve is drawn based on a couple of samples:

|                   |       |       |       |       |       |
|-------------------|-------|-------|-------|-------|-------|
| Frequency [rad/s] | 2     | 10    | 20    | 50    | 100   |
| Phase             | -102° | -141° | -165° | -195° | -219° |

In addition, one can also use

$$\begin{aligned} \arg G(i\omega) &\rightarrow -90^\circ, \omega \rightarrow 0 \\ \arg G(i\omega) &\rightarrow -270^\circ, \omega \rightarrow \infty \end{aligned}$$

From the Bode digram in Figure 5.4a it follows that  $\omega_c = 7.8$  rad/s,  $\varphi_m = 47^\circ$  and  $A_m = 11$ . However, these values are not used by the solution to this problem.

Figure 5.4b (the figure can also be found in Glad&Ljung) gives that the overshoot is acceptable if  $\zeta \geq 0.58$ . Choose for instance  $\zeta = 0.6$ . This results in a desired phase margin  $\varphi_{m,d} = 60^\circ$ . According to Figure 5.4c (the figure can also be found in Glad&Ljung), this also implies a desired gain crossover frequency:

$$\omega_{c,d} T_r = 1.33 \quad \Rightarrow \quad \omega_{c,d} = \frac{1.33}{T_r} = \frac{1.33}{0.1} = 13.3$$

At 13.3 rad/s a phase advance of approximately  $31^\circ$  is needed in order to get the desired phase margin. To this end, we use a lead compensator with old-fashion parameterization according to Problem 5.15 (it is not wrong to use the newer parameterization), with  $N = 4$  and  $b = \omega_{c,d}/\sqrt{N} = 6.7$ .  $K$  is adjusted to get the desired gain crossover frequency:

$$1 = K |F_{\text{lead}}(i\omega_{c,d})| \cdot |G(i\omega_{c,d})| = K\sqrt{N} \cdot 0.448 \quad \Rightarrow \quad K = 1.12$$

The transfer function from the reference input to the control error is given by

$$E(s) = \frac{1}{1 + F(s)G(s)} \theta_{\text{ref}}(s)$$

When  $\theta_{\text{ref}}(t)$  is a step signal, the final value theorem gives

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0$$

even without a lag compensator thanks to the integration in  $G$ . Here, the final value theorem may be used since the system by construction is stable (the phase margin is  $60^\circ$ ).

In order to handle errors for ramp references, introduce a lag compensator (also parameterized according to Problem 5.15) in the controller. Then  $|F_{\text{lag}}(0)| = M$ , and if  $\theta_{\text{ref}}(t) = 10 \cdot t$ , that is, if

$$\theta_{\text{ref}}(s) = \frac{10}{s^2}$$

one obtains

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + F(s)G(s)} \frac{10}{s^2} = \frac{10}{k_m \cdot K \cdot M} < 0.1$$

which gives  $M > \frac{1}{0.1K} = 9.0$ . Take  $M = 9.0$  to avoid excessively high low frequency loop gain. According to the rule of thumb, let  $a = 0.1 \cdot \omega_{c,d} = 1.33$ .

**Answer:**

$$F(s) = 1.12 \cdot 4 \cdot \frac{s + 6.7}{s + 27} \cdot \frac{s + 1.33}{s + 0.15}$$

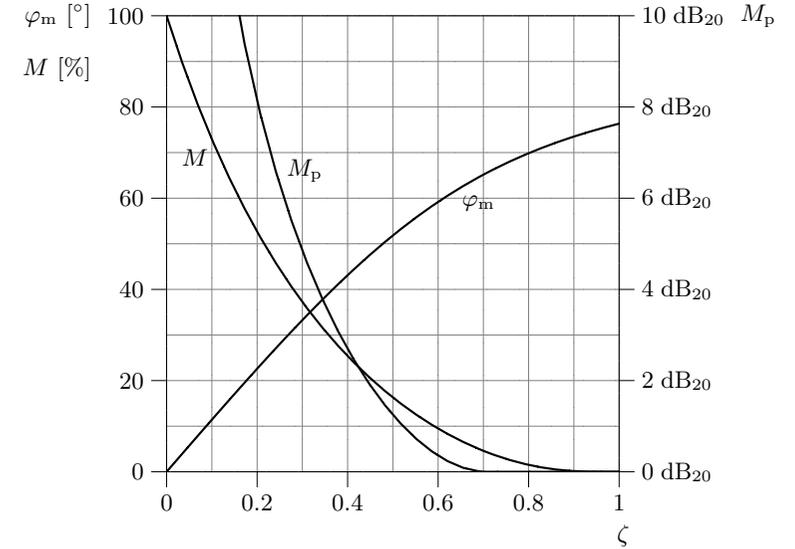


Figure 5.4b. Relations between overshoot,  $M$ , phase margin,  $\varphi_m$ , resonance gain,  $M_p$ , and relative damping,  $\zeta$ , for a second order system with no zeros and static gain 1.

5.5 *Notation.* The notation “A – B – C” is used to say that the system with open loop Bode plot in row A has its closed loop Bode plot in row B, and its step response in row C.

A good start is often to look at the static gain and the final value of the step responses. The static gain of the open loop system and the closed loop system are related as  $|G_c(0)| = \frac{|G_o(0)|}{|1+G_o(0)|}$ . Systems with the same static gain can then be separated by looking at stability margins, resonance peak, overshoot, bandwidth, and speed. Three of the combinations are easy to identify:

A – E – C: Finite but non-zero open loop static gain matches non-zero closed loop static gain less than 1. Infinite stability margins matches step response without overshoot.

B – C – E: Infinite open loop static gain matches closed loop static gain equal 1, which in turn matches a step response that settles at amplitude 1.

C – A – B: Zero static open loop gain matches zero closed loop gain, which in turn matches a step response that settles at amplitude 0. It is also possible to relate the Bode plots by their high frequency gains.

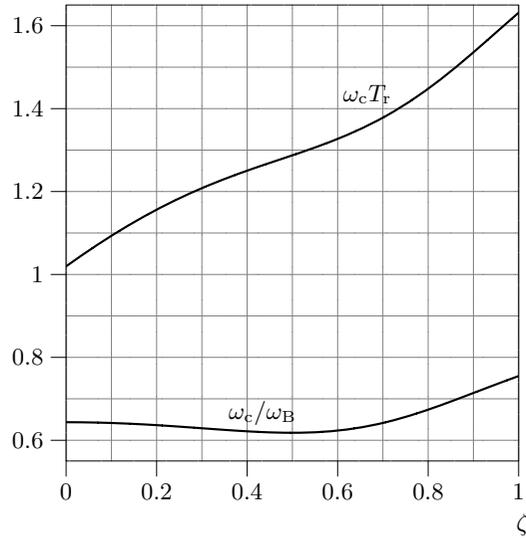


Figure 5.4c. Relations between gain crossover frequency,  $\omega_c$ , bandwidth,  $\omega_B$ , raise time,  $T_r$ , and relative damping,  $\zeta$ , for a second order system with no zeros and static gain 1.

The remaining open loop Bode plots are D and E. These should be matched with the closed loop gain curves B and D, and step responses A and D. Both open loop Bode plots show a static gain near 1, which will make it hard (albeit possible) to use that feature for identification. Easier is to approximately locate the (closed loop) resonance frequency, which will be near the frequency where the Nyquist curve minimizes its distance to  $-1$ . That is, the magnitude shall be near 1, and the phase near  $-180^\circ$  in the open loop Bode plot. This happens at a lower frequency in open loop Bode plot D than in E. The resonance peak in the closed loop gain curve B is located at a higher frequency than that in D. Finally, a higher resonance frequency gives faster oscillations in the step response, and the oscillations in step response A are much quicker than those in D. Alternatively the bandwidth's relation to response speed may be used; the bandwidth is higher in closed loop B than in D, and step response A is quicker than D. Anyway, the last two combinations are D-D-D, E-B-A.

5.6

$$G(s) = \frac{10}{s(1 + \frac{s}{20})(1 + \frac{s}{40})(1 + \frac{s}{100})}$$

gives

$$|G(i\omega)| = \frac{10}{\omega \sqrt{1 + (\frac{\omega}{20})^2} \sqrt{1 + (\frac{\omega}{40})^2} \sqrt{1 + (\frac{\omega}{100})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow \frac{10}{\omega}, \omega \rightarrow 0$$

and

$$\arg G(i\omega) = -90^\circ - \arctan \frac{\omega}{20} - \arctan \frac{\omega}{40} - \arctan \frac{\omega}{100}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote,  $|\frac{10}{10}| = 1$ , and the breakpoints and slopes of the asymptotes:

|                   |  |    |  |    |  |     |  |
|-------------------|--|----|--|----|--|-----|--|
| Frequency [rad/s] |  | 20 |  | 40 |  | 100 |  |
| Slope             |  | -1 |  | -2 |  | -3  |  |

The phase curve is drawn based on a couple of samples:

|                   |              |              |              |
|-------------------|--------------|--------------|--------------|
| Frequency [rad/s] | 10           | 20           | 50           |
| Phase             | $-136^\circ$ | $-173^\circ$ | $-236^\circ$ |

In addition, one can also use

$$\arg G(i\omega) \rightarrow -90^\circ, \omega \rightarrow 0$$

$$\arg G(i\omega) \rightarrow -360^\circ, \omega \rightarrow \infty$$

The Bode plot in Figure 5.6a gives that  $\omega_c = 8.9$  rad/s,  $\varphi_m = 48^\circ$  and  $A_m = 3.9$ . However, it is only the gain crossover frequency which directly interests us here; an increase of the speed with a factor of two and a preserved damping imply  $\omega_{c,d} = 18$  rad/s and  $\varphi_{m,d} = \varphi_m$ . From the figure, we have  $\varphi_\Delta = \arg G(i\omega_c) - \arg G(i\omega_{c,d}) = 35^\circ$ . The required phase lead is thus at least  $\varphi_\Delta + 6^\circ = 41^\circ$ . To this end, use a lead compensator (with standard notation of the parameters) with  $\beta = 0.21$  and  $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 0.12$ .  $K$  is adjusted to get the desired crossover frequency:

$$K |G(i\omega_{c,d})| \cdot |F_{lead}(i\omega_{c,d})| = 1 \Rightarrow K = \frac{\sqrt{\beta}}{0.37} = 1.2$$

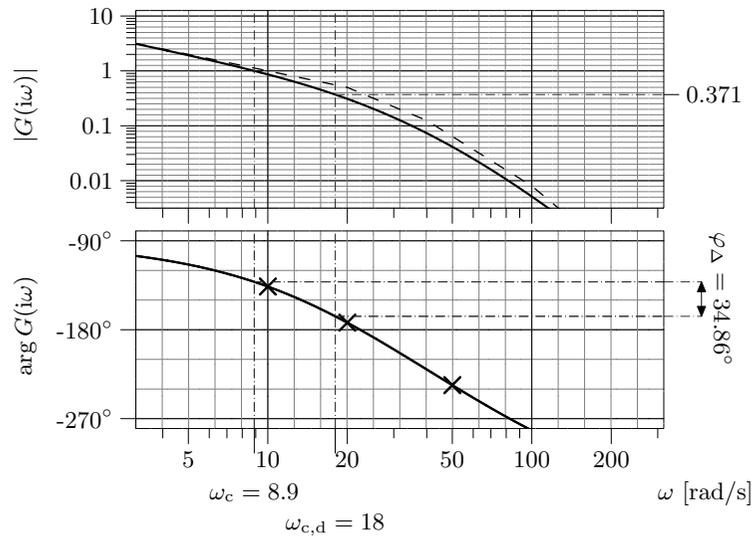


Figure 5.6a

The transfer function from the reference input to the control error is given by

$$E(s) = \frac{1}{1 + F(s)G(s)} \theta_{\text{ref}}(s)$$

When  $\theta_{\text{ref}}(t)$  is a step, the final value theorem gives

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0$$

even without a lag compensator thanks to the integration in  $G(s)$ . Here, the final value theorem may be used since the system by construction is stable (the phase margin is positive).

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then  $|F_{\text{lag}}(0)| = 1/\gamma$ , and if  $\theta_{\text{ref}}(t) = 10 \cdot t$ , that is, if

$$\theta_{\text{ref}}(s) = \frac{10}{s^2}$$

one obtains

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + F(s)G(s)} \frac{10}{s^2} = \frac{10\gamma}{k_m \cdot K} < 0.01$$

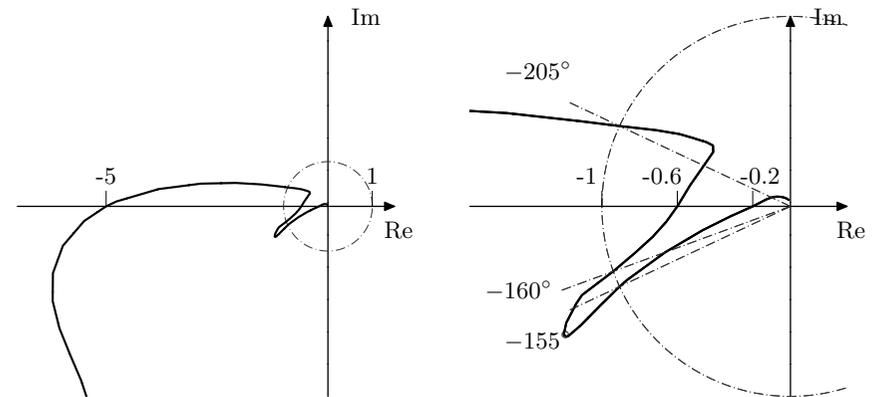


Figure 5.7a. Nyquist curve in two scales. Left: small scale. Right: big scale.

which gives  $\gamma < 0.01K = 0.012$ . Take  $\gamma = 0.012$  to avoid excessively high low frequency loop gain. According to the rule of thumb, let  $\tau_1 = 10/\omega_{c,d} = 0.56$ .

**Answer:**

$$F(s) = 1.2 \frac{0.12s + 1}{0.21 \cdot 0.12s + 1} \cdot \frac{0.56s + 1}{0.56s + 0.012}$$

5.7 Based on the Bode plot we plot the Nyquist curve, see Figure 5.7a. The system is stable when the point  $-1$  is not encircled by the Nyquist curve. This gives

$$K < \frac{1}{5} \quad \text{or} \quad \frac{1}{0.6} < K < \frac{1}{0.2}$$

5.8 A time delay of  $T$  seconds changes the phase curve with  $-\omega T$  rad at frequency  $\omega$ . The amplitude curve is not affected.

a) The crossover frequency is  $\omega = 1$  rad/s and the phase margin is 0.698 rad. This gives the stability condition

$$0.698 \text{ rad} - 1 \text{ rad/s} \cdot T > 0$$

that is,  $T < 0.698$  s.

- b) Plot the Nyquist curve as in Figure 5.8a. The point  $-1$  is not encircled if the phase is decreased at least  $40^\circ$  at  $\omega = 7$  rad/s but not more than  $80^\circ$  at  $\omega = 5$  rad/s. This gives the following conditions

$$7 \text{ rad/s} \cdot T > 40^\circ = 0.698 \text{ rad} \quad \text{and} \quad 5 \text{ rad/s} \cdot T < 80^\circ = 1.396 \text{ rad}$$

that is,  $0.1 \text{ s} < T < 0.28 \text{ s}$ .

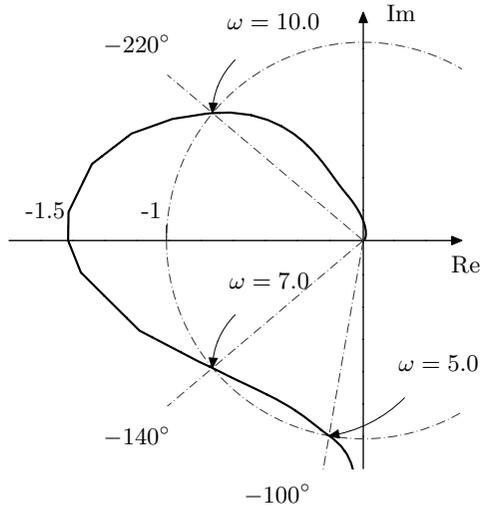


Figure 5.8a

- 5.9 a) The step response of  $G_A$  is

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{k_A}{a}(1 - e^{-at}) \rightarrow \frac{k_A}{a}, \quad t \rightarrow \infty$$

From the figure it is seen that  $k_A/a = 0.5$ . At time  $t = 1/a$  we have

$$y(1/a) = \frac{k_A}{a}(1 - e^{-1}) = 0.5 \cdot 0.63 = 0.315 = y(2)$$

Thus  $a = 0.5$ , which gives  $k_A = 0.25$ :

$$G_A(s) = \frac{0.25}{s + 0.5}$$

which is rewritten to make apparent the amplitude and phase

$$G_A(i\omega) = \frac{0.25}{\sqrt{\omega^2 + 0.25}} e^{-i \arctan 2\omega}$$

The corresponding Bode plot is shown in Figure 5.9a. To see how  $G_A$  modifies the Bode plot of  $G_m$ , consider for instance the frequency  $0.1$  rad/s. When computing the new gain, the logarithmic scale in the diagrams is used to do directly obtain the logarithm of the product of the two systems' gains:

$$|G_m(0.1i)| = 10^{0.15}$$

$$|G_A(0.1i)| = 10^{-0.31}$$

$$|G_A(0.1i)G_m(0.1i)| = 10^{0.15} \cdot 10^{-0.31} = 10^{0.15+(-0.31)} = 10^{-0.16}$$

The new phase is obtained by adding the arguments of the two transfer functions:

$$\arg G_m(0.1i) = -135^\circ$$

$$\arg G_A(0.1i) = -11^\circ$$

$$\arg G_A(0.1i)G_m(0.1i) = \arg G_A(0.1i) + \arg G_m(0.1i) = -146^\circ$$

Carrying out the procedure of "adding Bode plots" at a range of selected frequencies results in the Bode plot in Figure 5.9b, where  $G_o = G_A G_m$ .

- b) In Figure 5.9b it can be seen that the crossover frequency is  $0.078$  rad/s. Hence, let  $\omega_{c,d} = 0.4$  to obtain a 5 times as fast system. At the desired crossover frequency, the phase must be advanced by  $68^\circ$  to maintain the phase margin. To this end, employ two equal lead compensators (using standard notation of the parameters), each advancing the phase by  $34^\circ$ ; take  $\beta = 0.28$ , and  $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 4.7$ .

The controller gain is adjusted by the factor  $K$  to get the desired crossover frequency:

$$K |G(i\omega_{c,d})| \cdot |F_{\text{lead}}(i\omega_{c,d})|^2 = 1 \quad \Rightarrow$$

$$K = \frac{\sqrt{\beta^2}}{0.047} = 6.0$$

**Answer:**

$$F(s) = 6.0 \cdot \left( \frac{4.7s + 1}{1.3s + 1} \right)^2$$

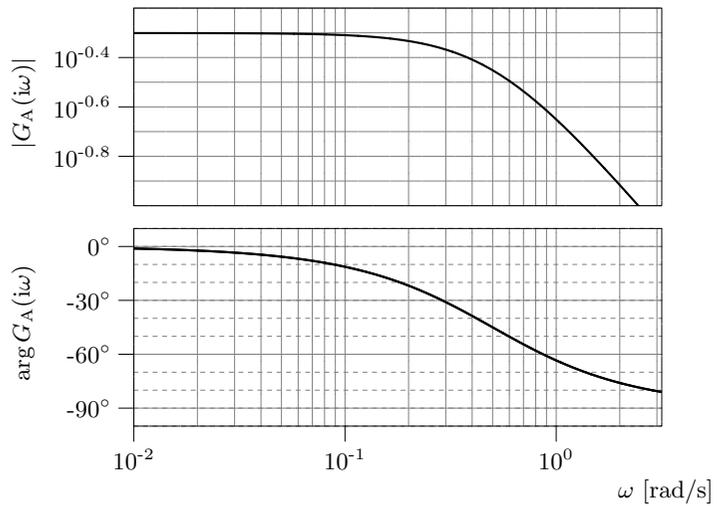


Figure 5.9a

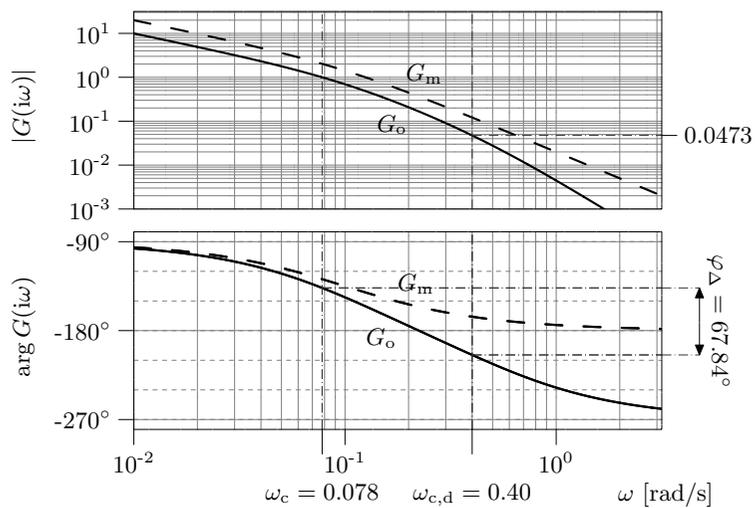


Figure 5.9b

5.10

$$G(s) = \frac{1}{s} G_1(s)$$

gives

$$|G(i\omega)| = \frac{|G_1(i\omega)|}{\omega}$$

$$\arg G(i\omega) = \arg G_1(i\omega) - 90^\circ$$

A P controller gives a phase margin of  $40^\circ$  when

$$\arg G(i\omega) = -140^\circ \Rightarrow \arg G_1(i\omega) = -50^\circ$$

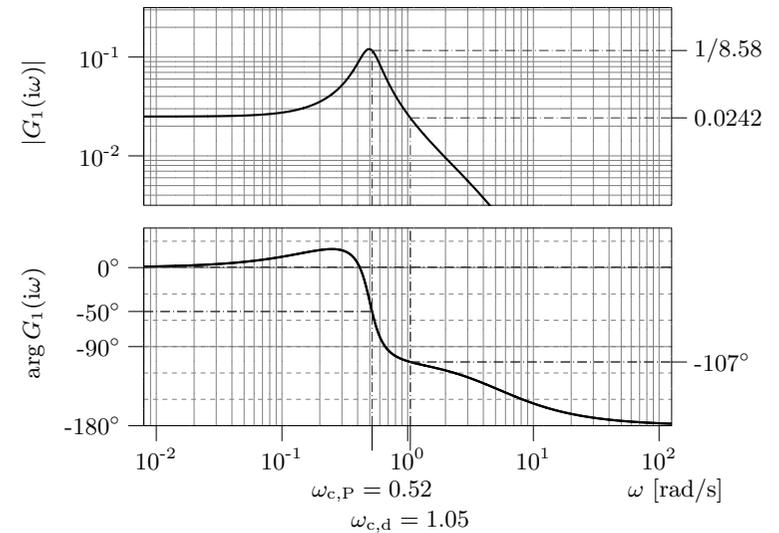


Figure 5.10a

From Figure 5.10a it is seen (although not easily) that this occurs at  $\omega_{c,P} = 0.52$  rad/s, which is also the highest possible gain crossover frequency possible to obtain with P control. The desired increase in speed by a factor of two is thus achieved by a new gain crossover  $\omega_{c,d} = 1.05$  rad/s. Figure 5.10a gives

$$\arg G_1(i\omega_{c,d}) = -107^\circ \Rightarrow \arg G(i\omega_{c,d}) = -197^\circ$$

A desired phase margin of  $40^\circ$  requires that the phase be advanced by  $57^\circ + 6^\circ = 63^\circ$ . To this end, employ a two equal lead compensators (using standard notation of parameters), each advancing the phase by  $32^\circ$ ; take  $\beta = 0.31$  and  $\tau_D = \frac{1}{\omega_{c,d}\sqrt{\beta}} = 1.72$ . The controller gain is adjusted by the factor  $K$  to get the desired crossover frequency:

$$K |F_{\text{lead}}(i\omega_{c,d})|^2 \cdot |G(i\omega_{c,d})| = 1 \Rightarrow K \frac{1}{\sqrt{0.31^2}} \frac{0.024}{1.05} = 1 \Rightarrow K = 13.3$$

In order to handle errors for ramp references, introduce a lag compensator (with the usual notation of parameters) in the controller. Then  $|F_{\text{lag}}(0)| = 1/\gamma$ , and  $|F(0)| = K/\gamma$ . To choose  $\gamma$ , consider the Laplace transform of the control error,

$$E(s) = \frac{1}{1 + F(s)G(s)} R(s)$$

If  $r(t) = A \cdot t$  (a ramp), that is, if

$$R(s) = \frac{A}{s^2}$$

one obtains

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + F(s)G_1(s)/s} \frac{A}{s^2} = \lim_{s \rightarrow 0} \frac{A}{s + F(s)G_1(s)} \\ &= \frac{A}{|F(0)| \cdot |G_1(0)|} \end{aligned}$$

This shows that the ramp error is inversely proportional to the static gain of the controller. According to Figure 5.10a, the highest possible controller gain when using a P controller and a phase margin of  $40^\circ$  is required, is  $8.6\omega_{c,P} = 4.5$  (remember that  $\frac{1}{s}$  contributes with  $\frac{1}{\omega_{c,P}}$  to the loop gain at  $\omega_{c,P}$ ). Hence, to reduce the ramp error to 1% of that of the P controller, the static gain of the new controller has to be at least 450. Therefore, take  $\gamma = K/450 = 0.0296$ , and, according to the rule of thumb, let  $\tau_1 = 10/\omega_{c,d} = 9.52$ .

**Answer:**

$$F(s) = 13.3 \cdot \left( \frac{1.72s + 1}{0.31 \cdot 1.72s + 1} \right)^2 \frac{9.52s + 1}{9.52s + 0.0296}$$

- 5.11 a) The Nyquist curve is drawn based on the following observations: First, as  $\omega \rightarrow 0$ ,  $|G(i\omega)|$  increases and  $\arg G(i\omega) \rightarrow -90^\circ$ . Then, as  $\omega \rightarrow \infty$ ,  $|G(i\omega)| \rightarrow 0$  and  $\arg G(i\omega)$  decreases. We also have,  $\omega_c = 0.78$  rad/s with  $\arg G(i\omega_c) = -133^\circ$ , and finally  $\omega_p = 3.2$  rad/s with  $|G(i\omega_p)| = 0.091$ . The resulting Nyquist curve is shown in Figure 5.11a.

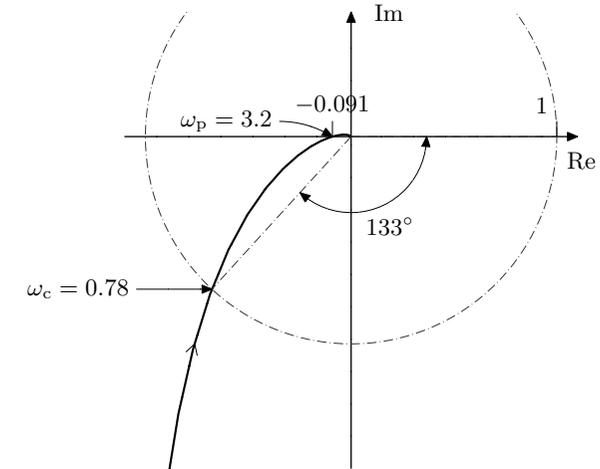


Figure 5.11a

- b) The gain margin is  $1/|G(i\omega_p)| = 11$ , which is also the highest possible proportional gain that preserves closed loop asymptotic stability.
- c) The Laplace transform of the control error is related to the reference as follows:

$$E(s) = \frac{1}{1 + KG(s)} R(s)$$

With

$$r(t) = 10t \Rightarrow R(s) = \frac{10}{s^2}$$

and using the final value theorem (from b we have that the system is stable), this yields

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{10}{2 \lim_{s \rightarrow 0} sG(s)}$$

For small  $\omega$  we have

$$G(s) \approx \frac{1}{s} \Rightarrow sG(s) \rightarrow 1, s \rightarrow 0 \Rightarrow \lim_{t \rightarrow \infty} e(t) = 5$$

d) Raising the gain curve in the Bode plot by  $K = 2$  results in

$$\omega_c = 1.24 \text{ rad/s} \quad \varphi_m = 32^\circ$$

The closed loop system becomes unstable when the phase margin is eaten up by the phase lag of the delay,

$$\arg e^{-i\omega T} = -\omega T$$

so in order to get an asymptotically stable closed loop system it is thus required that

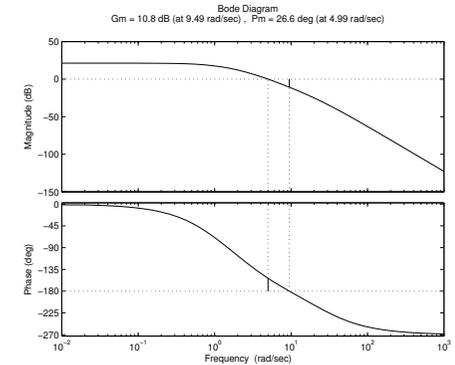
$$\omega_c T < 32^\circ \Rightarrow T < \frac{32^\circ}{1.24 \text{ rad/s}} = \frac{0.55 \text{ rad}}{1.24 \text{ rad/s}} = 0.44 \text{ s}$$

5.12 a) For this amplitude curve we cannot say anything about the stability since the system can contain an arbitrarily large time delay which could make the gain margin less than 1.

b) It is stable, since the gain is less than 1 for all frequencies; there is no risk that the Nyquist curve could encircle  $-1$  under these circumstances.

5.13 a) Enter the system and the regulator. Draw the Bode plot. This gives  $\omega_c = 5 \text{ rad/s}$ ,  $\omega_p = 9.5 \text{ rad/s}$ ,  $A_m = 3.5$  and  $\varphi_m = 27^\circ$ .

```
>> s = tf('s');
>> G = 725 / ...
 ((s + 1) * (s + 2.5) * (s + 25))
>> F = 1;
>> margin(F * G)
```



b) From a) we know that at  $\omega_{c,d} = 5 \text{ rad/s}$  the phase margin is  $27^\circ$ . In order to have  $\varphi_m \geq 60^\circ$  we need to increase the phase by approximately  $40^\circ$ , including  $6^\circ$  extra to compensate for a future lag compensator. This is obtained using a lead compensator (using standard notation of parameters) with  $\beta = 0.21$ . The phase compensation is located at the correct frequency by taking  $\tau_D = \frac{1}{\omega_{c,d} \sqrt{\beta}} = 0.43$ .

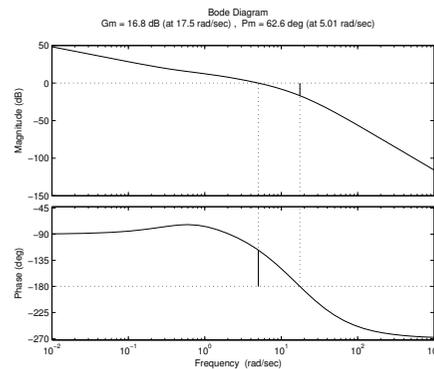
The controller gain is adjusted by the factor  $K$  to get the desired crossover frequency:

$$K \cdot \frac{1}{\sqrt{\beta}} \cdot |G(i5)| = K \cdot \frac{1}{\sqrt{0.21}} \cdot 1 = 1 \Rightarrow K = 0.46$$

The requirement  $e_0 = 0$ , that is, no steady state error for a unit step reference signal, is achieved by incorporating a lag compensator (using standard notation of parameters) with  $\gamma = 0$ , and, using the rule of thumb for the choice of  $\tau_1$ , we take  $\tau_1 = 10/5 = 2$ .

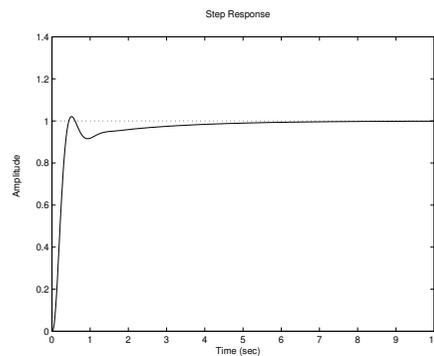
Generate a lead-lag regulator and make a Bode plot of the open loop system. Both the crossover frequency and the phase margin requirements are satisfied.

```
>> wc = 5;
>> b = 0.21;
>> tD = 1 / (wc * sqrt(b));
>> K = sqrt(b) / 1;
>> Flead = (tD * s + 1) / (b * tD * s + 1);
>> g = 0;
>> tI = 10 / wc;
>> Flag = (tI * s + 1) / (tI * s + g);
>> F = K * Flead * Flag;
>> margin(F * G)
```



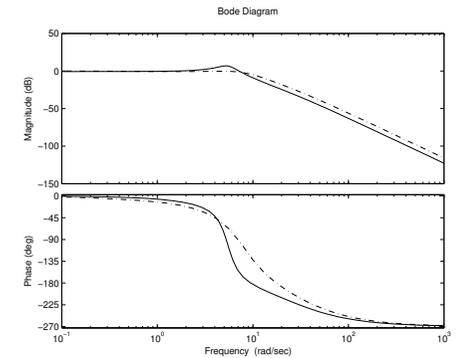
Plot the step response of the closed loop system.

```
>> Gc = feedback(F * G , 1);
>> step(Gc , 10)
```



c) Compute the transfer function of the closed loop system for  $F(s) = 1$ . Draw its Bode plot side by side with the Bode plot for the compensated system. (The curves of the compensated system are dash-dotted.)

```
>> Gc1 = feedback(G , 1);
>> bode(Gc1, '-', Gc, '-.')
```



Comparing the two Bode plots we see that the main difference is that the height of the resonance peak has been reduced, that is, the damping of the closed loop system has been increased due to the increased phase margin. We also see that the bandwidth is approximately the same, since we have not changed the gain crossover frequency.

d) Calculate the transfer function from the reference signal to the error:

$$E(s) = R(s) - F(s)G(s)E(s) \Rightarrow E(s) = \frac{1}{1 + F(s)G(s)}R(s)$$

Let

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Enter the transfer function  $S$ .  

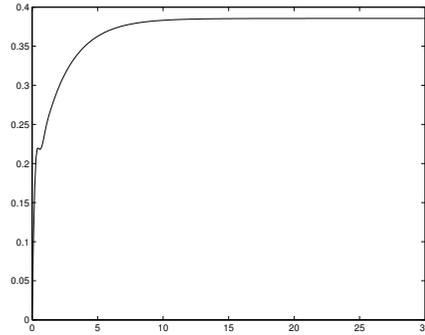
```
>> S = 1 / (1 + F * G);
```

Create a time vector between 0 and 30 with step 0.1, and a reference signal vector  $r(t) = t$ .  

```
>> t = (0 : 0.1 : 30).';
>> r = t;
```

Plot the result. Even though the steady state error for a step reference signal is zero (due to  $\gamma = 0$ ), the steady state error for a ramp reference signal is non-zero.

```
>> y = lsim(S, r, t);
>> plot(t, y)
```



5.14 The amplitude and phase at  $\omega = 0.2$  rad/s is 0.0162 and  $-140^\circ$ . We need a phase lift of  $20^\circ$  to obtain a phase margin of  $60^\circ$ . A lag part is needed to remove the steady state error. Hence we need  $6^\circ$  more in phase lift, all in all a  $26^\circ$  phase lift. This is obtain by employing lead and lag compensators (using standard notation of the parameters). First,  $N = 3$  and  $b = \omega_{c,d}/\sqrt{N} = 0.12$  give the required phase lead at the desired gain crossover frequency. Then  $K = \frac{1}{0.0162\sqrt{N}} = 35.7$  achieves that gain crossover frequency. Finally,  $a = 0.1\omega_{c,d} = 0.02$  and  $M = \infty$  remove the steady-state error.

The resulting controller is:

$$F(s) = 35.7 \cdot 3 \frac{s + 0.12}{s + 3 \cdot 0.12} \frac{s + 0.02}{s}$$

5.15 a) Combining the system's transfer function with the controller  $K$ , the loop gain becomes

$$G_o(s) = \frac{0.25K}{(\tau_1 s + 1)(\tau_2 s + 1)s}$$

which leads to the error coefficients

$$e_0 = \frac{1}{1 + \lim_{s \rightarrow 0} G_o(s)} = 0, \quad e_1 = \frac{1}{\lim_{s \rightarrow 0} s G_o(s)} = \frac{4}{K}$$

provided that  $G_c$  is stable. The Bode plot shows that stability of  $G_c$  under proportional control may be evaluated via the gain margin  $A_m$ , that is,  $G_c$  is stable if  $K < A_m$ . The Bode plot gives  $A_m = 4000$ , so the condition under which the error coefficients are defined is

$$K < 4000$$

b) The problem formulation suggests the use of a lead-lag compensator.

Let  $\omega_{c,d}$  denote the desired gain crossover frequency 100 rad/s. The Bode plot gives  $|G(i\omega_{c,d})| = 5 \cdot 10^{-4}$  and  $\arg G(i\omega_{c,d}) = -175^\circ$ . To obtain the desired phase margin, a phase lead of  $((-180^\circ) + 50^\circ + 6^\circ) - (-175^\circ) = 51^\circ$  is needed, where  $6^\circ$  has been added to ensure that the phase margin is kept even if a lag compensator is used. To this end, introduce a lead compensator in the controller:

$$F_{\text{lead}} = N \frac{s + b}{s + bN}$$

See the discussion of lead compensators in Glad&Ljung! To keep the high frequency gain of the controller as small as possible,  $N$  should be chosen as small as possible. The desired phase advance is obtained with  $N = 8$ . This phase lead is obtained at the desired crossover frequency if

$$b = \frac{\omega_{c,d}}{\sqrt{N}} = 35.4$$

The desired crossover frequency is obtained by adjusting the gain of the open loop system by introducing a factor,  $K$ , in the controller:

$$1 = K |F_{\text{lead}}(i\omega_{c,d})| \cdot |G(i\omega_{c,d})| = K \sqrt{N} \cdot 5 \cdot 10^{-4} \Rightarrow K = 707$$

Since the system contains an integrator, the step error coefficient  $e_0$  is zero. The ramp error coefficient requirement is

$$e_1 = \frac{1}{\lim_{s \rightarrow 0} s F(s) G(s)} < 0.001 \iff \frac{4}{\lim_{s \rightarrow 0} F(s)} < 0.001 \iff 4000 < \lim_{s \rightarrow 0} F(s)$$

but the controller  $KF_{\text{lead}}$  doesn't fulfill this requirement since

$$\lim_{s \rightarrow 0} KF_{\text{lead}}(s) = 707$$

Hence, the static gain of the controller must be increased by the factor  $\frac{4000}{707} = 5.7$ . To this end, introduce a lag compensator in the controller,

$$F_{\text{lag}} = \frac{s + a}{s + a/M}$$

with  $M = 5.7$  and  $a = 0.1\omega_{c,d} = 10$  (see the discussion of lag compensators in Glad&Ljung!).

The resulting controller is

$$F(s) = 707 \cdot 8 \frac{s + 35.4}{s + 282.8} \cdot \frac{s + 10}{s + 1.78}$$

## 6 Sensitivity and Robustness

6.1 The sensitivity function is the transfer function from  $v$  to  $y$ . The block diagram gives

$$Y(s) = \frac{1}{1 + \frac{K}{s(s+1)}} V(s) = \frac{s^2 + s}{\underbrace{s^2 + s + K}_{S(s)}} V(s)$$

$$|S(i\omega)| = \frac{\omega\sqrt{\omega^2 + 1}}{\sqrt{(K - \omega^2)^2 + \omega^2}}$$

For  $\omega = 1$  we get

$$|S(1i)| = \frac{\sqrt{2}}{\sqrt{(K - 1)^2 + 1}}$$

The amplitude of  $y(t)$  is less than the amplitude of  $v(t)$  if  $|S(1i)| < 1$ , that is,

$$\frac{\sqrt{2}}{\sqrt{(K - 1)^2 + 1}} < 1 \quad \Leftrightarrow \quad 2 < (K - 1)^2 + 1 \quad \stackrel{K \geq 0}{\Leftrightarrow} \quad K > 2$$

6.2 Determine the upper limit of the relative model error

$$G_{\Delta}(s) = \frac{G^0(s) - G(s)}{G(s)} = s \quad \Rightarrow \quad |G_{\Delta}(i\omega)| = \omega$$

The stability is then guaranteed if

$$|G_c(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{\omega} \quad \forall \omega$$

No steady state error for steps implies  $G_c(0) = 1$  and the bandwidth  $\omega_B$  is thus defined by the smallest value that satisfies

$$|G_c(i\omega)| < \frac{1}{\sqrt{2}}, \quad \omega > \omega_B$$

The curve  $1/\omega$  crosses  $1/\sqrt{2}$  at  $\omega = \sqrt{2}$ . Thus, the bandwidth must be less than  $\sqrt{2}$ . However, the curve  $|G_c(i\omega)|$  asymptotically approaches a line with slope  $-20 \text{ dB}_{20}/\text{decade}$ , which implies that  $\omega_B$  cannot be arbitrarily close to  $\sqrt{2}$ .

For example, if  $G_c$  is a first order system, then the breakpoint of the asymptote must be 1 rad/s if it shall coincide with  $1/\omega$ . The first order system with that asymptote is  $\frac{1}{1+s/1}$ , which has a bandwidth of 1 rad/s. If  $G_c$  would be a higher order system, the bandwidth could be made slightly higher, but the limited information about  $G_c$  excludes this possibility.

**Answer:** The maximum bandwidth is  $\omega_B = 1$ .

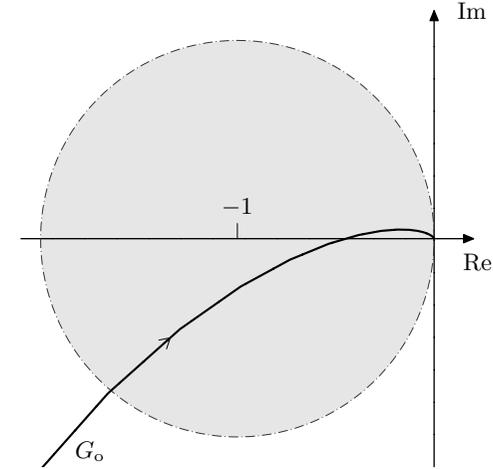


Figure 6.3a

6.3 The disturbance is amplified when the magnitude of the sensitivity function exceeds one, that is, when

$$\left| \frac{1}{1 + G_o(i\omega)} \right| > 1$$

that is

$$|1 + G_o(i\omega)| < 1$$

which corresponds to the part of  $G_o(i\omega)$  that is within a circle with center at  $-1$  and radius 1, see Figure 6.3a.

6.4 Let

$$g(\omega) = \frac{0.9}{\sqrt{1 + \omega^2}}$$

denote the upper bound on the norm of the relative model error. Robustness condition:

$$|T(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{g(\omega)} \quad \forall \omega$$

Now,

$$\begin{aligned} F(s)G(s) &= \frac{s+10}{s} \frac{1}{s+10} = \frac{1}{s} \Rightarrow \\ \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| &= \left| \frac{1}{i\omega + 1} \right| = \frac{1}{\sqrt{\omega^2 + 1}} \end{aligned}$$

so the robustness condition becomes

$$\begin{aligned} \forall \omega : \frac{1}{\sqrt{\omega^2 + 1}} &< \frac{\sqrt{\omega^2 + 1}}{0.9} \Leftrightarrow \\ \forall \omega : 0.9 &< \omega^2 + 1 \end{aligned}$$

which is satisfied.

**Answer:** Yes.

6.5 a) Using notation similar to that in Glad&Ljung, we have

$$G_{\Delta}(s) = e^{-sT} - 1$$

that is,  $G_{\Delta}(i\omega) = \cos \omega T - 1 - i \sin \omega T$ . This implies

$$|G_{\Delta}(i\omega)| = \sqrt{2 - 2 \cos \omega T}$$

and in particular

$$|G_{\Delta}(i\omega)| = \begin{cases} 0, & \text{when } \cos \omega T = 1 \\ 2, & \text{when } \cos \omega T = -1 \end{cases}$$

In Figure 6.5a,  $|G_{\Delta}(i\omega)|^{-1}$  is plotted as a function of  $\omega T$ .

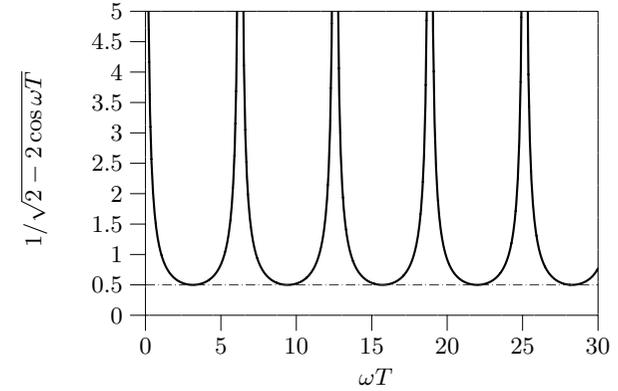


Figure 6.5a

b) The robustness criterion results in

$$\forall \omega : \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{|G_{\Delta}(i\omega)|}$$

Figure 6.5a therefore provides the answer.

**Answer:**

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right| < \frac{1}{2}$$

6.6 a) First identify the relative model error:

$$G^0(s) = G(s) + \tilde{G}(s) = G(s) \left( 1 + \frac{\tilde{G}(s)}{G(s)} \right)$$

that is,

$$G_{\Delta}(s) = \frac{\tilde{G}(s)}{G(s)}$$

The robustness criterion

$$\forall \omega : \left| \frac{1}{G_{\Delta}(i\omega)} \right| = \left| \frac{G(i\omega)}{\tilde{G}(i\omega)} \right| > \left| \frac{KG(i\omega)}{1 + KG(i\omega)} \right|$$

gives

$$\begin{aligned} |\tilde{G}(i\omega)| &< \frac{|\omega(i\omega + 5) + K|}{|Ki\omega(i\omega + 5)|} = \frac{2}{25} \cdot \sqrt{\frac{(25/2 - \omega^2)^2 + 25\omega^2}{\omega^2(\omega^2 + 25)}} = \\ &= \frac{2}{25} \cdot \sqrt{\frac{\omega^4 + (25/2)^2}{\omega^2(\omega^2 + 25)}} =: g(\omega) \end{aligned}$$

Because  $g(\omega) \rightarrow 2/25$  as  $\omega \rightarrow \infty$  stability cannot be guaranteed when  $\tilde{G}(s) = 1$ . Also note that the requirement that  $G^0(i\omega)F(i\omega) \rightarrow 0$  as  $\omega \rightarrow \infty$  fails, since  $G^0(i\omega) \rightarrow 1$ ,  $\omega \rightarrow \infty$ .

b) When  $\tilde{G}(s) = \alpha$  the closed loop system becomes

$$\frac{KG^0(s)}{1 + KG^0(s)} = \frac{K(1 + \alpha s(s + 5))}{s(s + 5) + K(1 + \alpha s(s + 5))}$$

with characteristic equation

$$s^2(2 + 25\alpha) + 5s(2 + 25\alpha) + 25 = 0$$

Rouths algorithm gives the condition

$$2 + 25\alpha > 0 \quad \Leftrightarrow \quad \alpha > -2/25$$

This is not contradictory since the robustness criterion is a *sufficient* but not *necessary* condition.

6.7 a) The characteristic equation can be determined for a generic nominal loop gain. Note that you are not required to derive the generic formula — just make sure that you are able to determine the correct polynomials  $P(s)$  and  $Q(s)$  below. Let

$$G_o(s) = \frac{b(s)}{a(s)}$$

denote the nominal loop gain. The true closed loop system becomes

$$G_c(s) = \frac{\frac{b(s)}{a(s)} \frac{\alpha}{s+\alpha}}{1 + \frac{b(s)}{a(s)} \frac{\alpha}{s+\alpha}} = \frac{b(s)\alpha}{a(s)(s+\alpha) + b(s)\alpha} = \frac{b(s)\alpha}{a(s)s + (a(s) + b(s))\alpha}$$

and has the same root locus with respect to  $\alpha$  as the open loop system

$$\frac{a(s) + b(s)}{a(s)s} = \frac{G_o + 1}{s}$$

has with respect to a proportional feedback. This can be used to draw the root locus using MATLAB. However, to draw the root locus by hand, we use that here  $G_o(s) = KG(s)$ , so

$$b(s) = 4 \quad a(s) = s(s + 1)$$

which lets us identify the polynomials  $P$  and  $Q$  in the characteristic equation  $P(s) + \alpha Q(s) = 0$  as

$$P(s) = a(s)s = s^2(s + 1) \quad Q(s) = a(s) + b(s) = s^2 + s + 4$$

◇ Starting points  $\Rightarrow$  zeros of  $P(s)$ : 0 (double), and  $-1$

End points  $\Rightarrow$  zeros of  $Q(s)$ :  $-\frac{1}{2} \pm i\frac{\sqrt{15}}{2}$

◇ Number of asymptotes:  $3 - 2 = 1$ .

Direction of asymptote:  $\frac{1}{1} \cdot \pi$ , that is, the negative real axis.

◇ Part of the real axis that belongs to the root locus:  $(-\infty, -1]$ .

◇ Intersection with the imaginary axis: Set  $s = i\omega$  and solve the characteristic equation:

$$-\omega^2(i\omega + 1) + \alpha(-\omega^2 + i\omega + 4) = 0$$

Isolate real and imaginary parts:

$$\begin{cases} -\omega^2(1 + \alpha) + 4\alpha = 0 \\ -\omega^3 + \alpha\omega = 0 \end{cases}$$

with solutions

$$(\alpha = 0, \omega = 0) \quad \text{or} \quad (\alpha = 3, \omega = \pm\sqrt{3})$$

The root locus is shown in Figure 6.7a, from which the conclusion immediately follows.

**Answer:** Asymptotically stable for  $\alpha > 3$ .

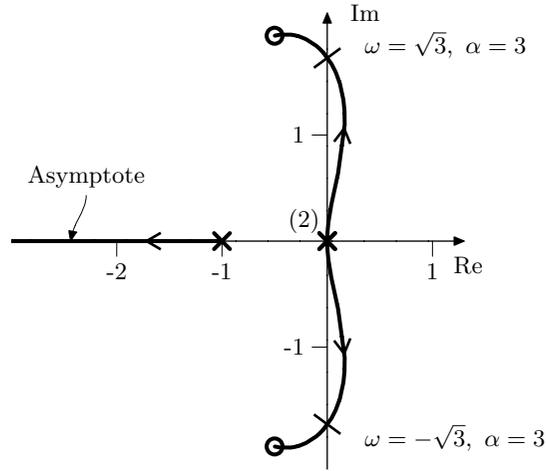


Figure 6.7a

b) Begin by identifying the relative model error:

$$G^0(s) = G(s) \frac{\alpha}{s + \alpha} = G(s) \left( 1 + \underbrace{\frac{\alpha}{s + \alpha} - 1}_{G_{\Delta}(s)} \right)$$

Thus

$$\frac{1}{|G_{\Delta}(i\omega)|} = \left| \frac{s + \alpha}{-s} \right| = \frac{\sqrt{\omega^2 + \alpha^2}}{\omega} =: f(\omega)$$

The robustness criterion  $\forall \omega : |G_c(i\omega)| < f(\omega)$  is fulfilled if the low frequency asymptote of  $f(\omega)$  exceeds the resonance peak at  $\omega = 2$ , where  $|G_c(i2)| = 2$ . This gives the condition

$$\frac{\sqrt{4 + \alpha^2}}{2} > 2 \quad \Leftrightarrow \quad \alpha > \sqrt{12}$$

**Answer:**  $\alpha > \sqrt{12}$

c) The robustness criterion gives a sufficient but not necessary condition, that is, the system can be stable even if the criterion is not satisfied. In this case for  $3 < \alpha < \sqrt{12}$ . With a root locus we obtain an exact

characterization of the stabilizing parameter values, that is, a necessary and sufficient condition.

6.8 Since the equation for  $G_c$  has the same “ $F$ ” in the numerator and the denominator, it follows that the complementary sensitivity function  $T$  and  $G_c$  are the same. It can be shown that both  $F(i\omega)G(i\omega)$  and  $F(i\omega)G^0(i\omega)$  tend to 0 as  $\omega \rightarrow \infty$ . The robustness criterion guarantees stability if  $|T(i\omega)| < 1/(\gamma\omega)$  since

$$|G_{\Delta}(i\omega)| < \gamma\omega \quad \Rightarrow \quad \frac{1}{\gamma\omega} < \frac{1}{|G_{\Delta}(i\omega)|}$$

The transfer function  $T$  has a resonance peak at  $\omega = 1$  (seen in Figure 6.8a, since  $T = G_c$ ) with  $|T(i1)| = 35$ , which leads to the condition

$$35 < \frac{1}{\gamma \cdot 1} \quad \Leftrightarrow \quad \gamma < \frac{1}{35}$$

Trivially,  $\gamma$  must also be positive.

**Answer:**  $0 \leq \gamma < \frac{1}{35}$

6.9 The closed loop system becomes

$$Y(s) = V(s) + G_o(s)(R(s) - N(s) - Y(s)) \quad \Rightarrow$$

$$Y(s) = \frac{G_o(s)}{1 + G_o(s)}(R(s) - N(s)) + \frac{1}{1 + G_o(s)}V(s)$$

where we can identify

$$T(s) = \frac{G_o(s)}{1 + G_o(s)} \quad S(s) = \frac{1}{1 + G_o(s)}$$

Notice that  $S(s) + T(s) = 1$ . In the problem formulation we have  $Y(s) = S(s)V(s)$  since the other inputs are zero. Hence, for  $v(t) = \sin t$ , we have

$$\mathcal{L}^{-1}\{SV\}(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)$$

and thus for  $n(t) = \sin t$ ,

$$Y(s) = -T(s)N(s) = -(1 - S(s))N(s) = S(s)N(s) - N(s) \quad \Rightarrow$$

$$y(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right) - \sin(t)$$

6.10 a) Putting

$$G^0(s) = G(s) \frac{1}{(s+1)} = G(s)(1 + G_{\Delta}(s))$$

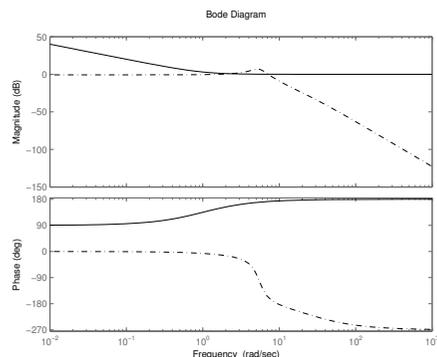
gives

$$G_{\Delta}(s) = -\frac{s}{s+1} \Leftrightarrow \frac{1}{G_{\Delta}(s)} = -\frac{s+1}{s}$$

b) Enter the system and the regulator from Problem 5.13.

```
>> s = tf('s');
>> G = 725 / ...
 ((s + 1) * (s + 2.5) * (s + 25));
>> wc = 5;
>> b = 0.21;
>> tD = 1 / (wc * sqrt(b));
>> K = sqrt(b) / 1;
>> Flead = (tD * s + 1) / (b * tD * s + 1);
>> g = 0;
>> tI = 10 / wc;
>> Flag = (tI * s + 1) / (tI * s + g);
>> F = K * Flead * Flag;
>> IDG = - (s + 1) / s;
>> T = feedback(1 * G, 1);
>> bode(IDG, 'k-', ...
 T, 'k-.');
```

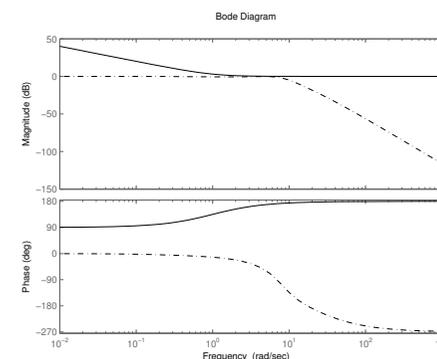
Enter the inverse relative model error and the complementary sensitivity function obtained when  $G(s)$  is controlled by  $F(s) = 1$ . Plot the amplitude curve of the inverse relative model error in the same diagram as the amplitude curve of the complementary sensitivity function.



Since the absolute value of the complementary sensitivity function goes above the inverse relative model error over a frequency interval, we cannot guarantee that the closed loop system obtained when  $G^0(s)$  is controlled by  $F(s)$  is asymptotically stable.

Enter the complementary sensitivity function obtained when  $G(s)$  is controlled by the lead-lag regulator designed in Problem 5.13. Plot the amplitude curve of the inverse relative model error in the same diagram as the amplitude curve of the complementary sensitivity function.

```
>> T = feedback(F * G, 1);
>> bode(IDG, 'k-', ...
 T, 'k-.');
```



In this case  $|T(i\omega)|$  stays below the inverse relative model error, and hence we can guarantee that the closed loop system obtained when the lead-lag regulator is applied to  $G^0(s)$  will be asymptotically stable.

6.11 The transfer function between the reference and the error is the sensitivity function. When the reference signal is a sinus the error signal will also be a sinus with the same frequency and with an amplitude modified by the gain of the transfer function at that frequency,  $|S(0.1i)| = -20 \text{ dB}_{20} = 0.1$ . This gives that the amplitude of the error is 0.2.

6.12 A way to see if the controller also stabilizes the system at 400 r/min is to look at the phase and amplitude margin of

$$F(s)G(s) = 35.7 \cdot 3 \frac{s + 0.116}{s + 0.116 \cdot 3} \frac{s + 0.02}{s} \frac{0.02}{s + 0.02} \frac{e^{-2s}}{1 + 20s}$$

A bode plot of this system is given in Figure 6.12a where it can be seen that the phase margin is  $9.54^\circ$  and that the amplitude margin is 1.3. The closed loop system is stable but the margin is small.

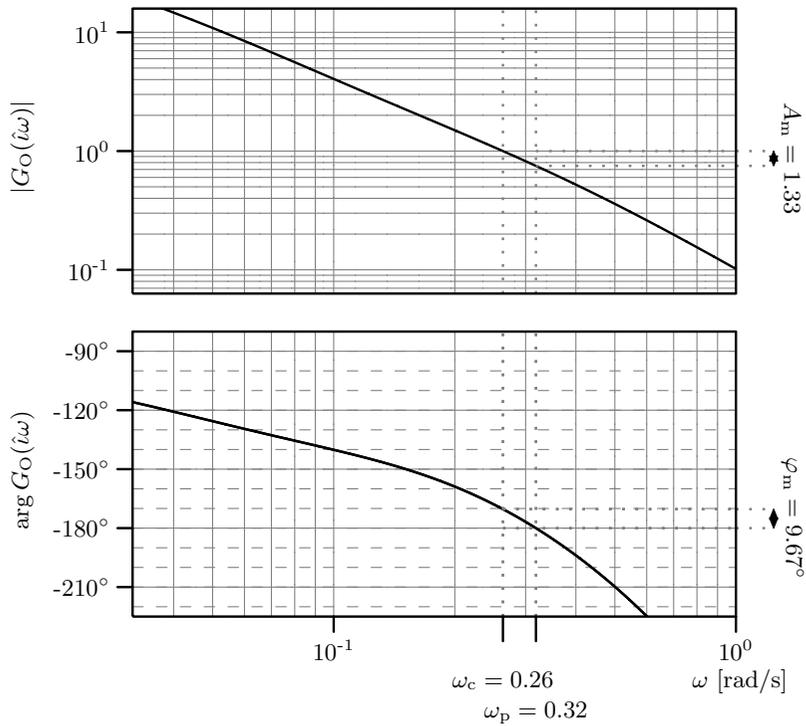


Figure 6.12a

6.13 The sensitivity function is given by

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

which in this case means

$$S(s) = \frac{(s + 1)^2}{(s + 1)^2 + K}$$

The demand that the amplification of the sensitivity function should be less than 1 at  $\omega = 1$  gives

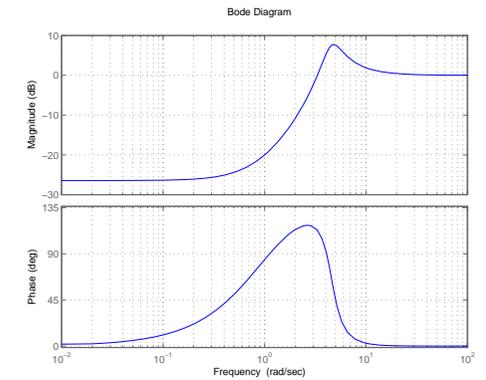
$$|S(i1)| = \frac{2}{\sqrt{4 + K^2}} \leq 0.1$$

that is,  $K \geq \sqrt{396} \approx 19.9$ .

To illustrate, the condition is verified in MATLAB.

Enter the system and create the sensitivity function. Plot with a grid.

```
>> s = tf('s');
>> G = 1 / (s + 1)^2;
>> K = 20;
>> S = minreal(1 / (1 + K * G));
>> bode(S);
>> grid;
```



# 7 Special Controller Structures

7.1 a) Derive the transfer function:

$$\theta(s) = \frac{1}{(1 + 30s)(1 + 3s)} \theta_m(s)$$

$$\theta_m(s) = \frac{G_{R2}(s)}{(1 + 10s) + G_{R2}(s)} W(s)$$

$G_{R2}(s) = K_2 = 9$  gives

$$\theta(s) = \frac{0.9}{(1 + \frac{s}{0.033})(1 + \frac{s}{0.33})(1 + s)} W(s) =: G(s)W(s)$$

Thus,

$$|G(i\omega)| = \frac{0.9}{\sqrt{1 + (\frac{\omega}{0.033})^2} \sqrt{1 + (\frac{\omega}{0.33})^2} \sqrt{1 + \omega^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow 0.9, \omega \rightarrow 0$$

and

$$\arg G(i\omega) = -\arctan \frac{\omega}{0.033} - \arctan \frac{\omega}{0.33} - \arctan \omega$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 0.9, and the breakpoints and slopes of the asymptotes:

|                   |   |       |    |      |    |   |    |
|-------------------|---|-------|----|------|----|---|----|
| Frequency [rad/s] |   | 0.033 |    | 0.33 |    | 1 |    |
| Slope             | 0 |       | -1 |      | -2 |   | -3 |

The phase curve is drawn based on a couple of samples:

|                   |       |      |       |       |       |
|-------------------|-------|------|-------|-------|-------|
| Frequency [rad/s] | 0.033 | 0.1  | 0.2   | 0.5   | 1.0   |
| Phase             | -52°  | -94° | -123° | -169° | -205° |

The Bode plot in Figure 7.1a gives that the gain crossover frequency and the phase margin are undefined, but we have a gain margin:

$$\omega_p = 0.61 \text{ rad/s} \quad A_m = 50.5$$

A gain margin of 2 is obtained when

$$K_1 \cdot \frac{1}{50.5} = \frac{1}{2}$$

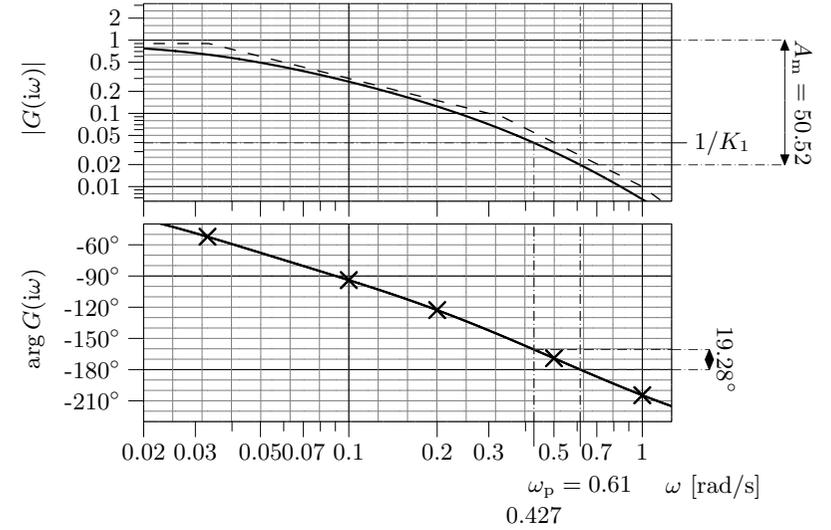


Figure 7.1a

that is,  $K_1 = 25.25$ . This results in a new gain crossover of 0.43 rad/s (and new phase margin of 19°). To find the steady state error, study how the Laplace transforms of the controll error relates to that of the reference:

$$E(s) = \frac{1}{1 + K_1 G(s)} \theta_{\text{ref}}(s)$$

which with  $\theta_{\text{ref}}(s) = \frac{a}{s}$  gives

$$\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \frac{a}{1 + K_1 \cdot 0.9} = 0.042 \cdot a$$

b) Without the internal feedback we get the transfer function defined by

$$\theta(s) = \frac{1}{(1 + \frac{s}{0.033})(1 + \frac{s}{0.1})(1 + \frac{s}{0.33})} W(s) =: G(s)W(s)$$

and thus

$$|G(i\omega)| = \frac{1}{\sqrt{1 + (\frac{\omega}{0.033})^2} \sqrt{1 + (\frac{\omega}{0.33})^2} \sqrt{1 + (\frac{\omega}{0.1})^2}}$$

with low frequency asymptote

$$|G(i\omega)| \rightarrow 1, \omega \rightarrow 0$$

and

$$\arg G(i\omega) = -\arctan \frac{\omega}{0.033} - \arctan \frac{\omega}{0.33} - \arctan \frac{\omega}{0.1}$$

The gain is drawn approximatively based on a known gain at some point of the low frequency asymptote, 1, and the breakpoints and slopes of the asymptotes:

|                   |   |       |    |     |    |      |    |
|-------------------|---|-------|----|-----|----|------|----|
| Frequency [rad/s] |   | 0.033 |    | 0.1 |    | 0.33 |    |
| Slope             | 0 |       | -1 |     | -2 |      | -3 |

The phase curve is drawn based on a couple of samples:

|                   |       |       |       |       |
|-------------------|-------|-------|-------|-------|
| Frequency [rad/s] | 0.033 | 0.1   | 0.2   | 0.4   |
| Phase             | -69°  | -134° | -174° | -212° |

The Bode plot in Figure 7.1b gives that, again, the gain crossover frequency and phase margin are undefined, but we have a gain margin:

$$\omega_p = 0.22 \text{ rad/s} \quad A_m = 19$$

A gain margin of 2 is obtained when  $K_1 \cdot \frac{1}{19} = \frac{1}{2}$ , which leads to  $K_1 = 9.5$ . This results in a new gain crossover of 0.15 rad/s (and a new phase margin of 21°). As above, we get the controll error for step references:

$$\lim_{t \rightarrow \infty} e(t) = \frac{1}{1 + 9.5 \cdot 1} a = 0.095a$$

We conclude that due to the internal feedback, the system in a) is faster (higher bandwidth) as well as more precise (smaller stationary error).

7.2 Consider the block diagram in Figure 7.2a. The change in tank volume per time unit is given by

$$A \frac{d}{dt} h(t) = x(t) - v(t)$$

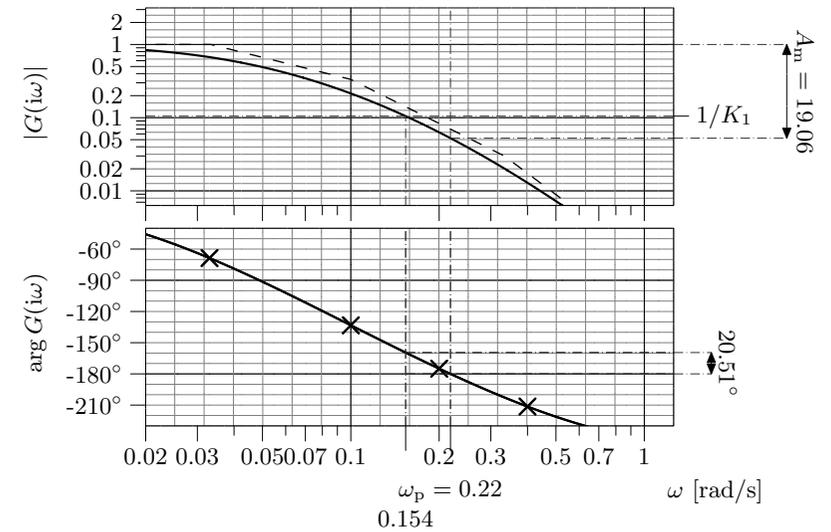


Figure 7.1b

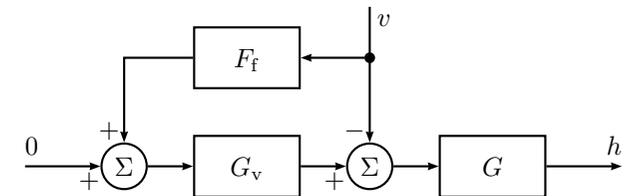


Figure 7.2a

or, equivalently,

$$A \cdot s \cdot H(s) = X(s) - V(s)$$

which gives

$$H(s) = \frac{1}{As} (X(s) - V(s))$$

Furthermore,

$$X(s) = G_v(s)U(s)$$

where

$$G_v(s) = \frac{1}{1 + s/2}$$

a) We let the input  $u(t)$  be a function of  $v(t)$  only, that is,

$$U(s) = F_f(s)V(s)$$

The level  $h(t)$  as a function of  $v(t)$  then becomes

$$H(s) = \frac{1}{As}(G_v(s)F_f(s) - 1)V(s)$$

If we choose

$$F_f(s) = \frac{1}{G_v(s)} = 1 + s/2$$

the level becomes independent of  $v(t)$ , but to get the controller Stu uses, we remove the derivative term:

$$F_f(s) = 1$$

The level as a function of  $v(t)$  then becomes

$$H(s) = \frac{1}{As} \left( \frac{1}{1 + s/2} - 1 \right) V(s) = -\frac{1}{2A} \frac{1}{1 + s/2} V(s)$$

With  $V(s) = 0.1/s$  this yields

$$H(s) = -\frac{0.1}{2A} \frac{1}{s(1 + s/2)} = -\frac{0.1}{2A} \left( \frac{1}{s} - \frac{1}{2 + s} \right)$$

that is

$$h(t) = -\frac{0.1}{A \cdot 2} (1 - e^{-2t})$$

which gives the steady state error  $-0.05/A$ .

b) We now choose the input  $u(t)$  to be a function of both  $h(t)$  and  $v(t)$ , that is, we add the term  $-Kh(t)$  to the control law from a). (See Figure 7.2b.) Thus

$$u(t) = -Kh(t) + v(t)$$

or, equivalently,

$$U(s) = -KH(s) + V(s)$$

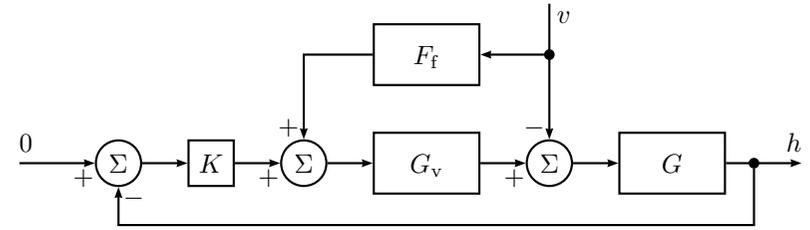


Figure 7.2b

This gives

$$\begin{aligned} AsH(s) &= G_v(s)(-KH(s) + V(s)) - V(s) \\ (As + KG_v(s))H(s) &= (G_v(s) - 1)V(s) \\ \frac{H(s)}{V(s)} &= \frac{-s/2}{A/2 \cdot s^2 + As + K} = \frac{-s}{A(s^2 + 2s + 2K/A)} \end{aligned}$$

To select  $K$ , we may compare\*

$$s^2 + 2s + 2K/A = 0$$

with the standard equation

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0$$

which gives

$$\omega_0^2 = 2 \cdot K/A \quad \zeta\omega_0 = 1$$

To obtain approximately 5% overshoot we choose  $\zeta = 0.707$ , and from

$$\sqrt{A/(2K)} = \zeta = 0.707$$

we get  $K = A$ . Hence,

$$\frac{H(s)}{V(s)} = \frac{-s}{A(s^2 + 2s + 2)}$$

If  $v(t)$  is a step of amplitude 0.1, the final level becomes

$$\lim_{t \rightarrow \infty} h(t) = 0$$

\*Note that any  $K > 0$  results in a stable closed loop system, and that the steady state error computations below are independent of the particular value of  $K$ . Hence, selecting  $K$  is not necessary for the solution of this problem.

that is, there will be no steady state error in the level for a step disturbance.

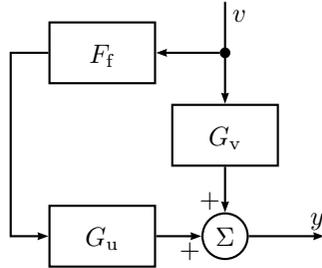


Figure 7.3a

7.3 a) A block diagram of the system is given in Figure 7.3a. The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{2}{s+3} \quad G_v(s) = \frac{3}{s+4}$$

Chose  $F_f$  such that  $(G_v + G_u F_f)V = 0$ :

$$F_f = -\frac{G_v}{G_u} = -\frac{3(s+3)}{2(s+4)}$$

Compute the controller.

```
>> s = tf('s');
>> Gu = 2 / (s + 3);
>> Gv = 3 / (s + 4);
>> F = - Gv / Gu;
```

b) If  $v(t) = 2 \sin \omega t$  then

$$u(t) = 2 |F_f(i\omega)| \sin(\omega t + \arg F_f(i\omega))$$

The amplitude is then

$$A(\omega) = 2 |F_f(i\omega)| = 2 \cdot \frac{3}{2} \sqrt{\frac{\omega^2 + 9}{\omega^2 + 16}} \leq 3$$

$$A(\omega) \rightarrow 3, \omega \rightarrow \infty$$

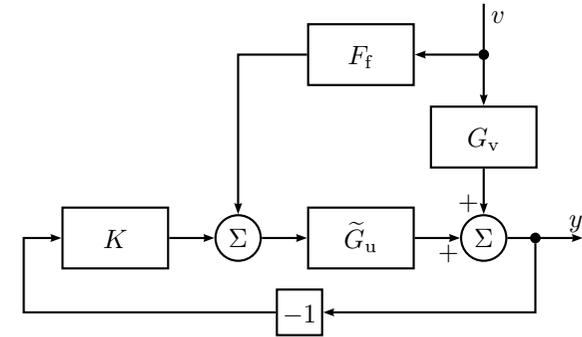


Figure 7.3b

c) A block diagram of the system with both feedforward and feedback is shown in Figure 7.3b. The output is now given by

$$Y = G_v V + \tilde{G}_u U = (G_v + \tilde{G}_u F_f)V - \tilde{G}_u K Y$$

where

$$\tilde{G}_u(s) = \frac{b}{s+3}$$

The transfer function from  $V$  to  $Y$  is given by

$$\begin{aligned} Y(s) &= \frac{G_v + \tilde{G}_u F_f}{1 + \tilde{G}_u K} V(s) = \frac{\frac{3}{s+4} - \frac{3b}{2(s+4)}}{1 + K \frac{b}{s+3}} V(s) \\ &= \frac{3(1 - b/2)(s+3)}{(s+4)(s+3) + Kb(s+4)} V(s) \end{aligned}$$

This is stable for  $K \geq 0$  and  $b \geq 0$ . The final value theorem can therefore be used (with  $V(s) = \frac{1}{s}$ ):

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} s Y(s) = \lim_{s \rightarrow 0} s \frac{3(1 - b/2)(s+3)}{(s+4)(s+3) + Kb(s+4)} \cdot \frac{1}{s} = \frac{9(1 - b/2)}{12 + 4Kb}$$

7.4 a) The output is given by

$$Y = (G_v + G_u F_f)V$$

where

$$G_u(s) = \frac{3}{s+1} \quad G_v(s) = \frac{4}{(s+2)(s+5)}$$

Chose  $F_f$  such that  $(G_v + G_u F_f)V = 0$ :

$$F_f(s) = -\frac{G_v(s)}{G_u(s)} = -\frac{4(s+1)}{3(s+2)(s+5)}$$

Create the system and the feedforward controller.

```
>> s = tf('s');
>> Gv = 4 / (s + 2) / (s + 5);
>> Gu = 3 / (s + 1);
>> F = - Gv / Gu;
```

b) The constant to replace  $F_f(s)$  is given by

$$\tilde{F}_f = F_f(0) = -\frac{4}{30}$$

The output is then given by

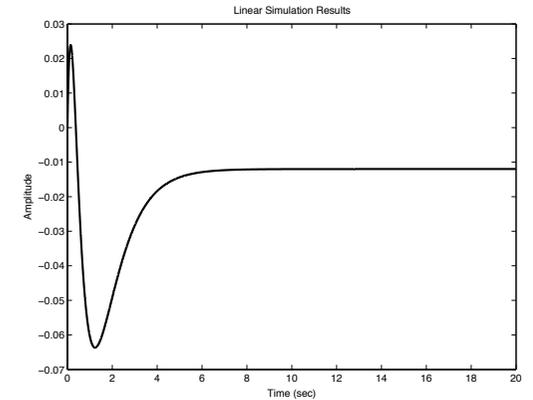
$$\begin{aligned} Y(s) &= \left( -\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)} \right) V(s) = \frac{40(s+1) - 4(s+2)(s+5)}{10(s+1)(s+2)(s+5)} V(s) \\ &= \frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)} V(s) \end{aligned}$$

Taking the Laplace transform of  $v(t) = -1 - 0.1t$  we get  $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$ . The final value theorem then gives (verify that the system is stable)

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} s \frac{-4s^2 + 12s}{10(s+1)(s+2)(s+5)} \left( -\frac{1}{s} - \frac{0.1}{s^2} \right) \\ &= \frac{12}{100} \cdot (-0.1) = -0.012 \end{aligned}$$

Create the system with the controller and create the disturbance signal.

```
>> F = -4/30;
>> G = F * Gu + Gv;
>> t = (0 : 0.001 : 20).';
>> v = -1 - 0.1*t;
>> lsim(G, v, t)
```



c) With the P controller the output is given by

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \left( -\frac{12}{30(s+1)} + \frac{4}{(s+2)(s+5)} \right) V(s)$$

which means that

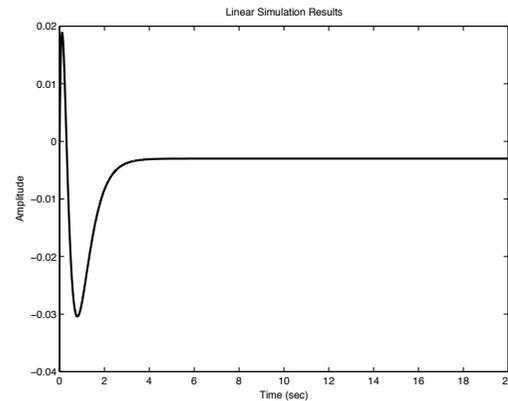
$$Y(s) = \frac{\frac{40(s+1) - 4(s+2)(s+5)}{10(s+1)(s+2)(s+5)}}{1 + \frac{3K}{s+1}} V(s) = \frac{-0.4s^2 + 1.2s}{(s+3K+1)(s+2)(s+5)} V(s)$$

Using the same disturbance,  $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$ , the final value theorem gives (verify that the system is stable)

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} s \frac{-0.4s^2 + 1.2s}{(s+3K+1)(s+2)(s+5)} \left( -\frac{1}{s} - \frac{0.1}{s^2} \right) \\ &= \frac{1.2}{(3K+1) \cdot 10} \cdot (-0.1) = -\frac{0.012}{3K+1} \end{aligned}$$

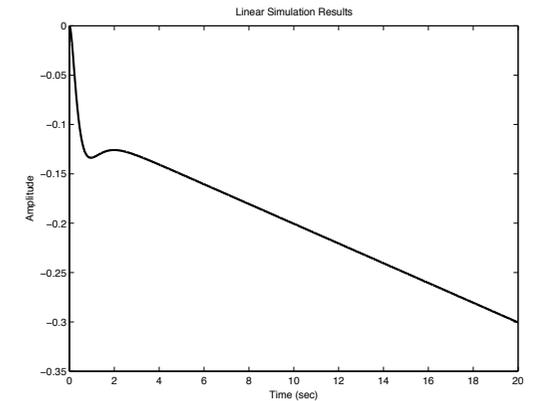
Create the new closed loop system with different values on  $K$ .

```
>> K = 1;
>> Gc = minreal(G / (1 + K * Gu));
>> lsim(Gc, v, t)
```



Simulate the output.

```
>> Gc = minreal(Gv / (1 + K * Gu));
>> lsim(Gc, v, t)
```



d) When only a P controller is used we have the following relationship between the disturbance and the output

$$Y(s) = -\frac{3}{(s+1)}KY(s) + \frac{4}{(s+2)(s+5)}V(s)$$

which means that

$$Y(s) = \frac{4(s+1)}{(s+2)(s+5)(s+3K+1)}V(s)$$

Again using the same disturbance,  $V(s) = -\frac{1}{s} - \frac{0.1}{s^2}$ , a careful inspection of  $Y(s)$  gives that there is no final value of  $y$ , hence the final value theorem does not apply.\* However, the possibility to simulate the system remains.

\*If it is assumed that the final value exists, a contradiction follows since then the final value theorem would apply, but give

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= \lim_{s \rightarrow 0} s \frac{4(s+1)}{(s+2)(s+5)(s+3K+1)} \left( -\frac{1}{s} - \frac{0.1}{s^2} \right) \\ &= -\lim_{s \rightarrow 0} \frac{4(s+1)}{(s+2)(s+5)(s+3K+1)} \frac{s+0.1}{s} = -\infty \end{aligned}$$

# 8 State Space Description

8.1 According to Solution 2.1 the differential equation for the motor is

$$\ddot{\theta} + \frac{1}{\tau}\dot{\theta} = Ku$$

where

$$\frac{fR_a + k_a k_v}{JR_a} = \frac{1}{\tau} \quad \frac{k_a}{JR_a} = K$$

Introduce the state variables  $x_1$  and  $x_2$  according to

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

This gives the state space equations

$$\begin{aligned} \dot{x}_1 &= \dot{\theta} = x_2 \\ \dot{x}_2 &= \ddot{\theta} = -\frac{1}{\tau}\dot{\theta} + Ku = -\frac{1}{\tau}x_2 + Ku \end{aligned}$$

In matrix form we get

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u \\ y &= (1 \quad 0) x \end{aligned}$$

where  $x^T = (x_1 \quad x_2)$ .

8.2 We start with the differential equations

$$\ell\ddot{\theta} + g \sin \theta + \ddot{z} \cos \theta = 0$$

The state variables

$$x_1 = \theta \quad x_2 = \dot{\theta}$$

input

$$u = \frac{\ddot{z}}{\ell}$$

and output

$$y = \theta$$

gives the (nonlinear) state space description

$$\begin{aligned} \dot{x}_1 &= x_2 =: f_1(x, u) \\ \dot{x}_2 &= \ddot{\theta} = -\frac{g}{\ell} \sin \theta - \frac{\ddot{z}}{\ell} \cos \theta = -\omega_0^2 \sin x_1 - u \cos x_1 =: f_2(x, u) \end{aligned}$$

where  $\omega_0^2 = g/\ell$ . We get that

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= (0 \quad 1) \\ \frac{\partial f_1}{\partial u} &= 0 \\ \frac{\partial f_2}{\partial x} &= (-\omega_0^2 \cos x_1 + u \sin x_1 \quad 0) \\ \frac{\partial f_2}{\partial u} &= -\cos x_1 \end{aligned}$$

Introduce  $x_{1\Delta} = x_1 - \pi$ ,  $x_{2\Delta} = x_2$ ,  $u_\Delta = u$ , and  $y_\Delta = y - \pi$ . Linearization around  $x_1 = \pi$ ,  $x_2 = 0$  and  $u = 0$  gives

$$\begin{aligned} \dot{x}_{1\Delta} &= x_{2\Delta} \\ \dot{x}_{2\Delta} &= \omega_0^2 x_{1\Delta} + u_\Delta \\ y_\Delta &= x_{1\Delta} \end{aligned}$$

8.3 Introduce the state variables

$$x_1 = y \quad x_2 = \dot{\theta} \quad x_3 = z$$

According to the figure, the variables are related as

$$\begin{aligned} X_1(s) &= Y(s) = \frac{1}{s}(M_1(s) + K_2 X_2(s)) \\ X_2(s) &= \dot{\theta}(s) = \frac{1}{s}(X_3(s) - X_1(s)) \\ X_3(s) &= Z(s) = \frac{1}{s}(K_1 I(s) - K_2 X_2(s)) \end{aligned}$$

Inverse Laplace transformation gives, in the time domain,

$$\begin{aligned}\dot{x}_1(t) &= K_2 x_2(t) + M_1(t) \\ \dot{x}_2(t) &= -x_1(t) + x_3(t) \\ \dot{x}_3(t) &= -K_2 x_2(t) + K_1 i(t)\end{aligned}$$

In matrix notation this becomes

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & K_2 & 0 \\ -1 & 0 & 1 \\ 0 & -K_2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} i(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} M_1(t) \\ y(t) &= (1 \ 0 \ 0) x(t)\end{aligned}$$

8.4 a)

$$\frac{d^3}{dt^3}y(t) + 6\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) + 6y(t) = 6u(t)$$

The state variables

$$x_1(t) = y \quad x_2(t) = \dot{y} \quad x_3(t) = \ddot{y}$$

gives

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ \dot{x}_3(t) &= \frac{d^3}{dt^3}y(t) = -6\ddot{y}(t) - 11\dot{y}(t) - 6y(t) + 6u(t) \\ &= -6x_3(t) - 11x_2(t) - 6x_1(t) + 6u(t)\end{aligned}$$

In matrix form we get

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} u(t) \\ y(t) &= (1 \ 0 \ 0) x(t)\end{aligned}$$

b)

$$\frac{d^3}{dt^3}y(t) + \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 3y(t) = 4\frac{d^2}{dt^2}u(t) + \frac{d}{dt}u(t) + 2u(t)$$

If we introduce  $x_1(t) = y(t)$  in the equation and collect all terms without differentiation on the right hand side we get

$$\frac{d^3}{dt^3}x_1(t) + \frac{d^2}{dt^2}x_1(t) + 5\frac{d}{dt}x_1(t) - 4\frac{d^2}{dt^2}u(t) - \frac{d}{dt}u(t) = -3x_1(t) + 2u(t)$$

that is

$$\frac{d}{dt} \left( \frac{d^2}{dt^2}x_1(t) + \frac{d}{dt}x_1(t) + 5x_1(t) - 4\frac{d}{dt}u(t) - u(t) \right) = -3x_1(t) + 2u(t)$$

Now introduce the expression within the parenthesis as a new state variable

$$x_2(t) = \frac{d^2}{dt^2}x_1(t) + \frac{d}{dt}x_1(t) + 5x_1(t) - 4\frac{d}{dt}u(t) - u(t)$$

that is

$$\dot{x}_2(t) = -3x_1(t) + 2u(t) \quad (8.1)$$

Repeating this procedure yields

$$\frac{d}{dt} \left( \frac{d}{dt}x_1(t) + x_1(t) - 4u(t) \right) = x_2(t) - 5x_1(t) + u(t) \quad (8.2)$$

and we can introduce

$$x_3(t) = \frac{d}{dt}x_1(t) + x_1(t) - 4u(t)$$

that is

$$\dot{x}_1(t) = x_3(t) - x_1(t) + 4u(t) \quad (8.3)$$

Equation (8.1), (8.2), and (8.3) define the state space equations

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -1 & 0 & 1 \\ -3 & 0 & 0 \\ -5 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} u(t) \\ y(t) &= (1 \ 0 \ 0) x(t)\end{aligned}$$

c) Partial fraction expansion of

$$Y(s) = \frac{2s+3}{s^2+5s+6}U(s)$$

gives

$$Y(s) = -\frac{1}{s+2}U(s) + \frac{3}{s+3}U(s)$$

Introducing the state variables

$$X_1(s) = -\frac{1}{s+2}U(s) \quad X_2(s) = \frac{3}{s+3}U(s)$$

gives

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) - u(t) \\ \dot{x}_2(t) &= -3x_2(t) + 3u(t) \end{aligned}$$

in the time domain. Furthermore, we have

$$y(t) = x_1(t) + x_2(t)$$

In matrix form

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 3 \end{pmatrix} u(t) \\ y(t) &= (1 \quad 1) x(t) \end{aligned}$$

### 8.5 The impulse response

$$g(t) = 2e^{-t} + 3e^{-4t}$$

gives the transfer function

$$G(s) = \frac{2}{s+1} + \frac{3}{s+4}$$

The output can then be written

$$Y(s) = \underbrace{\frac{2}{s+1}U(s)}_{X_1(s)} + \underbrace{\frac{3}{s+4}U(s)}_{X_2(s)}$$

Defining the state variables as above gives

$$\begin{aligned} sX_1(s) + X_1(s) &= 2U(s) \\ sX_2(s) + 4X_2(s) &= 3U(s) \end{aligned}$$

which in time domain can be written as

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + 2u(t) \\ \dot{x}_2(t) &= -4x_2(t) + 3u(t) \\ y(t) &= x_1(t) + x_2(t) \end{aligned}$$

### 8.6 The transfer function is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= (-1 \quad 2) \begin{pmatrix} s+2 & -1 \\ 0 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{(s+2)(s+3)} (-1 \quad 2) \begin{pmatrix} s+3 & 1 \\ 0 & s+2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{s}{(s+2)(s+3)} \end{aligned}$$

### 8.7

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

The state space equations have the general solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-s)}Bu(s) ds$$

The input signal  $u$  is constant, that is,  $u(t) = u_0$ , on the interval  $(t_0, t_0 + T)$ . This implies

$$x(t_0 + T) = e^{AT}x(t_0) + \left( \int_{t_0}^{t_0+T} e^{A(t_0+T-s)} ds \right) Bu_0$$

where

$$e^{AT} \quad \text{and} \quad \int_{t_0}^{t_0+T} e^{A(t_0+T-s)} ds B$$

are constant matrices.

8.8 a) Introduce the state variables  $x_1 = h$ ,  $x_2 = \int_0^t (h_{\text{ref}} - h) d\tau$  and  $x_3 = \int_0^t (h_{\text{ref}} - h) d\tau$ . This gives the following expressions for the control signals

$$\begin{aligned} u_1 &= h_{\text{ref}} - x_1 + x_2 \\ u_2 &= h_{\text{ref}} - x_1 + x_3 \end{aligned}$$

by using these expressions we can eliminate  $u_1$  and  $u_2$  form  $\dot{h} + h = u_1 + u_2$ . This gives

$$\dot{x}_1 = -x_1 + h_{\text{ref}} - x_1 + x_2 + h_{\text{ref}} - x_1 + x_3$$

By taking the Laplace transform on the expressions for  $x_2$  and  $x_3$  we obtain

$$\begin{aligned} X_2(s) &= \frac{H_{\text{ref}}(s) - H(s)}{s} \\ X_3(s) &= \frac{H_{\text{ref}}(s) - H(s)}{s} \end{aligned}$$

Inverse Laplace transformation gives

$$\begin{aligned} \dot{x}_2 &= h_{\text{ref}} - x_1 \\ \dot{x}_3 &= h_{\text{ref}} - x_1 \end{aligned}$$

In matrix notation this becomes

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} h_{\text{ref}}(t) \\ h(t) &= (1 \ 0 \ 0) x(t) \end{aligned}$$

b) The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 7 & -3 & -3 \end{pmatrix}$$

A vector which span the null space of a matrix must satisfy  $\mathcal{O}x = 0$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 1 \\ 7 & -3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This means in practise that you can't say if it is  $u_1$  or  $u_2$  or a combination of the two which fills the tank.

c) With  $h_{\text{ref}} = 0$  and  $u_1 = -h - n + \int_0^t -h - n d\tau$  we get

$$\begin{aligned} \dot{x}_2 &= -x_1 - n \\ \dot{x}_3 &= -x_1 \end{aligned}$$

and

$$\dot{x}_1 = -x_1 - x_1 - n + x_2 - x_1 + x_3$$

this gives in matrix form

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} n(t) \\ h(t) &= (1 \ 0 \ 0) x(t) \end{aligned}$$

8.9 The controllability matrix is

$$\mathcal{S} = (B \ AB) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Since  $\det \mathcal{S} = -1 \neq 0$  the system is controllable and it is possible to control the system from the origin to  $x^T = (1 \ 3)$  within 4 seconds.

8.10 a) The controllability matrix becomes

$$\mathcal{S} = (B \ AB \ A^2B) = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 3 & -9 \\ 2 & -6 & 18 \end{pmatrix}$$

and  $\det \mathcal{S} = 0$  since  $\text{rank } \mathcal{S} = 2$ . The controllable subspace is spanned by

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1.5 \\ -2 & -3 & -1.5 \\ 4 & 3 & 1.5 \end{pmatrix}$$

with  $\det \mathcal{O} = 0$ . Solving for the unobservable subspace

$$\mathcal{O}x = 0$$

gives (Gauss elimination)

$$\begin{aligned} x_1 + 3x_2 + 1.5x_3 &= 0 \\ 3x_2 + 1.5x_3 &= 0 \\ x_1 &= 0 \end{aligned}$$

Introducing  $x_3 = a$  gives  $x_2 = -0.5a$  and  $x^T = (0 \quad -0.5a \quad a)$ , that is, the silent (unobservable) subspace is spanned by

$$\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

b) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & -8 & 16 \\ -2 & 8 & -32 \end{pmatrix}$$

with  $\text{rank } \mathcal{S} = 2$ . The controllable subspace is spanned by, for example,

$$\begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -8 \\ 8 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & -6 & 0 \\ -9 & 12 & 0 \end{pmatrix}$$

Solving for the unobservable subspace  $\mathcal{O}x = 0$  gives

$$x = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$$

The unobservable subspace is spanned by

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

8.11 a)

$$\begin{aligned} \dot{x}_1 &= -x_1 + u &\Rightarrow x_1 &= 1 - e^{-t} \\ \dot{x}_2 &= 2x_2 + u &\Rightarrow x_2 &= 0.5(e^{2t} - 1) \end{aligned}$$

b) The system is not asymptotically stable since  $x_2 \rightarrow \infty$  as  $t \rightarrow \infty$ , but input-output stable because the transfer function has its pole in the complex left hand plane.

c)

$$\mathcal{S} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \quad \det \mathcal{S} = 3$$

The system is controllable.

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad \det \mathcal{O} = 0$$

The system is not observable.  $\mathcal{O}x = 0$  has solutions

$$x = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

This implies that the second component of the state vector cannot be seen in the output.

d) Because the second component of the state vector has unconstrained growth and this is not reflected in the output, the system will finally collapse.

8.12

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= (1 \quad 1) \begin{pmatrix} s-1 & 1 \\ -2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{s+1}{(s-1)^2 + 2} \end{aligned}$$

This gives poles in  $1 \pm i\sqrt{2}$  and zeros in  $-1$ .

8.13 a) For pendulum 1 we have

$$\ddot{z} \cos(\phi_1) + \alpha \ddot{\phi}_1 = \sin(\phi_1)$$

and for pendulum 2

$$\ddot{z} \cos(\phi_2) + \ddot{\phi}_2 = \sin(\phi_2)$$

Linearization gives

$$\ddot{z} + \alpha \ddot{\phi}_1 = \phi_1$$

$$\ddot{z} + \ddot{\phi}_2 = \phi_2$$

Consider  $\ddot{z}$  as an input to the system (the acceleration of the trolley  $\sim$  the force applied to the system). Introduce the state variables

$$x_1 = \phi_1 \quad x_2 = \dot{\phi}_1 \quad x_3 = \phi_2 \quad x_4 = \dot{\phi}_2$$

This gives the state space equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{\alpha} x_1 - \frac{u}{\alpha}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = x_3 - u$$

In matrix form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/\alpha \\ 0 \\ -1 \end{pmatrix} u$$

b) The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 0 & -1/\alpha & 0 & -1/\alpha^2 \\ -1/\alpha & 0 & -1/\alpha^2 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \quad \det \mathcal{S} = \frac{1}{\alpha^2} \left(1 - \frac{1}{\alpha}\right)^2$$

Thus, the system is controllable except for the case  $\alpha = 1$ , that is, when the two pendulums have the same lengths. If the pendulums have different lengths they react differently to the input, but if they have the same length there is no possibility to act upon them separately using the input.

8.14 The figure gives

$$X_1(s) = \frac{1}{(s+1)} U(s) \quad \Rightarrow \quad sX_1(s) = -X_1(s) + U(s)$$

and

$$X_2(s) = \frac{1}{(s+3)} (U(s) + X_1(s)) \quad \Rightarrow \quad sX_2(s) = -3X_2(s) + U(s) + X_1(s)$$

Inverse Laplace transformation gives

$$\dot{x}_1 = -x_1 + u$$

$$\dot{x}_2 = -3x_2 + x_1 + u$$

In matrix form this becomes

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

8.15 a) Mass balance gives

$$\frac{d(Vc_A)}{dt} = Vr_A + qc_{A,\text{in}} - qc_A$$

$$\frac{d(Vc_B)}{dt} = Vr_B + qc_B$$

By using  $r_A = -k_1 c_A^3$  and  $r_B = \frac{-r_A}{3}$  the following expression is obtained

$$V \frac{dc_A}{dt} = -Vk_1 c_A^3 + qc_{A,\text{in}} - qc_A$$

$$V \frac{dc_B}{dt} = \frac{Vk_1 c_A^3}{3} - qc_B$$

b) Linearization around  $c_A^*$ ,  $c_B^*$ , and  $c_{A,\text{in}}^*$  gives

$$\frac{d}{dt} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} = \begin{pmatrix} \frac{-q-3k_1 c_A^* V}{V} & 0 \\ k_1 c_A^* & \frac{-q}{V} \end{pmatrix} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix} + \begin{pmatrix} \frac{q}{V} \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} c_{A,\Delta} \\ c_{B,\Delta} \end{pmatrix}$$

8.16 a)

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} u$$

$$y = (0 \ 1) x$$

with

$$x = (x_1 \ x_2)^\top$$

b) The poles of the system are the eigenvalues of the A-matrix,

$$A = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} \end{pmatrix}$$

which are the zeros of the characteristic polynomial

$$\det(\lambda I - A) = \begin{vmatrix} \lambda + k_{21} & -k_{12} \\ -k_{21} & \lambda + k_{12} \end{vmatrix} = (\lambda + k_{21})(\lambda + k_{12}) - k_{21}k_{12}$$

$$= \lambda^2 + (k_{12} + k_{21})\lambda + k_{21}k_{12} - k_{21}k_{12} = \lambda(\lambda + k_{12} + k_{21})$$

i.e. the poles are 0 and  $-(k_{12} + k_{21})$ . Hence, the system contains an integrator (the pole at the origin) and is thus not stable.

c) The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k_{21} & -k_{12} \end{pmatrix}$$

with determinant  $-k_{21}$ . I.e. the system is observable (the determinant is non-zero) precisely when  $k_{21} \neq 0$ . If  $k_{21} = 0$  we are not able to observe state  $x_1$  by measuring  $x_2$ .

d) The nonlinear state space is given by

$$\dot{x}_1 = -k_{21}x_1 + \frac{V_{\max}x_2}{K_M + x_2}$$

$$\dot{x}_2 = -\frac{V_{\max}x_2}{K_M + x_2} + k_{21}x_1 + Ku$$

e) The relation is shown in Figure 8.16a.

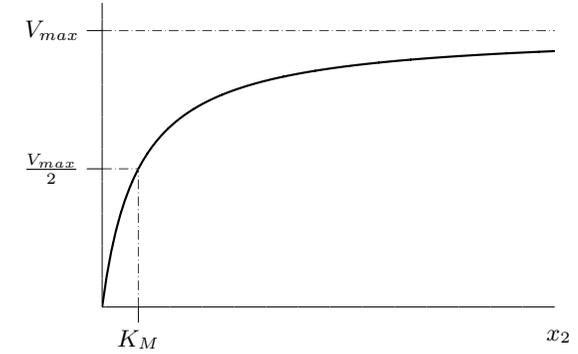


Figure 8.16a

f) We have the nonlinear state space description

$$\dot{x}_1 = -k_{21}x_1 + \frac{V_{\max}x_2}{K_M + x_2}$$

$$\dot{x}_2 = -\frac{V_{\max}x_2}{K_M + x_2} + k_{21}x_1 + Ku$$

and the stationary point

$$x_0 = \left( \frac{V_{\max}}{2k_{21}} \quad K_M \right)^\top, \quad u_0 = 0.$$

The A-matrix of the linearized system is given by the Jacobian of the dynamics (w.r.t  $x$ ) evaluated at the stationary point. The Jacobian is

$$J_f(x, u) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -k_{21} & \frac{V_{\max}K_M}{(K_M + x_2)^2} \\ k_{21} & -\frac{V_{\max}K_M}{(K_M + x_2)^2} \end{pmatrix}$$

which gives us

$$A = J_f(x_0, u_0) = \begin{pmatrix} -k_{21} & \frac{V_{\max}}{4K_M} \\ k_{21} & -\frac{V_{\max}}{4K_M} \end{pmatrix}.$$

Similarly, we get the B-matrix, with  $x = x_0$  and  $u = u_0$ , as

$$B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ K \end{pmatrix}.$$

Introduce  $x_\Delta = x - x_0$ ,  $u_\Delta = u - u_0$ . The linear approximation of the system around  $x_0$ ,  $u_0$  is then given by

$$\dot{x}_\Delta = Ax_\Delta + Bu_\Delta.$$

g)

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ 0 & k_{32} & -k_{23} \end{pmatrix} x + \begin{pmatrix} 0 \\ K \\ 0 \end{pmatrix} u$$

$$y = (0 \quad 1 \quad 0) x$$

h) The observability matrix is given by

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

Consider

$$A^2 = \begin{pmatrix} -k_{21} & k_{12} & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ 0 & k_{32} & -k_{23} \end{pmatrix} \begin{pmatrix} -k_{21} & k_{12} & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ 0 & k_{32} & -k_{23} \end{pmatrix}$$

$$= \begin{pmatrix} * & * & * \\ -k_{21}(k_{21} + k_{12} + k_{32}) & * & -k_{23}(k_{23} + k_{12} + k_{32}) \\ * & * & * \end{pmatrix}$$

where \* represent elements that are not of interest. The observability matrix thus becomes

$$\mathcal{O} = \begin{pmatrix} 0 & 1 & 0 \\ k_{21} & -(k_{12} + k_{32}) & k_{23} \\ O_1 & O_2 & O_3 \end{pmatrix}$$

with  $O_1 = -k_{21}(k_{21} + k_{12} + k_{32})$ ,  $O_2 = k_{21}k_{12} + (k_{12} + k_{32})^2 + k_{23}k_{32}$  and  $O_3 = -k_{23}(k_{23} + k_{12} + k_{32})$ . The determinant of  $\mathcal{O}$  can be expanded along the first row to obtain

$$\det \mathcal{O} = - \begin{vmatrix} k_{21} & k_{23} \\ -k_{21}(k_{21} + k_{12} + k_{32}) & -k_{23}(k_{23} + k_{12} + k_{32}) \end{vmatrix}$$

$$= k_{21}k_{23}(k_{23} + k_{12} + k_{32}) - k_{23}k_{21}(k_{21} + k_{12} + k_{32})$$

$$= k_{21}k_{23}(k_{23} - k_{21}).$$

I.e. the system is observable unless  $k_{21} = 0$  or  $k_{23} = 0$  or  $k_{23} = k_{21}$ . The first two cases mean that the corresponding state,  $x_1$  and  $x_3$  respectively, do not influence  $x_2$  and would then not be visible in the output. In the third case,  $k_{23} = k_{21}$ ,  $x_1$  and  $x_3$  do indeed influence  $x_2$  but they do it in the exact same way and we can therefore not distinguish between them.

8.17 a) With  $\dot{x} = 0$  and  $u \equiv 1$ ,

$$0 = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} - k_2 \end{pmatrix} x_0 + \begin{pmatrix} k_1 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_0 = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} - k_2 \end{pmatrix}^{-1} \begin{pmatrix} -k_1 \\ 0 \end{pmatrix} = \frac{1}{k_{21}(k_{12} + k_2) - k_{21}k_{12}} \begin{pmatrix} -k_{12} - k_2 \\ -k_{21} \end{pmatrix}$$

$$= \frac{1}{k_{21}k_2} \begin{pmatrix} k_1(k_{12} + k_2) \\ k_1k_{21} \end{pmatrix}$$

Thus, the stationary point is

$$x_0 = \begin{pmatrix} \frac{k_1(k_{12} + k_2)}{k_{21}k_2} & \frac{k_1}{k_2} \end{pmatrix}^\top.$$

b) The transfer function is computed as

$$G(s) = C(sI - A)^{-1}B = (0 \quad 1) \begin{pmatrix} s + k_{21} & -k_{12} \\ -k_{21} & s + k_{12} + k_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 \\ 0 \end{pmatrix}$$

$$= \frac{k_1k_{21}}{(s + k_{21})(s + k_{12} + k_2) - k_{12}k_{21}}.$$

c) The final value theorem gives

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sG(s)U(s) = G(0) = \frac{k_1k_{21}}{k_{21}(k_{12} + k_2) - k_{12}k_{21}} = \frac{k_1}{k_2}.$$

Since the output is defined as  $y = x_2$ , it approaches the stationary point  $x_{0,2}$ .

## 9 State Feedback

9.1 a) The control law

$$u = -Lx + y_{\text{ref}}$$

gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

and the poles of the closed loop system are given by the eigenvalues of  $A - BL$ .

$$A - BL = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \end{pmatrix} = \begin{pmatrix} -2 - l_1 & -1 - l_2 \\ 1 & 0 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (2 + l_1)s + 1 + l_2 = 0$$

Poles in  $\{-3, -5\}$  implies that we will have the equation

$$(s + 3)(s + 5) = s^2 + 8s + 15 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = 14$$

This gives the control law

$$u = -6x_1 - 14x_2 + y_{\text{ref}}$$

Similarly, poles in  $\{-10, -15\}$  gives

$$l_1 = 23 \quad l_2 = 149$$

corresponding to the control law

$$u = -23x_1 - 149x_2 + y_{\text{ref}}$$

One observes that the coefficients in the control law increase when the poles are placed further into the left half plane. In a physical system, this means that larger forces are required to realize to the control law.

b) Employ an observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

By combining the differential equations for the system and the observer we obtain an equation for the estimation error,  $\tilde{x} = x - \hat{x}$ ,

$$\dot{\tilde{x}} = Ax + Bu - A\hat{x} - Bu - K(Cx - C\hat{x}) = (A - KC)\tilde{x}$$

If  $K$  is chosen so that  $A - KC$  gets eigenvalues in the complex left hand plane, then  $\tilde{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . It is desirable that the estimation error approaches zero faster than the dynamics of the system. Thus, one should place the eigenvalues of the observer to the left of the poles of the closed loop system, for example, in  $-20$ . Regarding the influence of the pole placement, placing the poles too far into the left half plane will make the observer unnecessary sensitive to measurement noise. The characteristic equation is given by

$$\det(sI - A + KC) = s^2 + (2 + k_1)s + 1 - k_2 = 0$$

Two poles in  $-20$  corresponds to the equation

$$s^2 + 40s + 400 = 0$$

Identification of the coefficients gives

$$k_1 = 38 \quad k_2 = -399$$

The resulting observer becomes

$$\dot{\hat{x}} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \hat{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 38 \\ -399 \end{pmatrix} (y - (1 \ 0) \hat{x})$$

9.2 a) Introduce the state variables

$$x_1 = \dot{z} \quad x_2 = \theta \quad x_3 = \dot{\theta}$$

The figure gives the state equations

$$\begin{aligned} X_1(s) &= \frac{1}{s} K_2 X_2(s) \\ X_2(s) &= \frac{1}{s} X_3(s) \\ X_3(s) &= \frac{1}{s} K_1 U(s) \end{aligned}$$

Inverse Laplace transformation gives

$$\dot{x}_1(t) = K_2 x_2(t) \quad \dot{x}_2(t) = x_3(t) \quad \dot{x}_3(t) = K_1 u(t)$$

In matrix form we get

$$\dot{x}(t) = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u(t)$$

b) Since it is assumed that all states are measurable we apply a state feedback

$$u = -Lx + y_{\text{ref}}$$

which gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

where

$$A - BL = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ -K_1 l_1 & -K_1 l_2 & -K_1 l_3 \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^3 + K_1 l_3 s^2 + K_1 l_2 s + K_2 K_1 l_1 = 0$$

All three poles in  $-0.5$  implies that we will have the equation

$$(s + 0.5)^3 = s^3 + 1.5s^2 + 0.75s + 0.125 = 0$$

Identification of the coefficients gives

$$l_1 = \frac{1}{8K_1 K_2} \quad l_2 = \frac{3}{4K_1} \quad l_3 = \frac{3}{2K_1}$$

c) If only  $x_1$  is measurable we have

$$y = (1 \quad 0 \quad 0) x$$

Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^3 + k_1 s^2 + k_2 K_2 s + k_3 K_2 = 0$$

To get a similar behavior as in a), the poles of the observer are placed to the left of the poles of the closed loop system, for example, in  $-2$ . This pole placement corresponds to the equation

$$s^3 + 6s^2 + 12s + 8 = 0$$

Identification of the coefficients gives

$$k_1 = 6 \quad k_2 = 12/K_2 \quad k_3 = 8/K_2$$

9.3 Introduce the state variables

$$x_1 = \theta \quad x_2 = \omega$$

This gives the state equations

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -1/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} T$$

a) The feedback

$$u = -Lx + l_0 \theta_{\text{ref}} = -l_1 \theta - l_2 \omega + l_0 \theta_{\text{ref}}$$

gives

$$A - BL = \begin{pmatrix} 0 & 1 \\ -c_1 l_1 & -(c_1 l_2 + 1/\tau) \end{pmatrix}$$

The characteristic equation

$$\det(sI - A + BL) = s^2 + (l_2c_1 + \frac{1}{\tau})s + c_1l_1 = 0$$

Poles in  $1/\tau(-1 \pm i)$  corresponds to

$$(s + \frac{1-i}{\tau})(s + \frac{1+i}{\tau}) = s^2 + \frac{2}{\tau}s + \frac{2}{\tau^2} = 0$$

Identification of the coefficients gives

$$l_1 = \frac{2}{c_1\tau^2} \quad l_2 = \frac{1}{\tau c_1}$$

This gives the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2/\tau^2 & -2/\tau \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \end{pmatrix} l_0 \theta_{\text{ref}} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix} T$$

At steady state, that is, when  $\dot{x}_1 = \dot{x}_2 = 0$ , we should have  $\theta = \theta_{\text{ref}}$  when  $T = 0$ .  $\dot{x}_1 = 0$  implies that  $x_2 = 0$ , and  $\dot{x}_2 = 0$  then gives

$$\frac{-2}{\tau^2} x_1 + c_1 l_0 \theta_{\text{ref}} = 0$$

so that

$$l_0 = \frac{2}{c_1\tau^2}$$

The resulting control law becomes

$$u = -\frac{2}{c_1\tau^2}\theta - \frac{1}{\tau c_1}\omega + \frac{2}{c_1\tau^2}\theta_{\text{ref}}$$

b) Introduce the integrated control error as an extra state:

$$\dot{x}_3 = \theta_{\text{ref}} - \theta$$

The new state equations become

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1/\tau & 0 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} T + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\text{ref}}$$

Using the feedback law

$$u = -l_1\theta - l_2\omega - l_3x_3$$

we get the state derivative term

$$\begin{pmatrix} 0 \\ c_1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 0 & 0 & 0 \\ -c_1l_1 & -c_1l_2 & -c_1l_3 \\ 0 & 0 & 0 \end{pmatrix} x$$

and hence the closed loop system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ -c_1l_1 & -1/\tau - c_1l_2 & -c_1l_3 \\ -1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} T + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \theta_{\text{ref}}$$

The poles of the closed loop system are the eigenvalues of the “A” matrix, that is, they are given by the characteristic equation

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ -c_1l_1 & -1/\tau - c_1l_2 - \lambda & -c_1l_3 \\ -1 & 0 & -\lambda \end{pmatrix} = 0$$

Writing out and changing sign yields

$$\lambda^3 + (c_1l_2 + \frac{1}{\tau})\lambda^2 + c_1l_1\lambda - c_1l_3 = 0$$

Poles in  $\{\frac{1}{\tau}(-1 \pm i), \frac{1}{\tau}(-2)\}$  correspond to the equation

$$\lambda^3 + \frac{4}{\tau}\lambda^2 + \frac{6}{\tau^2}\lambda + \frac{4}{\tau^3} = 0$$

where the coefficients may be identified as:

$$l_1 = \frac{6}{c_1\tau^2} \quad l_2 = \frac{3}{c_1\tau} \quad l_3 = -\frac{4}{c_1\tau^3}$$

The resulting control law becomes (note that the static gain is 1 by construction, so there is no “ $l_0$ ” in this controller)

$$\begin{aligned} \dot{x}_3 &= \theta_{\text{ref}} - \theta \\ u &= -\frac{6}{c_1\tau^2}\theta - \frac{3}{c_1\tau}\omega + \frac{4}{c_1\tau^3}x_3 \end{aligned}$$

9.4 The feedback  $u = -Lx + y_{\text{ref}}$  gives the closed loop system

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

with characteristic equation

$$s^2 + (1 + l_1 + l_2)s + l_1 = 0$$

Poles in  $\{-2, -3\}$  implies that we will have the equation

$$(s + 3)(s + 2) = s^2 + 5s + 6 = 0$$

Identification of the coefficients gives

$$l_1 = 6 \quad l_2 = -2$$

and the control law becomes

$$u = -6x_1 + 2x_2 + y_{\text{ref}}$$

Introduce the observer

$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t) + K(y(t) - C\hat{x}(t))$$

It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system, for example, in  $-4$ . The characteristic equation of the observer is

$$s^2 + (1 + k_1 - k_2)s + k_1 = 0$$

and poles in  $-4$  corresponds to the equation

$$s^2 + 8s + 16 = 0$$

Identification of coefficients gives

$$k_1 = 16 \quad k_2 = 9$$

The complete system, that is, the closed loop system with reconstructed states, will have poles in  $\{-2, -3\}$ , and the observer will have poles in  $\{-4, -4\}$ .

9.5 The system has the observability matrix

$$\mathcal{O} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

that is,  $\det \mathcal{O} \neq 0$ . The system is observable and thus the poles of the observer may be placed arbitrarily.

9.6 The system is described in matrix form by

$$\dot{x}(t) = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t)$$

- a) Arbitrary values of the states can be obtained if the system is controllable. The controllability matrix becomes

$$\mathcal{S} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

and since  $\det \mathcal{S} = 1$  the system is controllable and an arbitrary temperature profile can be obtained.

- b) How the state decays depends on the poles of the closed loop system. Poles in  $-3$  will yield the desired result. The closed loop system,

$$\dot{x} = (A - BL)x + By_{\text{ref}}$$

$$A - BL = \begin{pmatrix} -2 - l_1 & 1 - l_2 & -l_3 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

has the characteristic equation

$$s^3 + (6 + l_1)s^2 + (10 + 4l_1 + l_2)s + 4 + 3l_1 + 2l_2 + l_3 = 0$$

Poles in  $-3$  implies that this coincide with the equation

$$(s + 3)^3 = s^3 + 9s^2 + 27s + 27 = 0$$

Identification of the coefficients gives

$$l_1 = 3 \quad l_2 = 5 \quad l_3 = 4$$

Thus, the control law is given by

$$u = -3x_1 - 5x_2 - 4x_3 + y_{\text{ref}}$$

- c) Check when the system is observable. The sensor at  $x_1$  corresponds to  $C = (1 \ 0 \ 0)$ , and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix} \quad \det \mathcal{O} = 1$$

The sensor at  $x_2$  corresponds to  $C = (0 \ 1 \ 0)$ , and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -4 & 6 & -4 \end{pmatrix} \quad \det \mathcal{O} = 0$$

The sensor at  $x_3$  corresponds to  $C = (0 \ 0 \ 1)$ , and results in

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -4 & 5 \end{pmatrix} \quad \det \mathcal{O} = -1$$

The system is hence observable when the sensor is placed at  $x_1$  or  $x_3$ , but not with the sensor placed at  $x_2$ . That is, the specifications may be fulfilled with the sensor placed at  $x_1$  or  $x_3$ . If the sensor is placed at  $x_1$ , the characteristic equation of the observer is given by

$$s^3 + (6 + k_1)s^2 + (10 + 4k_1 + k_2)s + 4 + 3k_1 + 2k_2 + k_3 = 0$$

Placing the poles in  $-4$  (which is somewhat faster than the nominal closed loop system) corresponds to the equation

$$(s + 4)^3 = s^3 + 12s^2 + 48s + 64 = 0$$

Identification of coefficients gives

$$k_1 = 6 \quad k_2 = 14 \quad k_3 = 14$$

9.7 From Solution 9.2 we have the state space description

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ K_1 \end{pmatrix} u(t) \\ y(t) &= (1 \ 0 \ 0) x(t) \end{aligned}$$

Introduce a reduced observer to estimate  $x_3$  from  $m_2$ . The last row in the state space description implies

$$\dot{\hat{x}}_3 = K_1 u + K(x_3 - \hat{x}_3) = K_1 u + K(\dot{x}_2 - \dot{\hat{x}}_3)$$

The estimation error becomes

$$\dot{\tilde{x}}_3 = x_3 - \hat{x}_3 = -K\tilde{x}_3$$

With a suitable choice of  $K$ , the estimation error can be made to decrease arbitrarily fast. To avoid differentiation of  $x_2$  we introduce

$$z = \hat{x}_3 - Kx_2$$

which implies

$$\dot{z} = \dot{\hat{x}}_3 - K\dot{x}_2 = -K(z + Kx_2) + K_1 u$$

This gives

$$\hat{X}_3(s) = \frac{K_1}{s + K} U(s) + \frac{K^2 s}{s + K} X_2(s)$$

which results in the block diagram in Figure 9.7a.

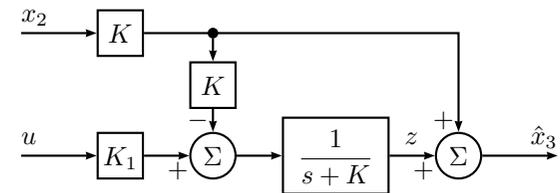


Figure 9.7a

9.8 a) The equations

$$\begin{aligned} T\dot{q} &= -q + k_1 u \\ A\dot{h} &= q - v \end{aligned}$$

with  $k_1 = 1$ ,  $T = 0.5$  and  $A = 1$  give, in state space form,

$$\begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v$$

The feedback

$$u = -l_1 q - l_2 h + r$$

gives the closed loop system

$$\begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} -2 - 2l_1 & -2l_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v$$

with characteristic equation

$$(s + 2 + 2l_1)s + 2l_2 = s^2 + (2 + 2l_1)s + 2l_2 = 0$$

Comparison with the desired characteristic equation

$$(s + 2)^2 = s^2 + 4s + 4 = 0$$

gives

$$l_1 = 1 \quad l_2 = 2$$

b) At steady state we have  $\dot{q} = 0$  and  $\dot{h} = 0$ . With  $v = 0.1$  and  $r = 0$  we get

$$\begin{aligned} 0 &= -4q - 4h \\ 0 &= q - 0.1 \end{aligned}$$

which gives  $h = -0.1$ .

c) In order to determine the feedforward controller we start from the description

$$Y(s) = G_1(s)R(s) + H(s)V(s)$$

The state space description

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{h} \end{pmatrix} &= \begin{pmatrix} -4 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ h \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} r + \begin{pmatrix} 0 \\ -1 \end{pmatrix} v \\ y &= (0 \quad 1) \begin{pmatrix} q \\ h \end{pmatrix} \end{aligned}$$

gives

$$\begin{aligned} H(s) &= \frac{1}{s^2 + 4s + 4} (0 \quad 1) \begin{pmatrix} s & -4 \\ 1 & s + 4 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= -\frac{(s + 4)}{s^2 + 4s + 4} \end{aligned}$$

and

$$\begin{aligned} G_1(s) &= \frac{1}{s^2 + 4s + 4} (0 \quad 1) \begin{pmatrix} s & -4 \\ 1 & s + 4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \frac{2}{s^2 + 4s + 4} \end{aligned}$$

To eliminate  $v$  completely we shall choose the feedforward controller

$$R(s) = F_f(s)V(s)$$

where

$$F_f(s) = -\frac{H(s)}{G_1(s)}$$

The computations above give

$$F_f(s) = \frac{(s + 4)}{2} = \frac{1}{2}s + 2$$

Removing the differentiation term yields  $F_f(s) = 2$  or

$$r = 2v$$

At steady state this gives

$$\begin{aligned} 0 &= -4q - 4h + 4v \\ 0 &= q - v \end{aligned}$$

that is  $h = 0$ .

d) Because  $k_1 \neq 1$  the feedback  $u = -q - 2h + 2v$  gives, at steady state,

$$\begin{aligned} 0 &= -2(1 + k_1)q - 4k_1h + 4k_1v \\ 0 &= q - v \end{aligned}$$

which gives

$$h = \frac{k_1 - 1}{2k_1}v$$

Because  $k_1 \neq 1$  we get a steady state control error. In order to determine when the expression for  $h$  is valid we consider the stability. The characteristic equation

$$s^2 + (2 + 2k_1)s + 4k_1 = 0$$

has both roots in the complex left hand plane for  $k_1 > 0$ , that is, the expression is valid for all  $k_1 > 0$ .

e) Introduce the integral of the height as a new state

$$z(t) = \int_0^t h(s) ds \Rightarrow \dot{z} = h$$

With the state vector

$$x(t) = \begin{pmatrix} q(t) & h(t) & z(t) \end{pmatrix}^T$$

this gives

$$\dot{x} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 2k_1 \\ 0 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} v$$

The state feedback  $u = -Lx$  gives

$$\dot{x} = \begin{pmatrix} -2 - 2k_1l_1 & -2k_1l_2 & -2k_1l_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} v$$

The third equation gives  $h = 0$  at steady state, independent of  $k_1$  provided  $L$  stabilizes the system.

9.9 The transfer function  $u$  to  $y$  is given by

$$Y(s) = C(sI - A)^{-1}BU(s) = \frac{1}{s^2}U(s)$$

In order to study the effect of the time delay we consider the block diagram in Figure 9.9a. The block diagram corresponds to the situation where the

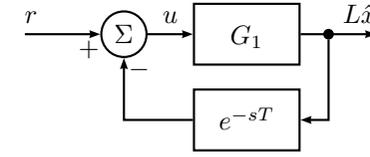


Figure 9.9a

observer uses the measured input (not the computed input). To determine the effect of the time delay, we study the loop gain,  $G_1(s)e^{-sT}$ , where  $G_1(s)$  is the transfer function from  $U(s)$  to  $Z(s) = L\hat{X}(s)$ .

The equation for the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

gives

$$\begin{aligned} \hat{X}(s) &= (sI - A + KC)^{-1}(BU(s) + KY(s)) \\ &= (sI - A + KC)^{-1}(BU(s) + K\frac{1}{s^2}U(s)) \end{aligned}$$

Using this together with  $Z(s) = L\hat{X}(s)$  gives

$$\begin{aligned} Z(s) &= G_1(s)U(s) \\ &= L(sI - A + KC)^{-1}(B + K\frac{1}{s^2})U(s) \\ &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} s+4 & -1 \\ 4 & s \end{pmatrix}^{-1} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \frac{1}{s^2} \right) U(s) \\ &= \frac{1+2s}{s^2}U(s) \end{aligned}$$

We shall analyze the stability using the Nyquist curve\* for  $G_o = G_1(s)e^{-sT}$ , that is,

$$G_1(i\omega)e^{-i\omega T} = \frac{1+i2\omega}{-\omega^2}e^{-i\omega T}$$

The crossover frequency is obtained from

$$|G_1(i\omega_c)e^{-i\omega_c T}| = \frac{\sqrt{1+4\omega_c^2}}{\omega_c^2} = 1$$

\*Using a Bode plot instead of the Nyquist curve would perhaps be more straightforward. However, for no particular reason, we use the Nyquist curve here.

or

$$\omega_c = \sqrt{2 + \sqrt{5}}$$

The phase of  $G_o$  is

$$\arg(G_1(i\omega)e^{-\omega T}) = -\pi + \arctan 2\omega - \omega T$$

In order to obtain a stable closed loop system it is required that

$$-\pi + \arctan 2\omega_c - \omega_c T > -\pi$$

which gives

$$T < \frac{\arctan 2\omega_c}{\omega_c} = 0.65 \text{ s}$$

9.10 a) The observability matrix:

$$\mathcal{O} = \begin{pmatrix} 2 & 1 \\ -2+a & 0 \end{pmatrix} \quad \det \mathcal{O} = 2 - a$$

The system is observable (and the poles of the observer can be placed arbitrarily) when  $a \neq 2$ .

$$A - KC = \begin{pmatrix} -1 - 2k_1 & 1 - 2k_2 \\ 1 - k_1 & -2 - k_2 \end{pmatrix}$$

We desire that the eigenvalues be  $\{-5, -10\}$ . Use that the determinant is the product of the eigenvalues and the trace\* is the sum of the eigenvalues:

$$\begin{aligned} 5k_1 + 3k_2 + 1 &= 50 \\ -2k_1 - k_2 - 3 &= -15 \end{aligned}$$

which gives

$$k_1 = -13 \quad k_2 = 38$$

b) The equation for the estimation error is

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t) - Kv(t)$$

The transfer function from  $v$  to  $\tilde{x}_1$  is

$$-C_1(sI - A + KC)^{-1}K = \frac{13s - 12}{s^2 + 15s + 50}$$

where  $C_1 = (1 \ 0)$ .

\*The trace of a matrix is the sum of its diagonal elements.

9.11 a) According to the initial value theorem we have that

$$y(0) = \lim_{s \rightarrow \infty} sG(s)U(s)$$

For a step input, that is,  $U(s) = 1/s$ , we get

$$\dot{y}(0) = \lim_{s \rightarrow \infty} s \cdot sG(s)U(s) = \lim_{s \rightarrow \infty} \frac{s(1 - s/\alpha)}{(1 + \frac{s}{\beta})^2} = -\frac{\beta^2}{\alpha}$$

Hence  $\dot{y}(0)$  decreases as  $\alpha$  decreases, that is, as the zero of the system approaches the origin.

b) No. This problem is caused by a RHP zero and it is impossible to move the zeros with state feedback.

9.12 A very fast closed loop system:

- implies that the poles are far into the LHP which implies a need for generating large input signals.
- easily becomes unstable in case of model uncertainties.
- becomes sensitive to measurement noise.
- has a sensitivity function with a large peak.

9.13 a) The system  $G(s) = C(sI - A + BL)^{-1}B$  has poles where

$$\det(sI - A + BL) = s^2 + (5 - l_1 + 2l_2)s + 5 + 6l_2 = 0$$

The poles in  $-2 \pm i$  implies the characteristic equation

$$(s + 2 + i)(s + 2 - i) = s^2 + 4s + 5 = 0$$

Identification of coefficients gives

$$l_1 = 1 \quad l_2 = 0$$

b) The closed loop system is given by

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} r(t) \\ y(t) &= (1 \ 1) x(t)\end{aligned}$$

The condition  $y(t) = 0$  gives  $x_1 + x_2 = 0$ , and hence  $\dot{x}_1 = -\dot{x}_2$ . From the state equations we get

$$\begin{aligned}-2x_1 + x_2 - r &= x_1 + 2x_2 - 2r && \Leftrightarrow \\ -3x_1 &= x_2 - r\end{aligned}$$

Together with  $x_1 + x_2 = 0$  we get  $x_1 = -x_2 = r/2$  and

$$\dot{r} = 2\dot{x}_1 = 2(-2x_1 + x_2 - r) = -5r$$

Since  $r(t) = e^{\alpha t}$  we have  $\alpha = -5$ . Moreover, for  $y(t)$  to be zero for all  $t$ , the system must start in the initial condition  $x_1(0) = -x_2(0) = r(0)/2$ .

We hence have the state space representation

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= (0 \ 1) x(t)\end{aligned}$$

From the last equation we have  $x_2(t) = y(t)$ , that is,  $x_2$  is the motor angle. From the first equation we have  $\dot{x}_2(t) = x_1(t)$ , that is,  $x_1$  is the angular velocity.

9.14 a) Enter the transfer function and generate the state space model.

```
>> s = tf('s');
>> G = ss(1 / (s * (s + 1)))
a =
 x1 x2
x1 -1 -0
x2 1 0

b =
 u1
x1 1
x2 0

c =
 x1 x2
y1 0 1

d =
 u1
y1 0
```

Continuous-time model.

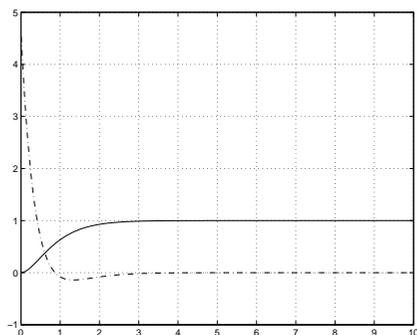
b) Compute feedback gains. This time, the gain  $l_0$  is computed by explicitly constructing a system with  $l_0 = 1$  first, and then correcting by the inverse of that system's static gain. Note that if we don't need  $l_0$ , this approach simplifies to  $G_c = G_{c0} / \text{dcgain}(G_{c0})$ . However, we do need  $l_0$  in order to compute the control signal.

```
>> L = place(G.a, G.b, [-2.2 -2.1]);
>> Gc0 = ss(G.a - G.b * L, G.b, G.c, 0);
>> l_0 = 1 / dcgain(Gc0);
>> Gc = Gc0 * l_0;
```

Note that although  $l_0 * G_{c0}$  and  $G_{c0} * l_0$  implement the same transfer function, the states of the two implementations will differ. To obtain the correct states the factor  $l_0$  must be inserted at the input side of  $G_{c0}$ , that is, as the right operand when the systems are connected in series using the operator  $*$ .

Calculate the step response and the corresponding control signal of the closed loop system. To calculate the control signal magnitude use  $[ y, t, x ] = \text{step}( G_c )$ . The function `step` will in this case return  $y$ , the output of the closed loop system,  $t$  the time vector, and  $x$  the states of the system. To compute the control signal, use that  $u(t) = l_0 r(t) - Lx(t)$ , where  $r(t) = 1$ . Then plot the result.

```
>> [y, t, x] = step(Gc, 10);
>> u = l_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
```

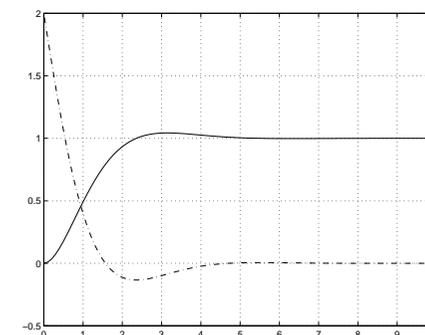


Compute a new feedback. This time, we compute the gain  $l_0$  by using the formula for the static gain of the system with  $l_0 = 1$  (put  $s = 0$  in the generic expression for the transfer function).

Calculate the step response and the corresponding control signal. Plot the result.

```
>> L = place(G.a, G.b, [-1+i -1-i]);
>> l_0 = 1 / ...
 (G.c * inv(-G.a + G.b*L) * G.b);
>> Gc = ss(G.a - G.b * L, G.b * l_0, ...
 G.c, 0);

>> [y, t, x] = step(Gc, 10);
>> u = l_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
```



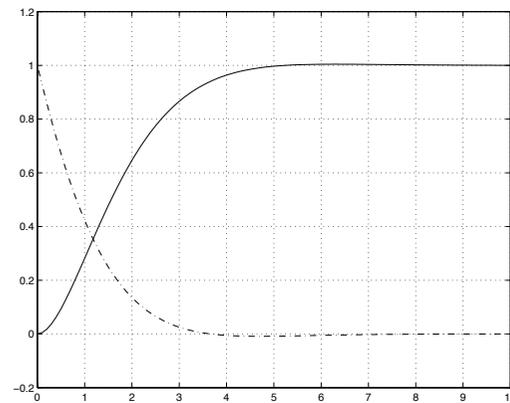
The step responses have approximately the same rise and settling times. By choosing the closed loop poles complex, and hence allowing a small overshoot in the step response, we have however reduced the maximum value of the input signal significantly.

c) Case (i): Compute the feedback gain  $L$ ,  $l_0$ , and the closed loop system.

```
>> L = lqr(G.a, G.b, diag([0 1]), 1);
>> l_0 = 1 / ...
 (G.c * inv(-G.a + G.b*L) * G.b);
>> Gc = ss(G.a - G.b * L, G.b * l_0, ...
 G.c, 0);
```

Simulate the system and plot the result.

```
>> [y, t, x] = step(Gc, 10);
>> u = l_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
```



Compute the closed loop poles. This time, via the eigenvalues of the “A” matrix.

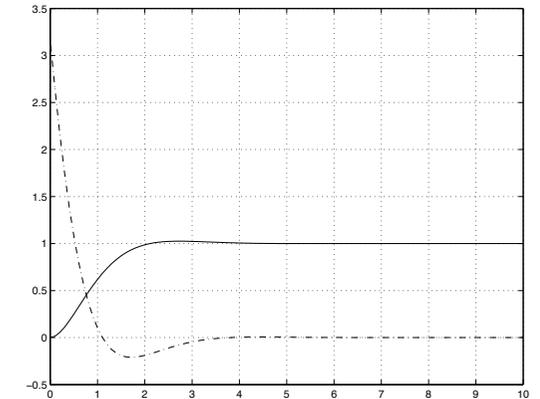
```
>> eig(Gc.a)
ans =
 -0.8660 + 0.5000i
 -0.8660 - 0.5000i
```

Case (ii): Repeat, this time with larger weight on the motor angle.

```
>> L = lqr(G.a, G.b, diag([0 10]), 1);
>> l_0 = 1 / ...
 (G.c * inv(-G.a + G.b*L) * G.b);
>> Gc = ss(G.a - G.b * L, G.b * l_0, ...
 G.c, 0);
```

Simulate the system and plot the result. The step response is now significantly faster.

```
>> [y, t, x] = step(Gc, 10);
>> u = l_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
```



Compute the closed loop poles. This time using a dedicated command from the toolbox. The poles are now further away from the origin and the relative damping is slightly reduced.

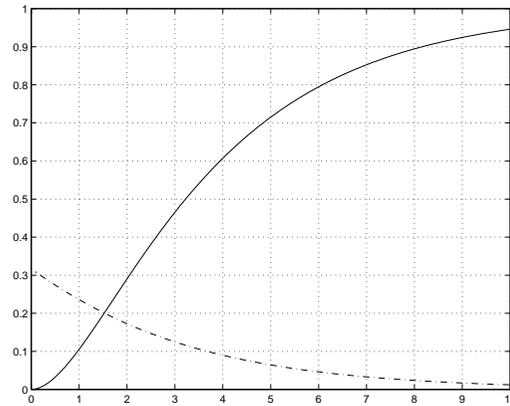
```
>> pole(Gc)
ans =
 -1.3532 + 1.1537i
 -1.3532 - 1.1537i
```

Case (iii): Repeat, this time with smaller weight on the motor angle.

```
>> L = lqr(G.a, G.b, diag([0 0.1]), 1);
>> l_0 = 1 / ...
 (G.c * inv(-G.a + G.b*L) * G.b);
>> Gc = ss(G.a - G.b * L, G.b * l_0, ...
 G.c, 0);
```

Simulate the system and plot the result. The step response is now much slower.

```
>> [y, t, x] = step(Gc, 10);
>> u = 1_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
```



Compute the closed loop poles. We now get two real closed loop poles, where the pole in  $-0.34$  causes the slow step response.

```
>> pole(Gc)
ans =
 -0.9420
 -0.3357
```

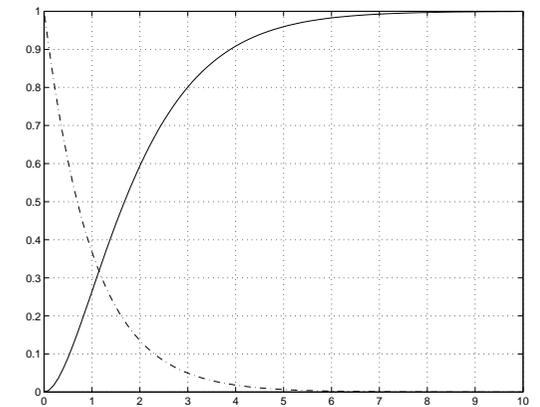
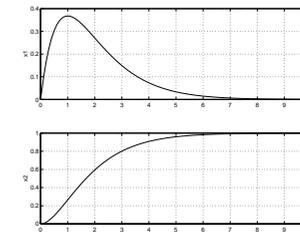
d) If we start from case (ii) and increase the matrix called R in the call to `lqr`, the closed loop system gradually becomes slower, since we put an increasing weight on the control signal magnitude. When we reach  $R = 10$  we get exactly the same result as for case (i). Since it is the “ratio” between Q and R that determines the closed loop properties we get the same feedback gain if we scale Q and R by the same scalar.

e) Compute feedback gains, adjust static gain, and compute closed loop system.

```
>> L = lqr(G.a, G.b, diag([1 1]), 1);
>> 1_0 = 1 / ...
 (G.c * inv(-G.a + G.b*L) * G.b);
>> Gc = ss(G.a - G.b * L, G.b * 1_0, ...
 G.c, 0);
```

Simulate the system and plot the result. Then we also plot the states,  $x_1$  and  $x_2$ , in two different diagrams.

```
>> [y, t, x] = step(Gc, 10);
>> u = 1_0 - x * L.';
>> plot(t, y, '-.', ...
 t, u, '-.');
>> grid
>> figure
>> subplot(2, 1, 1);
>> plot(t, x(:,1));
>> grid; ylabel('x1');
>> subplot(2, 1, 2);
>> plot(t, x(:,2));
>> grid; ylabel('x2');
```



Increasing the weight on the angular velocity forces the motor to move slower, and then also the step response becomes slower.

### 9.15 Introduce the state variables

$$x_1 = q \quad x_2 = \dot{q}$$

This gives the state space description

$$\dot{x} = \begin{pmatrix} -0.05 & 0 \\ 0.05 & -0.02 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

a) The system has the controllability matrix

$$S = (B \quad AB) = \begin{pmatrix} 1 & -0.05 \\ 0 & 0.05 \end{pmatrix} \quad \det S = 0.05$$

Thus, the system is controllable.

b) The control law

$$u = -Lx$$

gives the closed loop system

$$\dot{x} = (A - BL)x$$

and the poles of the closed loop system is given by the eigenvalues of  $A - BL$ .

$$A - BL = \begin{pmatrix} -0.05 - l_1 & -l_2 \\ 0.05 & -0.02 \end{pmatrix}$$

The characteristic equation is given by

$$\det(sI - A + BL) = s^2 + (0.07 + l_1)s + 0.001 + 0.02l_1 + 0.05l_2 = 0$$

Both poles in  $-0.1$  implies that we shall have the equation

$$(s + 0.1)^2 = s^2 + 0.2s + 0.01 = 0$$

Identification of the coefficients gives

$$l_1 = 0.13 \quad l_2 = 0.128$$

This gives the control law

$$u = -0.13x_1 - 0.128x_2$$

c) It is desirable that the estimation error converges to zero faster than the dynamics of the system. Thus, we should place the eigenvalues of the observer to the left of the poles of the closed loop system. To avoid large amplification of the measurement noise the poles of the observer should not be placed to far into the left hand plane.

d) Only  $y = x_2$  is measurable. Employ the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

The characteristic equation is

$$\det(sI - A + KC) = s^2 + (0.07 + k_2)s + 0.05k_1 + 0.05k_2 + 0.001$$

Both poles in  $-0.2$  implies that we shall have the equation

$$s^2 + 0.4s + 0.04 = 0$$

Identification of the coefficients gives

$$k_1 = 0.45 \quad k_2 = 0.33$$

9.16 Are the specifications 1–4 fulfilled?

1. The bandwidth is  $\omega_B \approx 1.1 < 5$  which is seen from the gain curve of the closed loop system.

**The bandwidth requirement is not fulfilled.**

2. Stability despite model errors and disturbances?

We have  $Y(s) = \kappa e^{-\tau s} G(s)U(s) + E(s)$  instead of  $Y(s) = G(s)U(s)$ . The factor  $\kappa$  thus represents the gain uncertainty, while the factor  $e^{-\tau s}$  represents a phase uncertainty. These uncertainties are also present (with the same magnitudes) in the loop gain  $G_o = FG$ .

Looking in the Nyquist curve of  $G_o$ , where amplitudes near 1 are easiest to read, one can see that there is always just one intersection with  $|G_o(i\omega)| = 1$ , independently of the present uncertainties in gain and phase. Thus the stability criterion based on the Bode plot applies.

The uncertain phase lag is  $\omega\tau$  at the frequency  $\omega$ . Thus the maximum negative phase lag occurs for  $\tau^* = 0.3$  s.

Next, we must find the worst case gain crossover frequency in order to see if the worst case phase lag causes instability by reducing the phase margin

below 0. Study the amplitude and phase curves for the loop gain  $G_o(s)$ . Since the phase of  $G_o$  is decreasing, higher gain crossover will always be more critical since it both means a smaller phase margin to begin with, and also a bigger phase lag due to the worst case time delay.

From the gain curve of  $G_o$  it is clear that higher values of  $\kappa$  are more critical since those give the higher gain crossovers. By very careful inspection of the gain curve, one can see that the most critical value,  $\kappa^* = 1.1$ , leads to  $\omega_c^* \approx 2.3 \text{ rad/s} < 3 \text{ rad/s}$ , and  $\varphi_m^* > 55^\circ$ .

Combining the worst case  $\kappa$  (leading to the  $\omega_c^*$  and  $\varphi_m^*$  above) with the worst case and  $\tau^* = 0.3 \text{ s}$  results in a total worst case phase margin of at least  $55^\circ - \omega_c^* \tau^* = 55^\circ - 3 \text{ rad/s} \cdot 0.3 \text{ s} = 55^\circ - 0.9 \text{ rad} \approx 3^\circ > 0$ . Thus the system is guaranteed to be stable.

### The system is stable despite the model errors.

Remark: The robustness criterion  $\forall \omega : |Q(i\omega)| < \frac{1}{|G(i\omega)|}$  is sufficient but not necessary to show stability.

- Both the Bode plot and the Nyquist curve of the loop gain tells us that the loop gain does not contain an integration which could remove static errors. This implies the model errors will influence the static gain. The details of this argument follow.

With  $u = F_r r - F_y y$ , the closed loop system is

$$G_c(s) = \frac{F_r(s)G(s)}{1 + F_y(s)G(s)}$$

The real closed loop system is

$$G_c^0(s) = \frac{F_r(s)\kappa e^{-\tau s}G(s)}{1 + F_y(s)\kappa e^{-\tau s}G(s)}$$

Since the system is stable (see 2) the final value theorem gives the final value of the step response as

$$\lim_{t \rightarrow \infty} y^0(t) = \lim_{s \rightarrow 0} s \cdot G_c^0(s) \cdot \frac{1}{s} = \frac{F_r(0)\kappa G(0)}{1 + F_y(0)\kappa G(0)}$$

which cannot be 1 for all possible values of  $\kappa$ .

**The gain will be different from 1 for some possible value of  $\kappa$ .**

- If  $e(t)$  is measurement noise, then the complementary sensitivity function,  $T(s)$ , should be checked. If  $e(t)$  is process noise, then the sensitivity function,  $S(s)$ , should be checked. Both  $T(s)$  and  $S(s)$  have peaks  $> 1$  at exactly  $\omega = 10 \text{ rad/s}$ , which implies the both measurement and process noise are amplified.

**The (measurement) noise is amplified by the system.**

- a) The linearized system is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u =: Ax + Bu$$

Using the state feedback law  $u = -Lx = -l_1 x_1 - l_2 x_2$  gives

$$\dot{x} = A + B(-Lx) = (A - BL)x = \begin{pmatrix} l_1 & 1 + l_2 \\ -1 - l_1 & -3 - l_2 \end{pmatrix} x$$

The poles of this closed loop system are given by the eigenvalues of  $A - BL$ , which are the roots of the characteristic polynomial

$$\begin{aligned} P(s) &= \det(sI - (A - BL)) = \det \begin{pmatrix} s - l_1 & -1 - l_2 \\ 1 + l_1 & s + 3 + l_2 \end{pmatrix} \\ &= (s - l_1)(s + 3 + l_2) - (1 + l_1)(-1 - l_2) \\ &= s^2 + (-l_1 + l_2 + 3)s - 2l_1 + l_2 + 1 \end{aligned}$$

To place the poles in  $\{-2, -4\}$ ,  $P(s)$  must be the polynomial

$$(s + 2)(s + 4) = s^2 + 6s + 8$$

This gives the system of equations

$$\begin{aligned} -l_1 + l_2 + 3 &= 6 \\ -2l_1 + l_2 + 1 &= 8 \end{aligned}$$

which has the solution

$$l_1 = -4 \quad l_2 = -1$$

The state feedback law thus becomes  $u = -Lx = 4x_1 + x_2$ .

b) If only  $x_2$  is measured, the output equation is given by

$$y = x_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} x =: Cx$$

Given  $y$  ( $x_2$ ) and  $u$ ,  $x_1$  can be estimated if the system is observable. The observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \quad \det \mathcal{O} = 1$$

Hence the system is observable and  $x_1$  can be estimated using an observer.

It is essential that the input  $u$  is known since  $u$  is required in the observer design to get an asymptotically vanishing state estimation error.

c) If  $u$  is unknown but constant, we can introduce a third state  $x_3 = u$  which has the dynamics  $\dot{x}_3 = 0$ . Introducing  $z^T = (x_1 \ x_2 \ x_3)$ , the system dynamics can be rewritten as

$$\begin{aligned} \dot{z} &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} z =: \tilde{A}z \\ y &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} z =: \tilde{C}z \end{aligned}$$

The observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -3 & 1 \\ 3 & 8 & -2 \end{pmatrix} \quad \det \mathcal{O} = 1$$

(Tip:  $\det(\mathcal{O}) \neq 0$  can be established without computing the determinant, by checking that the rows of  $\mathcal{O}$  are linearly independent.) The fact that the system is observable means that  $x_1$  (and also  $u$ ) can be estimated from measurements of  $x_2$  using an observer of the form

$$\dot{\hat{z}} = \tilde{A}\hat{z} + K(y - \tilde{C}\hat{z}) = (\tilde{A} - K\tilde{C})\hat{z} + Ky$$

where the observer gain  $K$  is selected so that the observer poles, that is, the eigenvalues of  $\tilde{A} - K\tilde{C}$ , are all in the left half plane.

9.18 a) The system is described by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

A P controller corresponds to  $u = K(r - x_1)$ , this means that the closed loop system is given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 - K & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ K \end{pmatrix} r + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

The poles to the closed loop system are given by

$$\det \begin{pmatrix} s & -1 \\ 1 + K & s \end{pmatrix} = 0$$

which leads to  $s^2 + 1 + K = 0$ . The poles are pure complex and thus the system doesn't have a well defined stationary error or speed of response.

b) A linear combination of  $r$  and  $x_2$  is given by

$$u = l_0 r - l_2 x_2$$

with this controller the closed loop is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & l_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ l_0 \end{pmatrix} r + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w$$

The poles to the closed loop system are given by

$$\det \begin{pmatrix} s & -1 \\ 1 & s + l_2 \end{pmatrix} = 0$$

which means  $s^2 + l_2 s + 1 = 0$ . The poles can be placed with  $l_2$  as

$$s = \frac{-l_2}{2} \pm \sqrt{\frac{l_2^2 - 4}{4}}$$

We have that  $\dot{x} = 0$  at stationary which gives that  $x_2 = -w$  and  $x_1 = l_0 r - l_2 x_2$  if  $w = 0$ . If we select  $l_0 = 1$  then the stationary error will be zero. If  $w \neq 0$  and  $l_0 = 1$  then there will be stationary error of size  $l_2 w$ .

c) Introduce a new state  $x_3 = w$  to estimate the unknown signal. The extended system is described by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x \end{aligned}$$

Create an observer to estimate the states

$$\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky$$

the poles of the observer can be placed with

$$\det(sI - (A - KC)) = 0$$

which gives  $s^3 + k_2s^2 + (1 - k_1)s - k_3 = 0$ . Place the poles for example in  $-2$ , that is, seek the polynomial  $s^3 + 6s^2 + 12s + 8 = 0$ . Comparison gives

$$k_1 = -11 \quad k_2 = 6 \quad k_3 = -8$$

Now, let  $u = l_0r - l_2\hat{x}_2 - l_3\hat{x}_3$ . At stationary we have  $\hat{x}_3 = w = -\hat{x}_2$ , so with  $l_3 = l_2$  we have  $x_1 = l_0r$ , and with  $l_0 = 1$  there will be no error.

# 11 Implementation

11.1 Inverse Laplace transformation of

$$U(s) = KN \frac{s+b}{s+bN} E(s)$$

gives the differential equation

$$\dot{u}(t) + bNu(t) = KN\dot{e}(t) + bKNe(t) \quad (11.1)$$

At time  $t - T$  we have

$$\dot{u}(t - T) + bNu(t - T) = KN\dot{e}(t - T) + bKNe(t - T) \quad (11.2)$$

By replacing  $\dot{u}(t)$  and  $\dot{e}(t)$  in (11.1) and (11.2) with  $\Delta_t u(t)$  and  $\Delta_t e(t)$ , respectively, and then adding the equations we get

$$\begin{aligned} \Delta_t u(t) + \Delta_t u(t - T) + bNu(t) + bNu(t - T) \\ = KN\Delta_t e(t) + KN\Delta_t e(t - T) + bKNe(t) + bKNe(t - T) \end{aligned}$$

Tustins formula

$$\frac{1}{2}(\Delta_t u(t) + \Delta_t u(t - T)) = \frac{1}{T}(u(t) - u(t - T))$$

now gives

$$\begin{aligned} \frac{2}{T}(u(t) - u(t - T)) + bN(u(t) + u(t - T)) \\ = \frac{2}{T}KN(e(t) - e(t - T)) + bKNe(t) + e(t - T) \end{aligned}$$

Inserting the numerical values,  $K = 2$ ,  $T = 0.1$ ,  $N = 10$  and  $b = 0.1$ , we get

$$\begin{aligned} 20(u(t) - u(t - T)) + (u(t) + u(t - T)) \\ = 400(e(t) - e(t - T)) + 2(e(t) + e(t - T)) \end{aligned}$$

which gives

$$u(t) = \frac{19}{21}u(t - T) + \frac{402}{21}e(t) - \frac{398}{21}e(t - T)$$

that is

$$u(t) = 0.905u(t - T) + 19.14e(t) - 18.95e(t - T)$$

11.2 a) Consider the differential equation

$$\dot{y}(t) = u(t)$$

during the sampling interval  $kT \leq t < kT + T$ . The input is constant during the sampling interval,  $u(t) = u_k$ , which gives

$$\dot{y}(t) = u_k$$

By integrating the left- and right-hand sides from  $t = kT$  to  $t = kT + T$  we get

$$y(kT + T) - y(kT) = \int_{kT}^{kT+T} u_k dt = Tu_k$$

With the notation  $y_{k+1} = y(kT + T)$  and  $y_k = y(kT)$  this gives

$$y_{k+1} - y_k = Tu_k$$

b) The feedback

$$u_k = -Ky_k$$

gives

$$y_{k+1} = (1 - KT)y_k \quad y_0 = y^0$$

that is

$$y_k = (1 - KT)^k y^0$$

The closed loop system is asymptotically stable if

$$|y_k| \rightarrow 0, \quad t \rightarrow \infty$$

This gives the condition

$$|1 - KT| < 1$$

or, equivalently,  $0 < K < \frac{2}{T}$ .

11.3 a) Because the prefilter is linear, the signal prior to sampling may be written

$$y(t) = y_0(t) + y_1(t)$$

where  $y_1(t)$  stems from the disturbance  $u_1(t) = \sin \omega_2 t$ . After all transients have disappeared, we get

$$y_1(t) = A \sin(\omega_2 t + \Phi)$$

where

$$A = |G(i\omega_2)| = \frac{1}{\sqrt{1 + (\omega_2 T_1)^2}}$$

$$\Phi = \arg G(i\omega_2) = -\arctan \omega_2 T_1$$

Let us introduce the notation  $\omega_1 = \omega_s - \omega_2$  where  $\omega_s$  denotes the sampling frequency,  $\omega_s = 2\pi/T$ . When  $y_1(t)$  is sampled with the sampling interval  $T$ , we get

$$\begin{aligned} y_1(kT) &= A \sin(\omega_2 kT + \Phi) = A \sin((\omega_s - \omega_1)kT + \Phi) \\ &= A \sin(2k\pi - \omega_1 kT + \Phi) = A \sin(-\omega_1 kT + \Phi) \\ &= -A \sin(\omega_1 kT - \Phi) = A \sin(\omega_1 kT + \pi - \Phi) \\ &= A \sin(\omega_1 kT + \varphi) \end{aligned}$$

that is

$$A = \frac{1}{\sqrt{1 + (\omega_2 T_1)^2}}$$

$$\omega_1 = \frac{2\pi}{T} - \omega_2$$

$$\varphi = \pi + \arctan \omega_2 T_1$$

b) The bandwidth of the filter is obtained from the relation

$$|G(i\omega_B)| = \frac{1}{\sqrt{1 + (\omega_B T_1)^2}} = \frac{1}{\sqrt{2}}$$

which gives  $\omega_B = 1/T_1$ . The signal  $u_0$  is in the interval  $0 \leq \omega < \pi/T$ , and this gives the specification

$$\frac{\pi}{T} \leq \frac{1}{T_1}$$

The limiting case

$$\frac{\pi}{T} = \frac{1}{T_1}$$

gives  $T_1 = T/\pi$ . Inserting this in the expression for  $A$  in a), we get the answer

$$A = \frac{1}{\sqrt{1 + (\omega_2 T/\pi)^2}}$$

# Liten reglerteknisk ordlista

This version: January 2011



# 1 Engelsk-svensk

|                              |                                   |
|------------------------------|-----------------------------------|
| actuator                     | ställdon                          |
| amplitude                    | amplitud                          |
| attenuation                  | dämpning                          |
| bandwidth                    | bandbredd                         |
| bond graph                   | bindningsgraf                     |
| closed loop system           | slutet system                     |
| control law                  | styrslag                          |
| controllability              | styrbarhet                        |
| controller                   | regulator                         |
| convolution                  | faltning                          |
| correlation analysis         | korrelationsanalys                |
| credibility                  | trovärdighet                      |
| crossover frequency          | skärfrekvens                      |
| damping                      | dämpning                          |
| damping ratio                | relativ dämpning                  |
| describing function          | beskrivande funktion              |
| discrete event systems       | händelseorienterade system        |
| distributed parameter models | fördelade parametriska modeller   |
| disturbance rejection        | störningsundertryckning           |
| eigenvalue                   | egenvärde                         |
| feedback                     | återkoppling                      |
| feedforward                  | framkoppling                      |
| flow                         | flöde                             |
| gain                         | förstärkning                      |
| gain crossover frequency     | (amplitud)skärfrekvens            |
| gain margin                  | amplitudmarginal                  |
| impulse response             | impulssvar                        |
| initial value                | begynnelsevärde                   |
| loop gain                    | kretsförstärkning, öppna systemet |
| lumped models                | aggregerade modeller              |
| magnitude                    | amplitud                          |
| observability                | observerbarhet                    |
| observer                     | observatör                        |
| open-loop system             | öppet system, kretsförstärkning   |

|                           |                              |
|---------------------------|------------------------------|
| overfit                   | överanpassning               |
| overshoot                 | översläng                    |
| parsimonious              | sparsam                      |
| peak frequency            | resonansfrekvens             |
| peak resonance            | resonanstopp                 |
| phase crossover frequency | fasskärfrekvens              |
| phase lag                 | fasretarderande              |
| phase lead                | fasavancerande               |
| phase margin              | fasmarginal                  |
| ramp function             | ramp                         |
| rank                      | rang                         |
| reset windup              | integratoruppvridning        |
| resonant frequency        | resonansfrekvens             |
| rise time                 | stigtid                      |
| root locus (pl. loci)     | rotort                       |
| sensitivity function      | känslighetsfunktion          |
| sensor                    | givare                       |
| settling time             | insvängningstid, lösningstid |
| sinusoidal                | sinusformad                  |
| stability robustness      | stabilitetsrobusthet         |
| state                     | tillstånd                    |
| state feedback            | tillståndsåterkoppling       |
| static gain               | statisk förstärkning         |
| steady state              | stationärt tillstånd         |
| step function             | steg                         |
| step response             | stegsvar                     |
| subspace                  | underrum                     |
| time delay                | tidsfördröjning              |
| transfer function         | överföringsfunktion          |
| unit step                 | enhetsteg                    |
| unstable                  | instabil                     |
| validity                  | giltighet                    |
| whitening filter          | blekningsfilter              |

## 2 Svensk-engelsk

|                                 |                              |                         |                             |
|---------------------------------|------------------------------|-------------------------|-----------------------------|
| (amplitud)skärffrekvens         | gain crossover frequency     | observatör              | observer                    |
| aggregerade modeller            | lumped models                | observerbarhet          | observability               |
| amplitud                        | amplitude                    | ramp                    | ramp function               |
| amplitud                        | magnitude                    | rang                    | rank                        |
| amplitudmarginal                | gain margin                  | regulator               | controller                  |
| bandbredd                       | bandwidth                    | relativ dämpning        | damping ratio               |
| begynnelsevärde                 | initial value                | resonansfrekvens        | peak frequency              |
| beskrivande funktion            | describing function          | resonansfrekvens        | resonant frequency          |
| bindningsgraf                   | bond graph                   | resonanstopp            | peak resonance              |
| blekningsfilter                 | whitening filter             | rotort                  | root locus (pl. loci)       |
| dämpning                        | damping, attenuation         | sinusformad             | sinusoidal                  |
| egenvärde                       | eigenvalue                   | slutet system           | closed loop system          |
| enhetsteg                       | unit step                    | skärffrekvens           | gain crossover frequency    |
| faltning                        | convolution                  | sparsam                 | parsimonious                |
| fasavancerande                  | phase lead                   | stabilitetsrobusthet    | stability robustness        |
| fasmarginal                     | phase margin                 | stationärt tillstånd    | steady state                |
| fasretarderande                 | phase lag                    | statisk förstärkning    | static gain                 |
| fasskärffrekvens                | phase crossover frequency    | steg                    | step function               |
| flöde                           | flow                         | stegsvar                | step response               |
| framkoppling                    | feedforward                  | stigtid                 | rise time                   |
| fördelade parametriska modeller | distributed parameter models | styrbarhet              | controllability             |
| förstärkning                    | gain                         | styrslag                | control law                 |
| giltighet                       | validity                     | ställdon                | actuator                    |
| givare                          | sensor                       | störningsundertryckning | disturbance rejection       |
| händelseorienterade system      | discrete event systems       | tidsfördröjning         | time delay                  |
| impulssvar                      | impulse response             | tillstånd               | state                       |
| instabil                        | unstable                     | tillståndsåterkoppling  | state feedback              |
| insvängningstid, lösningstid    | settling time                | trovärdighet            | credibility                 |
| integratoruppvridning           | reset windup                 | underrum                | subspace                    |
| korrelationsanalys              | correlation analysis         | återkoppling            | feedback                    |
| krets förstärkning              | loop gain, open loop system  | öppet system            | open-loop system, loop gain |
| känslighetsfunktion             | sensitivity function         | överanpassning          | overfit                     |
|                                 |                              | överföringsfunktion     | transfer function           |
|                                 |                              | översläng               | overshoot                   |