

Exercises

This version: January 2011

1 Mathematics

1.1 Determine the one-sided Laplace transform of the following signals.

a) $u(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$, where A is a constant.

b) $u(t) = \begin{cases} 0, & t < 0 \\ At, & t \geq 0 \end{cases}$, where A is a constant.

c) $u(t) = e^{-2t}$ for $t \geq 0$.

d) $u(t) = \cos 5t$ for $t \geq 0$.

Express the following in $U(s)$, the Laplace transform of $u(t)$.

e) $\dot{u}(t)$

f) $\dot{u}(t)$, when $u(t) = 0$ for $t \leq 0$.

g) $\ddot{u}(t)$

h) $\ddot{u}(t)$, when $u(t) = \dot{u}(t) = 0$ for $t \leq 0$.

i) $u(t - T)$

1.2 Consider the differential equation

$$\dot{y}(t) + 2y(t) = u(t)$$

a) If $u(t)$ is constant then $\dot{y}(t) \approx 0$ when time goes to infinity. What value will $y(t)$ approach as $t \rightarrow \infty$ if $u(t) = 5$?

b) Determine the transfer function relating $U(s)$ and $Y(s)$ for the differential equation above.

1.3 Determine the general solution of the differential equation

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = 3\sin(2t)$$

1.4 Below, differential equations that describe dynamic systems are given together with system inputs and initial conditions. Use the Laplace transform to determine the system outputs.

a)

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sigma(t)$$

$$\sigma(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$\frac{dy}{dt}(0) = y(0) = 0$$

b)

$$\dot{y}(t) + y(t) = u(t)$$

$$u(t) = 1 + \sin t$$

$$y(0) = 0$$

1.5 Write the following complex numbers in polar form, that is, determine their absolute value and argument.

a) $1 + i$

b) $\frac{1+i}{5i(1+\sqrt{3}i)}$

Write the following complex numbers on rectangular form:

c) $2e^{i\frac{\pi}{3}}$

d) $5e^{-i\pi}$

1.6 A system has amplification 100. What is the amplification expressed in decibel (dB_{20})? What is the amplification corresponding to 20 dB_{20} , -3 dB_{20} , 0 dB_{20} , -10 dB_{20} , and 10 dB_{20} respectively?

1.7 Verify that the following rule for inversion of 2×2 matrices holds.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

1.8 Determine eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -6 & 2 & 1 \end{pmatrix}$$

1.9 Determine a transformation matrix T , such that $T^{-1}AT$ is a diagonal matrix, where

$$A = \frac{1}{3} \begin{pmatrix} 6 & 0 & -3 \\ -1 & 5 & -1 \\ -2 & -2 & 7 \end{pmatrix}$$

1.10 Characterize the range space and null space of the matrix A by specifying bases for them. Find the rank of the matrix.

$$A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

1.11 What are the time functions corresponding to the Laplace transforms below? What values will the time functions approach as time goes to infinity?

a)

$$F(s) = \frac{1}{s^2 + s}$$

b)

$$F(s) = \frac{1}{s^2 - 1}$$

c)

$$F(s) = \frac{1}{(s + 1)^2}$$

1.12 The water level, y , in a tank is modelled by the differential equation

$$\dot{y}(t) + y(t) = z(t)$$

where z denotes the inflow. The inflow is a function of a valve position, which in turn is controlled by the electric control signal u . The relation between control signal and flow is given by the differential equation

$$\ddot{z}(t) + \dot{z}(t) + z(t) = u(t)$$

What differential equation relates the water level y to the control signal u ?

2 Dynamic Systems

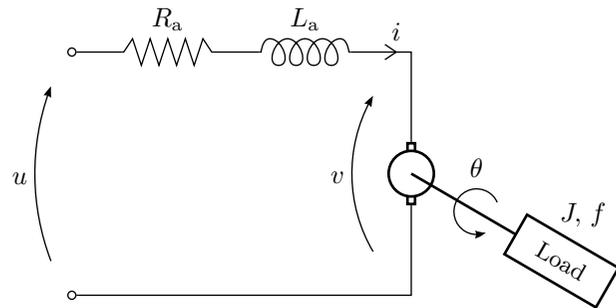


Figure 2.1a

2.1 A common component in a control system is a DC-motor. A schematic picture of the motor is shown in Figure 2.1a. The motor is characterized by a number of physical relationships as will now be explained. The rotating axis is described by

$$J\ddot{\theta} = -f\dot{\theta} + M,$$

where θ is the angle of rotation, M is the torque, J is the moment of inertia of the load and f is the frictional coefficient.

The interplay between rotor and stator is given by

$$M = k_a i \quad \text{and} \quad v = k_v \dot{\theta}$$

where i is the anchor current, k_a a proportional constant characteristic for the motor, v is voltage induced by the rotating axis and k_v is a proportional constant. The input voltage u is the control signal and θ is the output.

- Use the equations above and Kirchhoff's voltage law to write a differential equation that relates u and θ . The inductance L_a can be neglected.
- Determine the transfer function of the system from u to θ .
- Study the behavior of the system by calculating θ when u is a step.

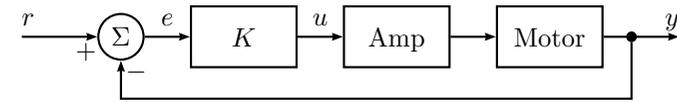


Figure 2.2a

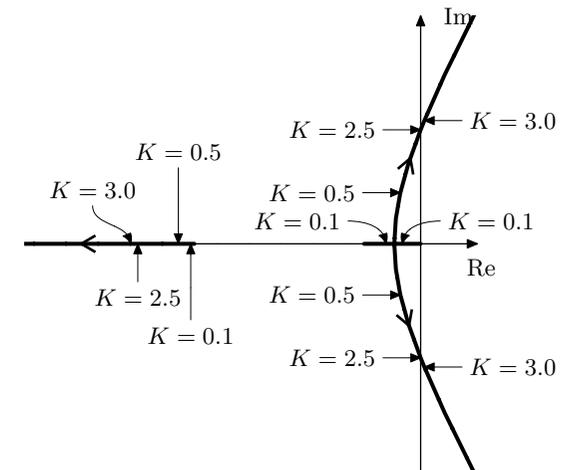


Figure 2.2b

2.2 A servo system for positioning of a tool in a tooling machine is depicted in Figure 2.2a. In Figure 2.2b, the poles of the closed loop system are plotted for different values of the gain K . Find (without calculations), for each of the step responses in Figure 2.2c, the corresponding value of K used when generating the step response.

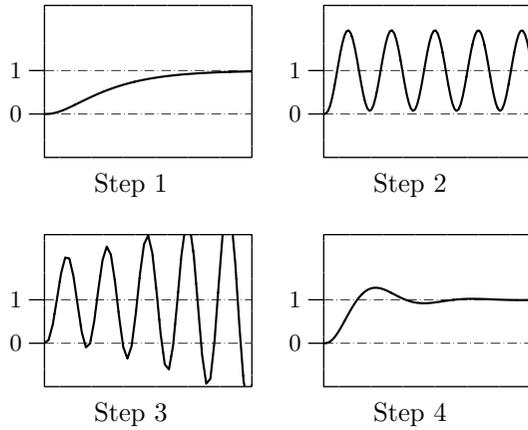


Figure 2.2c. All comparable axes have equal scaling.

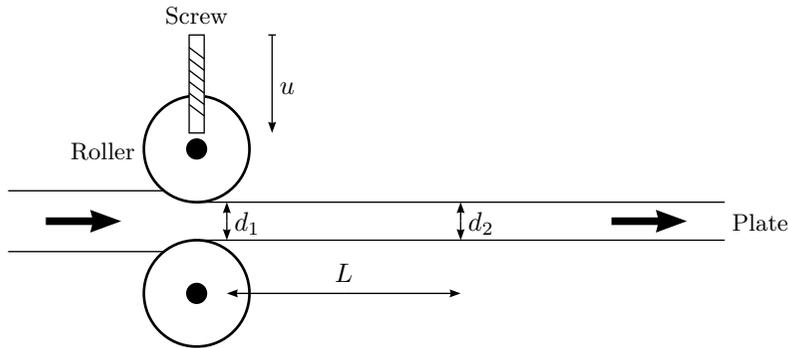


Figure 2.3a

2.3 Consider the simple model of the roller depicted in Figure 2.3a. To obtain a simple model we describe the relationship between the position of the screw and the thickness of the sheet d_1 directly after the rollers as a first order transfer function

$$G(s) = \frac{\beta}{1 + sT}$$

To determine the constants β and T we register the effect of a sudden change in the position of the screw. The units used in the model are chosen such that a unit step will make an appropriately sized input for identification purposes,

and that is the input used in the experiment for which the resulting thickness profile $d_1(t)$ is shown in Figure 2.3b. In production the thickness cannot be measured directly behind the rollers for practical reasons, and instead the thickness $d_2(t)$ is measured L length units after the rollers. Find the transfer function from the position of the screws to the thickness d_2 . The sheet moves with speed V .

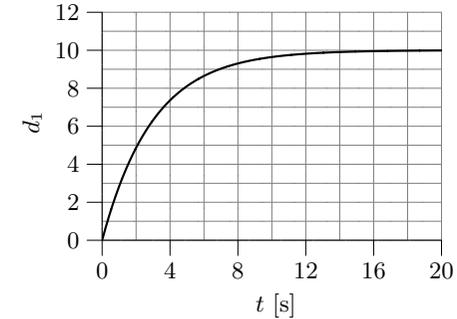


Figure 2.3b

2.4 The step response of the following system

$$G(s) = \frac{1}{s^2 + s + 1}$$

is shown as the dashed line in each part of Figure 2.4a.

a) The step response of the system

$$G(s) = \frac{1}{s^2 + as + 1}$$

is shown as the solid line in the left of Figure 2.4a. Determine if $a > 1$ or $a < 1$.

b) The step response of the system

$$G(s) = \frac{b^2}{s^2 + bs + b^2}$$

is shown as the solid line in the right of Figure 2.4a. Find b .

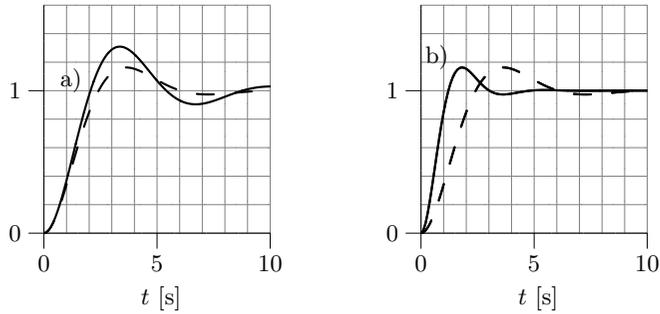


Figure 2.4a. Dashed: original system. Solid left: part a). Solid right: part b).

2.5 Pair the step responses and pole-zero diagrams in Figure 2.5a.

2.6 Consider the systems

$$G_A(s) = \frac{1}{s^2 + 2s + 1} \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1} \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

- Use MATLAB to plot the step responses of the systems. Find T_r (rise time), T_s (settling time) and M (overshoot) for the five step responses.
- Compute the poles of the systems $G_A(s)$, $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ respectively.
- How is the location of the poles related to the properties of the step responses?

2.7 Consider a system with the transfer function

$$G(s) = \frac{\alpha s + 1}{s^2 + 2s + 1}$$

Compute and plot the step response of the system for some different values of α in the range $-10 < \alpha < 10$. How are the properties of the step response affected by the location of the zero of the system?

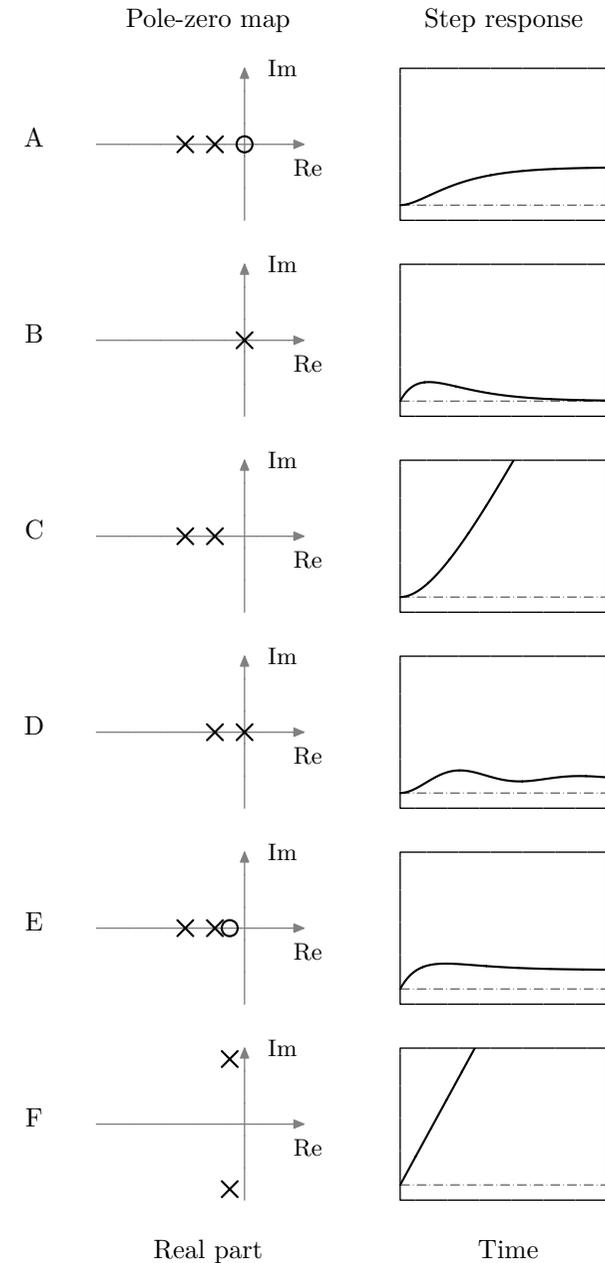


Figure 2.5a. All comparable diagrams have equal scaling. In the pole-zero maps, imaginary and real parts have equal scaling, \times marks poles, and \circ marks zeros.

2.8 Consider a system described by the model $Y(s) = G(s)U(s)$ as shown in Figure 2.8a. Given $G(s)$, how is the step response computed? How can the step response be determined using experiments?

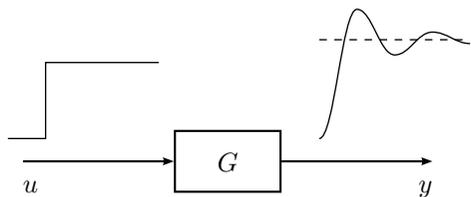


Figure 2.8a

2.9 Figure 2.9a shows the step response of a system $Y(s) = G(s)U(s)$. The input step has amplitude 1. Use the figure and determine

- Steady state value.
- Overshoot M in % of the final value.
- Rise time T_r .
- Settling time T_s .

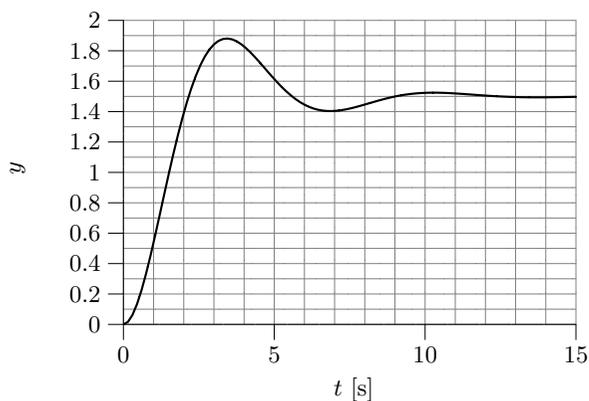


Figure 2.9a

2.10 Figure 2.10a shows the step responses of four different systems. Combine each step response with a transfer function from the alternatives below.

Transfer function	Poles	Zeros	$ G(0) $
$G_1(s) = \frac{100}{s^2+2s+100}$	$-1 \pm 10i$		1
$G_2(s) = \frac{1}{s+2}$	-2		1/2
$G_3(s) = \frac{10s^2+200s+2000}{(s+10)(s^2+10s+100)}$	$-10, -5 \pm 8.7i$	$-10 \pm 10i$	2
$G_4(s) = \frac{200}{(s^2+10s+100)(s+2)}$	$-2, -5 \pm 8.7i$		1
$G_5(s) = \frac{600}{(s^2+10s+100)(s+3)}$	$-3, -5 \pm 8.7i$		2
$G_6(s) = \frac{400}{(s^2-10s+100)(s+2)}$	$-2, 5 \pm 8.7i$		2

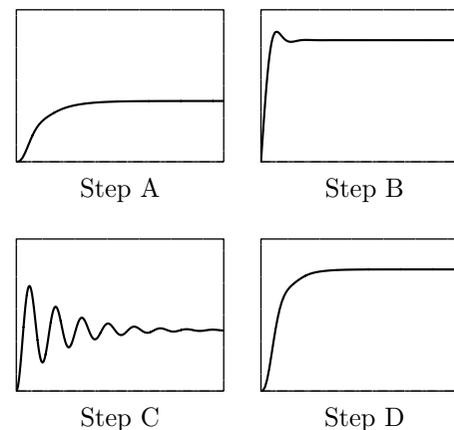


Figure 2.10a. All comparable axes have equal scaling.

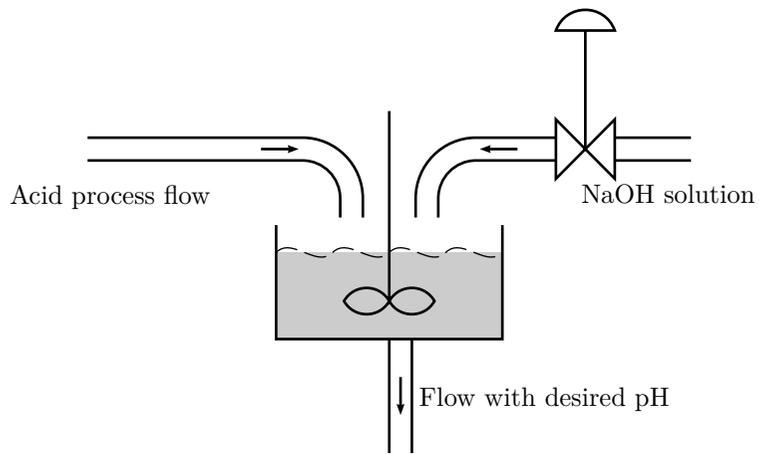


Figure 2.11a

2.11 In the continuously stirred tank, see Figure 2.11a, an acid process flow is neutralized by adding a concentrated NaOH solution. The acid process flow has a tendency to vary its pH with time, which gives undesired variation of the pH in the outflow. In an effort to reduce these variations one has decided to use control.

- Classify the different signals as input, output, and disturbance signal.
- Draw a block diagram of the system with a control strategy.

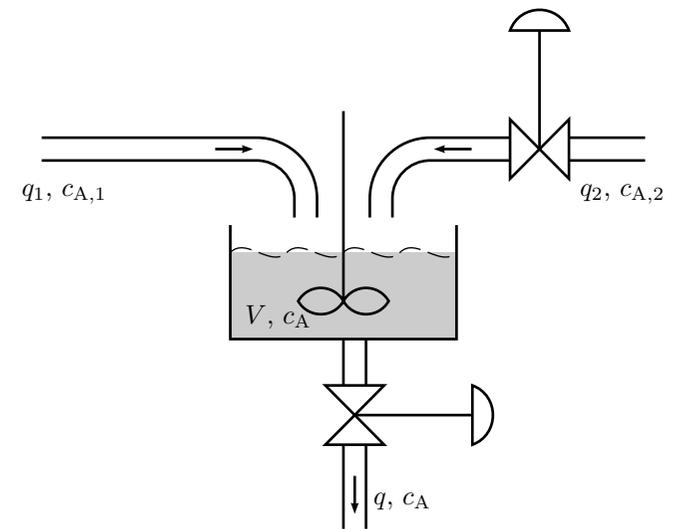


Figure 2.12a

2.12 Two flows with different concentrations of a chemical component A are mixed in a continuously stirred tank, see Figure 2.12a. The volume can be assumed constant, $V = 1 \text{ m}^3$.

- During a stationary period the values of the concentrations and flows are, $c_{A,1}^* = 1.0 \text{ kmol/m}^3$, $q_1^* = 1.0 \text{ m}^3/\text{min}$, $c_{A,2}^* = 4.0 \text{ kmol/m}^3$ and $q_2^* = 0.5 \text{ m}^3/\text{min}$. What are $c_A(t)$ and $q(t)$ during this period?
- Write down the dynamical balance equation for component A. State all your assumptions. Is the dynamical model linear?
- Assume that $c_{A,1}$ changes value from 1.0 kmol/m^3 to 1.2 kmol/m^3 at $t = 0$. Show that the expression for $c_A(t)$ can be written as

$$c_A(t) = k_0 + k_1 (1 - e^{-t/\tau})$$

Determine the constants k_0 , k_1 and τ .

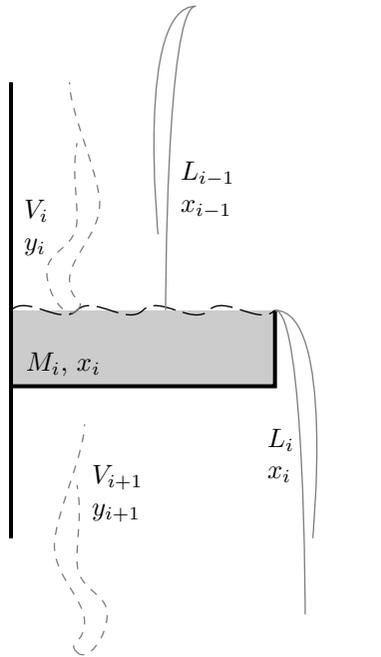


Figure 2.13a

2.13 The distillation plate, see Figure 2.13a, has the following variables:

L_i - Liquid flow from plate i (kmol/min)

V_i - Steam flow from plate i (kmol/min)

M_i - Amount of liquid on plate i (kmol)

x_i - Mole fraction of the most volatile component in the liquid on plate i .

y_i - Mole fraction of the most volatile component in the steam from plate i .

Assumptions:

- Perfect mixing of liquid on the plate.
- Binary separation
- y_i is in equilibrium with x_i .

- The equilibrium is described by

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}$$

- a) Write the two differential equations that describe $M_i(t)$ and $x_i(t)$.
- b) Linearize the model under the assumption that the change of the mass on the plate is zero.

3 Feedback Systems

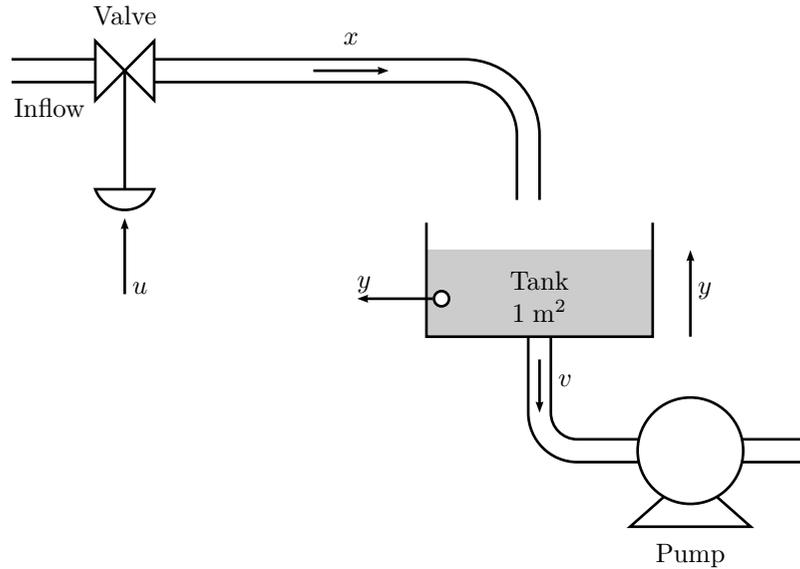


Figure 3.1a

3.1 Consider a part of a process industry, consisting of a valve and a tank. The system is described by Figure 3.1a and the following information:

- The valve is operated electrically, and the input voltage is denoted u . The resulting liquid flow is denoted x .
- The level in the tank is denoted y , and the flow out from the tank, v , is determined by a pump located further down the process. The cross section area of the tank is 1 m^2 .
- The pipes are always filled with incompressible liquid, so there are no transport delays in the system.

a) Use mass balance, that is, the fact that the change in tank volume per time unit is proportional to the difference between inflow and outflow, to determine a transfer function $G_t(s)$ for the tank.

b) The transfer function of the valve is assumed to be

$$G_v(s) = \frac{k_v}{1 + Ts}$$

To find k_v and T a unit step change in u has been applied. The step response, that is, the resulting liquid flow, is shown in Figure 3.1b. Determine the constants k_v and T .

c) Draw a block diagram of the system, in which the valve and the tank are represented by one block each.

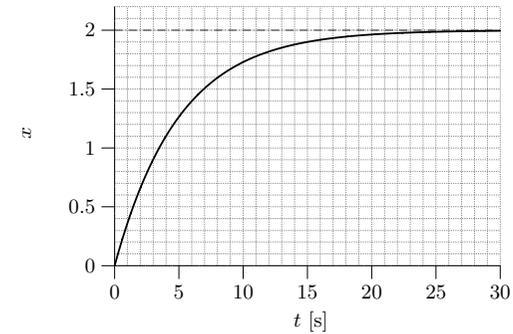


Figure 3.1b

3.2 Consider the system studied in Problem 3.1. The aim is now to control the tank level automatically, and therefore PID feedback control is introduced according to Figure 3.2a. This means that the valve input is determined according to the expression

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau + K_D \frac{de(t)}{dt} \quad (3.1)$$

where $e(t)$ denotes the control error, that is,

$$e(t) = r(t) - y(t)$$

and $r(t)$ denotes the reference level, that is, desired tank level.

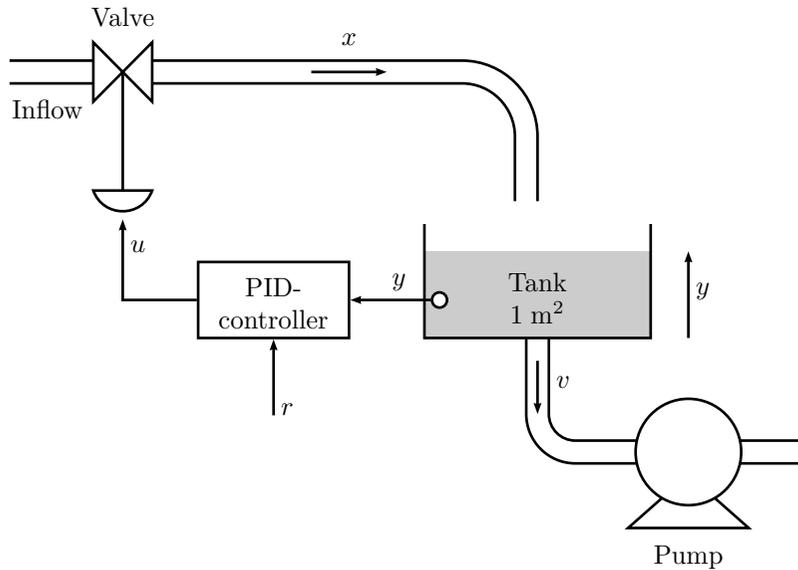


Figure 3.2a

- a) Verify, using Laplace transform, that the relationship in (3.1) can be expressed

$$U(s) = F(s)(R(s) - Y(s))$$

where

$$F(s) = K_P + K_I \frac{1}{s} + K_D s$$

- b) Let the PID controller be represented by a block and draw a block diagram of the entire feedback control system.
- c) Use the block diagram from above together with the transfer functions $G_v(s)$, $G_t(s)$, and $F(s)$, and derive the transfer functions of the closed loop system, that is, the transfer functions from the reference level and the outflow to the tank level.

3.3 Consider the level control system studied in Problems 3.1 and 3.2.

- a) Assume that the tank level is controlled using proportional control, that is, $F(s) = K_P$. Compute the poles of the closed loop system for $K_P =$

0.02 and $K_P = 1$ respectively. How does the choice of K_P affect the properties of the closed loop system?

- b) Assume that the reference level is constant, $r(t) = 5$, and that the outflow is constant, $v(t) = 2$. Determine the steady state tank level when the proportional feedback $F(s) = K_P$ is used. Is it possible for the output signal to reach the desired level? What happens with the steady state level and other properties of the closed loop system if K_P is chosen very large?
- c) Assume that a PI controller is used, that is,

$$u(t) = K_P e(t) + K_I \int e(\tau) d\tau$$

What can be said about the steady state tank level in this case? What is the possible benefit of introducing the integrating part in the feedback?

- d) Finally, assume that a PD controller is used, that is,

$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}$$

Assume also that $K_P = 1$ and calculate a value of K_D so the damping ratio of the closed loop poles will be greater than $1/\sqrt{2}$. This corresponds to placing the poles in the grey area in Figure 3.3a. What is the benefit of including the derivative part in the feedback?

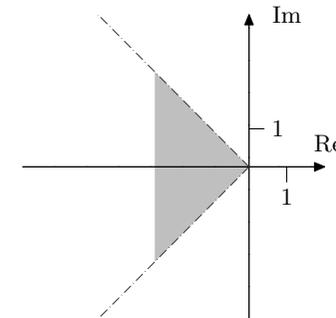


Figure 3.3a

3.4 Figure 3.4a shows the block diagram for a hydraulic servo system in an automatic lathe (Swedish: *svarv*). The output signal $y(t)$ represents the position of

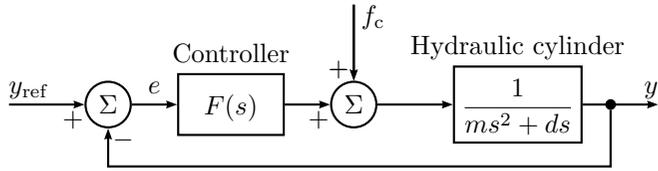


Figure 3.4a

the lathe tool (Swedish: *svarvstål*), $y_{\text{ref}}(t)$ is the desired position of the lathe tool, m is the mass of the tool slide (Swedish: *verktygsslid*) and the hydraulic piston (Swedish: *kolv*), d is the viscous damping of the tool slide, $F(s)$ is the transfer function for the controller, f_c is the cutting power.

- How large, in steady state, is the error $e(t)$ between the actual value of the lathe tool and its desired reference value, when there is a step disturbance in the cutting power $f_c(t)$? The controller is assumed to be an amplifier with a constant gain $F(s) = K$.
- How is this error changed if the amplifier is replaced by a PI controller with transfer function $F(s) = K_1 + K_2/s$?

3.5 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- Suppose $G(s)$ is controlled by a proportional controller with gain K_P , that is,

$$U(s) = K_P(R(s) - Y(s)).$$

Use MATLAB to compute the closed loop system, and to plot the step response of the closed loop system. Choose some values for K_P in the range 0.1 to 10. How are the properties of the step response affected by K_P ? What happens with the steady state error when K_P increases? Is it possible to obtain a well damped closed loop system and small steady state error using proportional control?

- Let us now introduce integration in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s})(R(s) - Y(s)).$$

Put $K_P = 1$ and try some values of K_I in the range $0 < K_I < 2$. How are the step response and the steady state error affected by the introduction of the integrating part and the value of K_I ?

- Finally we will introduce the differentiating part in the regulator and use

$$U(s) = (K_P + K_I \frac{1}{s} + \frac{K_D s}{sT + 1})(R(s) - Y(s)).$$

Since true differentiation is difficult to implement, the derivative part is approximated by $\frac{K_D s}{1 + sT}$. (This will low-pass filter the error signal before differentiation.) Put $K_P = 1$, $K_I = 1$ and $T = 0.1$ and try some values of K_D in the range $0 < K_D < 3$. How does the D-part affect the step response of the closed loop system?

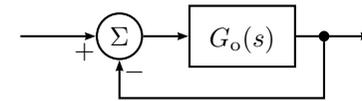


Figure 3.6a

3.6 Draw a root locus with respect to K for the system in Figure 3.6a, with $G_o(s)$ given below. For which values of K are the systems stable? What conclusions on the principal shape of the step response can be drawn from the root locus?

- A Ferris wheel (Swedish: *Pariserhjöl*):

$$G_o(s) = \frac{K(s + 2)}{s(s + 1)(s + 3)}$$

- A Mars rover:

$$G_o(s) = \frac{K}{s(s^2 + 2s + 2)}$$

- A magnetic floater:

$$G_o(s) = \frac{K(s + 1)}{s(s - 1)(s + 6)}$$

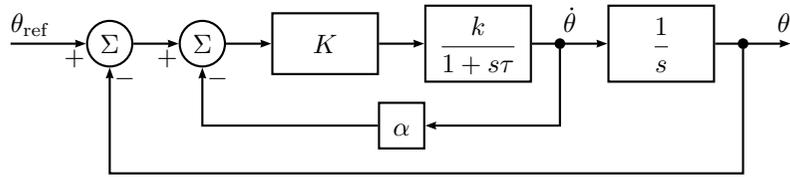


Figure 3.7a

3.7 Consider the servo system in Figure 3.7a with a DC-motor. Suppose that the angular velocity can be measured with a tachometer and let the control law be as in the block diagram. Let $\tau = 0.5$ and $k = 2$.

- Draw the root locus with respect to K for the system without the tachometer feedback (that is, $\alpha = 0$).
- Draw the root locus with respect to K for $\alpha = 1$.
- Draw the root locus with respect to K for $\alpha = 1/3$.
- Let $K = 1$ and draw the root locus with respect to α .

Discuss, using the results from a), b), c), and d), what is gained by using the tachometer.

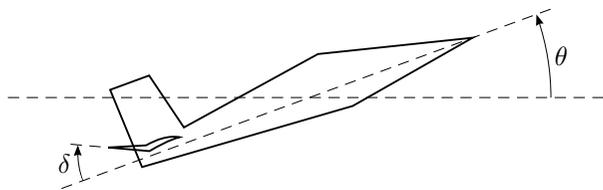


Figure 3.8a

3.8 Consider an aircraft where the pitch angle θ is controlled by the elevator deflection (Swedish: *höjdroderutslag*) δ , see Figure 3.8a. Let ω be the angular velocity,

$$\omega = \dot{\theta}.$$

If we consider small deviations from a reference value θ_0 , we get the transfer function from δ to ω for a specific aircraft as

$$G_2(s) = \frac{s + 1}{(s + 4)(s - 3)}$$

This model is valid when the aircraft is flown with a large θ_0 . The elevator (Swedish: *höjdroder*) is driven by a hydraulic servo amplifier with the transfer function

$$G_1(s) = \frac{10}{s + 10}$$

from elevator command δ_{ref} to δ .

- What happens with ω if one gives a constant elevator command δ_{ref} ? Motivate!
- The angular velocity ω is measured and a control law is used so that the input δ_{ref} to the servo amplifier is

$$K(\omega_{\text{ref}} - \omega).$$

Draw root locus with respect to K . For which values of K is the system stable?

- Is there any value of K such that the closed loop system is stable and all poles are real?

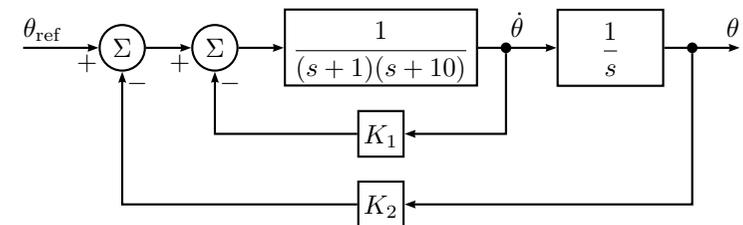


Figure 3.9a

3.9 The block diagram in Figure 3.9a shows a cascade controlled DC-motor where $K_1 > 0$ and $K_2 > 0$.

- Draw root locus with respect to K_2 for the characteristic equation of the closed loop system. For which $K_2 > 0$ are the closed loop system asymptotically stable?

- b) How is the stability requirement on K_2 affected by the size of the velocity feedback K_1 ?

3.10 We want to control the temperature of an unstable chemical reactor. The transfer function is

$$\frac{1}{(s+1)(s-1)(s+5)}$$

- a) Use a proportional controller and draw a root locus with respect to the amplification K . Calculate which K in the compensator that stabilizes the system.
 b) Use a PD controller. The control law is given by

$$u = K\left(e + T_D \frac{de}{dt}\right)$$

where e is the error. Let $T_D = 0.5$ and draw a root locus with respect to K . For which values of K does the controller stabilize the system?

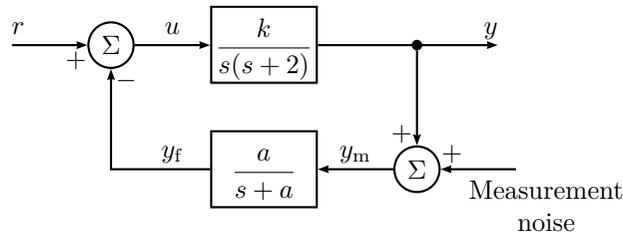


Figure 3.11a

3.11 Consider the system in Figure 3.11a. In a realistic situation what you really measure is not $y(t)$ but a signal $y_m(t)$ which is the sum of $y(t)$ and measurement noise. To avoid that the control is based on noisy measurements one uses $y_f(t)$ instead of $y_m(t)$. The signal $y_f(t)$ is $y_m(t)$ filtered through the low pass filter

$$\frac{a}{s+a}$$

- a) First we assume that the measurement noise is negligible. Choose $k = 6$. Draw a root locus for the closed loop system with respect to the time constant of the low pass filter $1/a$. Find for which $a > 0$ the system is stable.

- b) Use $k = 6$ and assume that the noise is a high frequency sinusoid. The amplitude of y_f when

$$y_m(t) = \sin(10t)$$

is used as a measurement of how effective the noise reduction is. What is the smallest value you can obtain (after transients) by choosing a suitable a . We also want y to tend to the steady state value 1 when $r(t)$ is a unit step.

3.12 Figure 3.12a shows the root locus for the characteristic equation of a P-controlled process G with respect to the gain K . In Figure 3.12b four step responses for the closed loop system with different values of K are shown. Match the plots in Figure 3.12b with the K -values below. Justify your answer.

$$K = 4 \quad K = 10 \quad K = 18 \quad K = 50$$

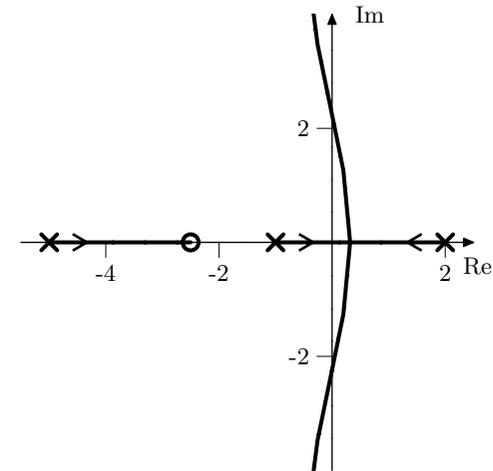


Figure 3.12a. Starting points are marked \times and end points \circ .

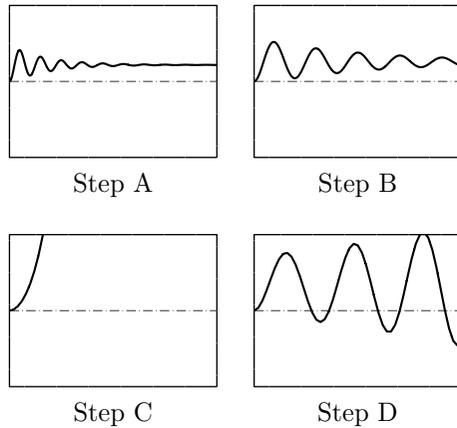


Figure 3.12b. All comparable axes have equal scaling.

3.13 Consider a system with the transfer function

$$G(s) = \frac{s^{n-1} + b_1 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

that has all zeros strictly in the left half plane. Show that such a system always can be stabilized by

$$u(t) = -Ky(t)$$

if K is selected large enough.

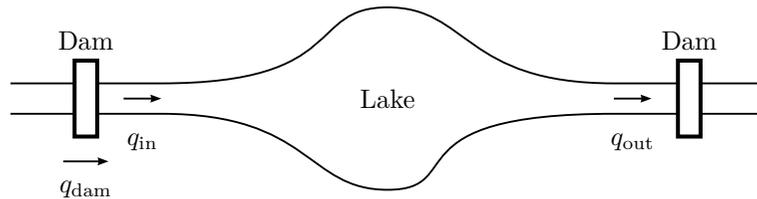


Figure 3.14a

3.14 We want to control the level in a lake by controlling the flow using flood gates, see Figure 3.14a. The relationship between changes in the level of the lake,

h_Δ , and changes in the flows, $q_{in,\Delta}$ and $q_{out,\Delta}$, is given by

$$\frac{d}{dt}(Ah_\Delta) = q_{in,\Delta} - q_{out,\Delta}$$

where A is the area of the lake. In order to try to keep the level of the lake constant the flows through the lake are controlled at the inflow such that

$$q_{dam,\Delta} = K(h_{ref,\Delta} - h_\Delta)$$

where $h_{ref,\Delta}$ is the reference value. The reservoir that controls the outflow is controlled so that q_{out} is constant, that is, $q_{out,\Delta} = 0$. Since it takes time before a change in q_{dam} gives a result in q_{in} we have

$$q_{in,\Delta} = q_{dam,\Delta}(t - T)$$

where $T = 0.5$ hours. How large can the quotient K/A be at the most before the system becomes unstable?

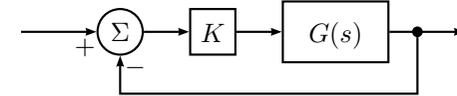


Figure 3.15a

3.15 A system $G(s)$ is controlled using feedback with a proportional controller according to Figure 3.15a.

- For $K = 1$, the open loop system $KG(s)$ has the Nyquist diagram according to (i), (ii), (iii), or (iv) in Figure 3.15b. Is the closed loop system stable in each case? $G(s)$ has no poles in the right half plane.
- If $K > 0$, for which values of K are the different closed loop systems stable?

3.16 a) Draw the Nyquist curve for an integrator $G(s) = 1/s$.

b) Draw the Nyquist curve for the double integrator $G(s) = 1/s^2$.

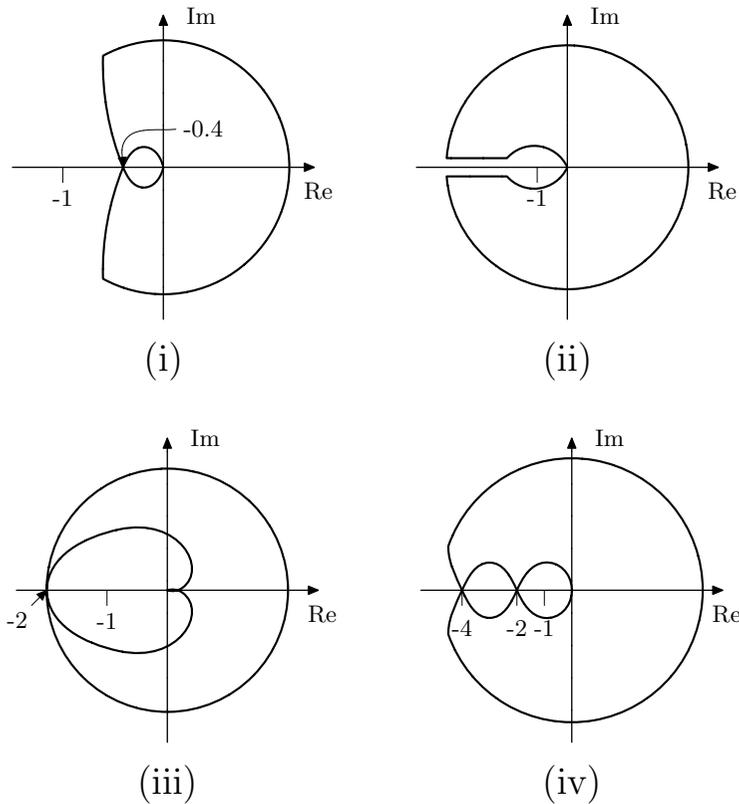


Figure 3.15b

3.17 The system $G(s)$ is asymptotically stable and has the Nyquist curve in Figure 3.17a. It is controlled using feedback according to Figure 3.17b.

- For what values of K ($K > 0$) is the closed loop system asymptotically stable?
- Determine the steady state error, e , as a function of K if y_{ref} is a unit step.
- Assume that G is controlled using an I controller according to Figure 3.17c. For what values of K is the closed loop system stable?

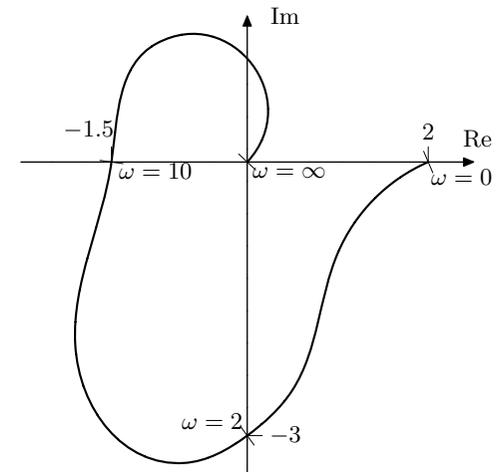


Figure 3.17a

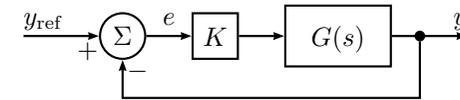


Figure 3.17b

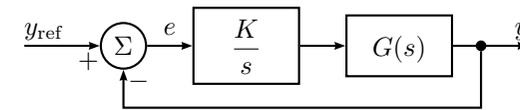


Figure 3.17c

3.18 Consider the DC-motor

$$\tau \ddot{y}(t) + \dot{y}(t) = u(t)$$

It is controlled by

$$u(t) = K(r(t - T) - y(t - T))$$

Here τ and T are positive constants. K is slowly increased until the system oscillates with the angular frequency $\omega = 1$. K is then set to 33% of this value. After a while the system starts to oscillate again, now with the angular frequency $\omega = 0.5$. This is due to the fact that the time delay T has changed

to T_1 . Can the parameters τ , T , and T_1 be decided from these data? If so determine τ , T and T_1 .

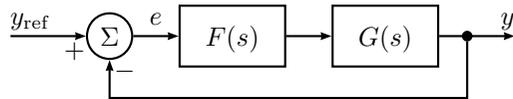


Figure 3.19a

3.19 A system $G(s)$ is to be controlled using the regulator

$$F(s) = \frac{K}{s+1}$$

according to the Figure 3.19a. The controller has positive gain K that, however, is not completely known. For what values of K is the closed loop system asymptotically stable? The Nyquist curve for $G(s)$ is given in Figure 3.19b.

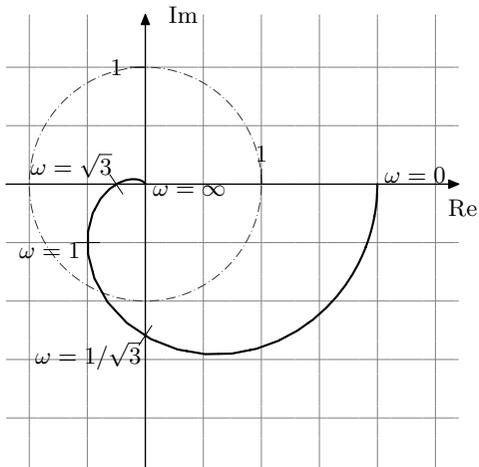


Figure 3.19b

3.20 The Nyquist curve for the system $G(s) = \frac{B(s)}{A(s)}$ can be seen in Figure 3.20a. Determine which one of the root loci in Figure 3.20b that matches

$$A(s) + KB(s) = 0$$

for this system.

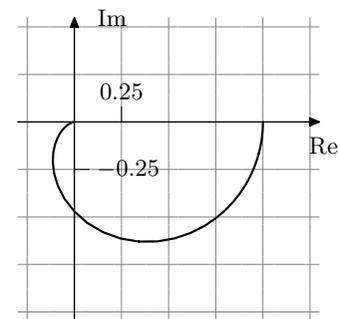


Figure 3.20a

3.21 The system $\frac{1}{s(s+1)}$ with input u and output y has the controller

$$U(s) = F(s)(R(s) - Y(s)) \quad F(s) = \frac{b_0 s^2 + b_1 s + b_2}{s}$$

How should the coefficients of F be chosen to achieve pure P, pure I and pure D control respectively?

3.22 The equations for the P, PI, and PID controllers to be used in this problem are given in Problem 3.5.

a) Let the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s)$$

be controlled by a proportional controller with gain K_P . Use MATLAB to plot the root locus with respect to K_P of the characteristic equation of the closed loop system. For which values of $K_P > 0$ is the closed loop system asymptotically stable?

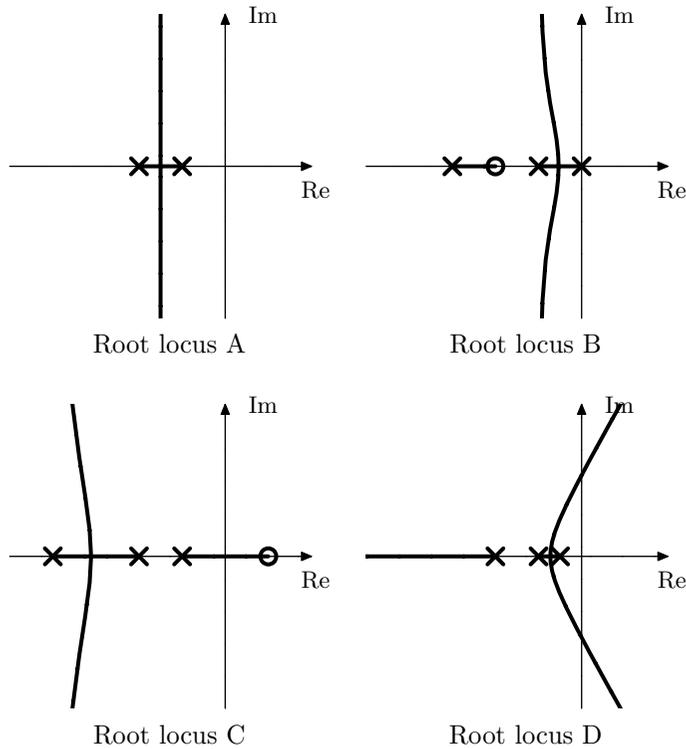


Figure 3.20b. Starting points ($K = 0$) are marked \times , and end points ($K \rightarrow \infty$) are marked \circ . All diagrams have equal scaling.

In Problem 3.5, we found that the step response was slow but well damped for small values of K_P , while it became faster but more oscillatory when K_P was increased. For large values of K_P the system became unstable. We also found that the steady state error was reduced when K_P was increased. Can these results be interpreted using the plot of the root locus?

- b) Now assume that the system is controlled by a PI controller where $K_P = 1$. Plot the root locus of the characteristic equation, with respect to K_I , and determine for which values of $K_I > 0$ the closed loop system is asymptotically stable.

Problem 3.5 showed that an integrator eliminates the steady state error.

A small value of K_I gives a large settling time, while a too large value gives an oscillatory, and perhaps unstable closed loop system. Give an interpretation of these results using the root locus.

- c) Finally, let the system be controlled by a PID controller where $K_P = 1$, $K_I = 1$ and $T = 0.1$. Plot a root locus of the closed loop characteristic equation, with respect to $K_D > 0$, and relate the behavior of the root locus to the simulation result in Problem 3.5, that is, that the derivative part increases the damping of the closed loop system, but a too large K_D will give an oscillation with a higher frequency, and finally an unstable closed loop system.

3.23 Consider the system

$$Y(s) = G(s)U(s) = \frac{0.2}{(s^2 + s + 1)(s + 0.2)}U(s).$$

- a) Use MATLAB to plot the Nyquist curve of the open loop system when $G(s)$ is controlled by a proportional regulator. Try some different values of K_P and find for which K_P the closed loop system is asymptotically stable. Compare your results with those from Problem 3.22a.
- b) Assume now that the system is controlled by a PI controller where $K_P = 1$. Investigate how K_I affects the Nyquist curve and determine for which values of K_I the closed loop system is asymptotically stable. Do you get the same results as in Problem 3.22b?
- c) Finally test a PID controller with $K_P = 1$, $K_I = 1$ and $T = 0.1$ (cf Problem 3.5). How is the Nyquist curve affected by the value of K_D ?

3.24 a) Assume that the system

$$Y(s) = G(s)U(s) = \frac{0.4}{(s^2 + s + 1)(s + 0.2)}U(s)$$

is controlled by a proportional controller where $K_P = 1$. Use MATLAB to make a Bode plot of the open loop system and determine ω_c (gain crossover frequency), ω_p (phase crossover frequency), φ_m (phase margin) and A_m (gain margin) respectively. Compute the closed loop system and plot the step response.

- b) Now let $K_P = 2.5$. How does the change of K_P affect ω_c , ω_p , φ_m , and A_m ? Simulate the step response of the closed loop system and plot the result. How have the properties of the step response changed?

- c) How much can K_P be increased before the closed loop system becomes unstable? How does this value relate to the value of A_m that was obtained for $K_P = 1$? Compute and plot the step response of the closed loop system for this value of K_P . How does the closed loop system behave in this case?

3.25 A system is controlled by a PID controller,

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)E(s)$$

In Figure 3.25a four step responses from a unit step for the parameter triples

- i) $K_P = 1 \quad K_I = 0 \quad K_D = 0$
- ii) $K_P = 1 \quad K_I = 1 \quad K_D = 0$
- iii) $K_P = 1 \quad K_I = 0 \quad K_D = 1$
- iv) $K_P = 1 \quad K_I = 1 \quad K_D = 1$

are shown. Match each one of the parameter triples to one of the step responses. Justify your answer!

3.26 Assume that a DC-motor of the type described in Problem 2.1 is controlled by a proportional controller, that is, $u(t) = K_P(\theta_{\text{ref}} - \theta)$.

- a) Write down a block diagram for the control system. Compute the closed loop transfer function and determine how the poles of the closed loop system depend on the control gain K_P . Discuss what this means for the behavior of the system for different values of K_P .
- b) Determine the transfer function from the reference signal to the error. Let the reference signal be a step and a ramp respectively and determine what the control error will be in steady state in these two cases.
- c) Let the controller be a PI controller. What will the steady state error be in this case if the reference signal is a ramp?

3.27 Determine the transfer function for the loop gain and the closed loop system for the control system given by the block diagram in Figure 3.27a.

3.28 Figure 3.28a shows a block diagram of a control system. Determine the transfer function

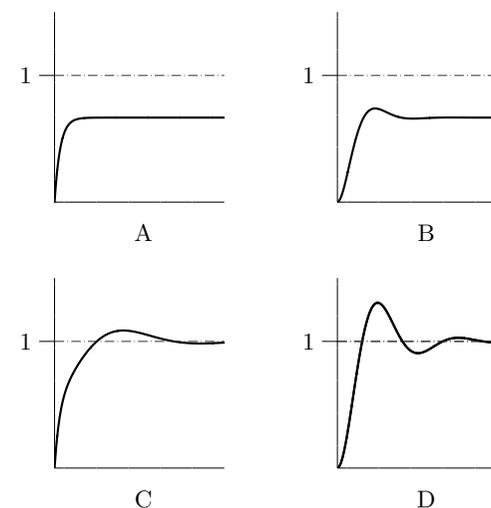


Figure 3.25a. Four step responses. All comparable axes have equal scaling.

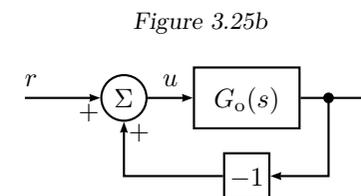


Figure 3.27a

- a) of the loop gain,
- b) of the closed loop system from $R(s)$ to $Y(s)$,
- c) from the disturbance $N(s)$ to the output $Y(s)$,
- d) from the reference signal $R(s)$ to the error signal $E(s)$.

3.29 Consider again the control system in Figure 3.28a, with $n = 0$ and

$$G(s) = \frac{1}{(s+1)(s+3)}$$

- a) Assume $F(s) = K$. Determine the steady state control error when $r(t)$ is a step.

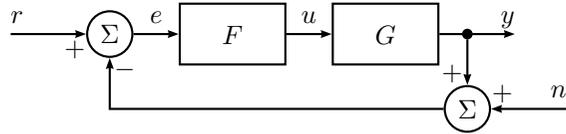


Figure 3.28a

- Determine a regulator $F(s)$ such that the steady state error is zero when $r(t)$ is a step.
- Assume $F(s) = 1$. Determine the poles and zeros of the closed loop system.

3.30 The system

$$Y(s) = \frac{1}{(s/0.6 + 1)(s + 1)}U(s)$$

is controlled using PID feedback

$$U(s) = (K_P + K_I \frac{1}{s} + K_D s)(R(s) - Y(s))$$

Figure 3.30a shows the step responses for the following four combinations of coefficient values. Combine the step responses and coefficients.

- (1) $K_P = 4 \quad K_I = 0 \quad K_D = 0$
- (2) $K_P = 4 \quad K_I = 3 \quad K_D = 0$
- (3) $K_P = 4 \quad K_I = 1 \quad K_D = 0$
- (4) $K_P = 4 \quad K_I = 0 \quad K_D = 4$

3.31 Continuously stirred tanks have an extensive use in chemical processes. They are often supplied with some sort of heat exchange system. If the tank is used for a chemical or biochemical reaction it is often important to keep a certain temperature to obtain desired productivity. A continuously stirred tank with a common type of heat exchange system is shown in Figure 3.31a. The tank is surrounded with a heating/cooling layer in which a liquid flows through in order to heat or chill the liquid in the tank.

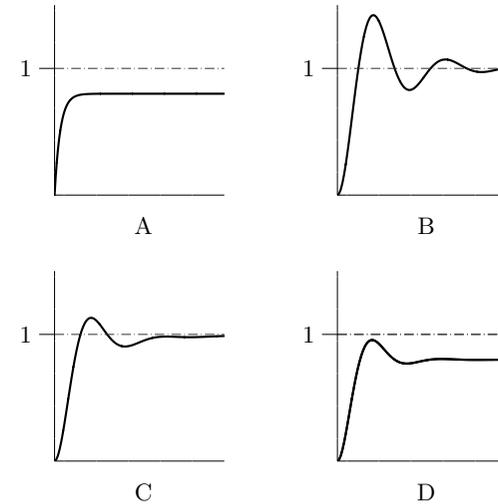


Figure 3.30a. Four step responses. All comparable axes have equal scaling.

- Determine the important signals of the system. Suggest a control strategy based on feedback and draw the block diagram.
- To be able to construct and evaluate different controllers it is necessary to have a process model. Determine a dynamical model for this system. Assume that the volumes in the tank and in the heat exchange system are constant.
- Linearize the model. (Stationary values: $T_t^* = 50.0 \text{ }^\circ\text{C}$, $T_c^* = 75.0 \text{ }^\circ\text{C}$.)
- The parameters of the model are as follows

$$\begin{aligned} \rho_t = \rho_c = 1000.0 \text{ kg/m}^3 \quad c_{t,\text{in}}^p = c_c^p = 4.2 \text{ kJ/kg}^\circ\text{C} \quad U = 672 \text{ kJ/}^\circ\text{Cmin} \\ F_t = 0.1 \text{ m}^3/\text{min} \quad F_c = 0.2 \text{ m}^3/\text{min} \quad V_t = 1.0 \text{ m}^3 \quad V_c = 1.0 \text{ m}^3 \\ T_{t,\text{in}} = 10.0 \text{ }^\circ\text{C} \quad T_{c,\text{in}} = 95.0 \text{ }^\circ\text{C} \end{aligned}$$

Determine the transfer function from flow in the heat exchange system to the temperature in the tank.

- Let the system be controlled by a P controller. Draw the root locus for the system.

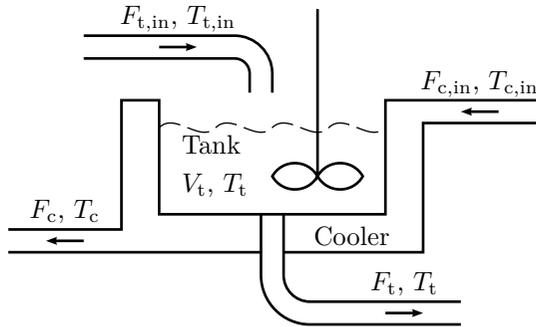


Figure 3.31a. Process consisting of a tank and a cooler. Input flows have a “in” subscript, while outputs have no such subscript since they are also the same as the quantities found inside the tank or cooler.

3.32 Consider a continuously stirred tank with a cooling system. In the tank a component A reacts to form component B in an exothermic reaction. This reaction is unstable, but possible to stabilize with feedback. A model for the purpose of control has been established

$$Y(s) = \frac{-1}{s^2 + 2s - 3} U(s)$$

where $y(t)$ is the temperature in the tank and $u(t)$ is the cooling flow.

- a) Show that the system is unstable.
- b) Prove that the system can be stabilized by a P controller.

3.33 Bacterial growth is described by the equation $\dot{y} = \mu y$ where y is the amount of bacteria and μ is a positive constant. Assume that we have a control signal u available that affects the speed of growth so that

$$\dot{y} = \mu y + u$$

One can then use a P controller $u = K(r - y)$ where r is a reference signal. For which K -values will the system approach an equilibrium?

3.34 While working in space an astronaut has to be able to move. Necessary force is obtained by letting out gas from thrusters. For such a positioning control system the control law

$$u = K_1(r - y) - K_1 K_2 \frac{dy}{dt}$$

is used, where u = thruster force, r = set point for the astronaut's position and y = the actual value of the astronaut's position. Draw a block diagram of the system, and use physics (Newton's law) to make a model of the astronaut. Also determine K_1 and K_2 such that

- If the set point $r(t) = t$ there has to be a time T_0 so that $|r(t) - y(t)| < 1$ for $t \geq T_0$.
- The damping ratio of the closed loop system will be 0.7. The mass of the astronaut is assumed to be 100 kg, equipment included.

4 Frequency Description

4.1 A mercury thermometer can be described with high accuracy as a first order linear time invariant dynamic system. The input is the real temperature and the output is the thermometer reading. In order to decide the transfer function in a thermometer it is placed in liquid where the temperature is varied as a sinusoid. The obtained result is shown in Figure 4.1a. Find the transfer function of the thermometer.

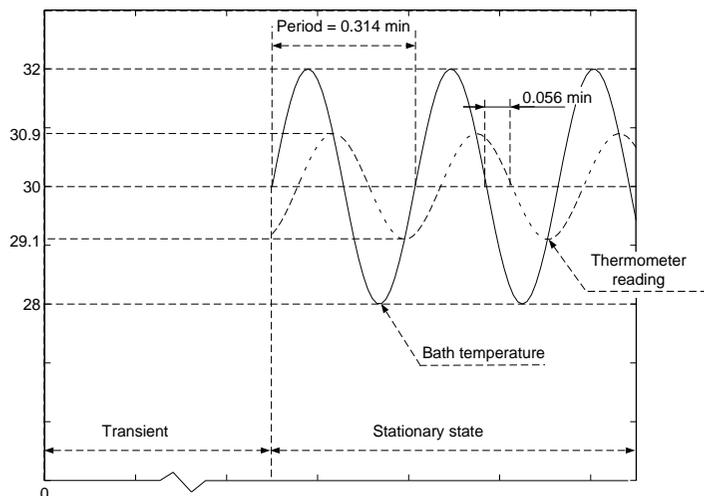


Figure 4.1a

4.2 We want to keep a ship on a given course, Ψ , with an automatic control system using the rudder angle δ . See Figure 4.2a. If ω denotes the angular velocity of the ship,

$$\omega = \dot{\Psi} \quad (4.1)$$

the following differential equation is valid for small values of ω and δ ,

$$T_1 \dot{\omega} = -\omega + K_1 \delta \quad (4.2)$$

where $T_1 = 100$ and $K_1 = 0.1$. The desired course, Ψ_{ref} , and the measured course, Ψ , are fed in to the auto pilot, which gives the signal u to the rudder

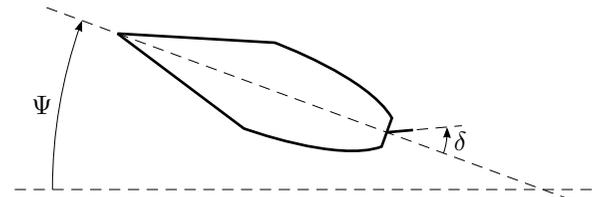


Figure 4.2a

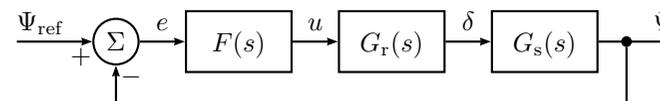


Figure 4.2b

engine. Figure 4.2b shows a block diagram of the auto pilot. The auto pilot has the transfer function

$$F(s) = K \frac{1 + \frac{s}{a}}{1 + \frac{s}{b}}, \quad a = 0.02, b = 0.05$$

while G_r is given by

$$G_r(s) = \frac{1}{1 + sT_2}, \quad T_2 = 10$$

and $G_s(s)$ is defined by (4.1) and (4.2).

- a) Make a Bode plot for the transfer function FG_rG_s , for $K = 0.5$.

- b) At the testing of the auto pilot we do the following experiment. The gain of the auto pilot K is increased until the control system oscillates with constant amplitude. At what value of K does this occur? What is the period time of the oscillation?
- c) Ψ_{ref} is allowed to vary as a sinusoid

$$\Psi_{\text{ref}}(t) = A \sin \alpha t$$

where $A = 5^\circ$ and $\alpha = 0.02$. When the movements of the ship have stabilized we have

$$\Psi(t) = B \sin(\beta t + \varphi)$$

What values do B , β , and φ have if $K = 0.5$?

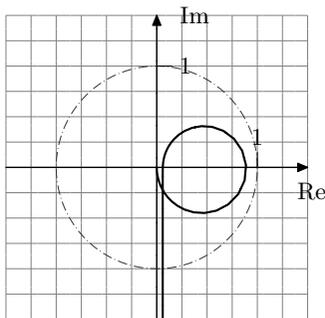


Figure 4.3a

- 4.3 a) In Figure 4.3a the Nyquist curve for a system is shown. Draw the Bode plot for the same system. The scale on the ω -axis is not important, as long as the amplitude and phase curve are in agreement.
- b) Draw a diagram for the poles and zeros for the system. The relative placement is important, not the scale.

- 4.4 Figure 4.4a shows the step responses (when the input is a unit step) and Bode gain plots of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each step response, find the corresponding Bode gain plot (amplitude curve). Motivate your answer by pointing out a set of unique features for each system.

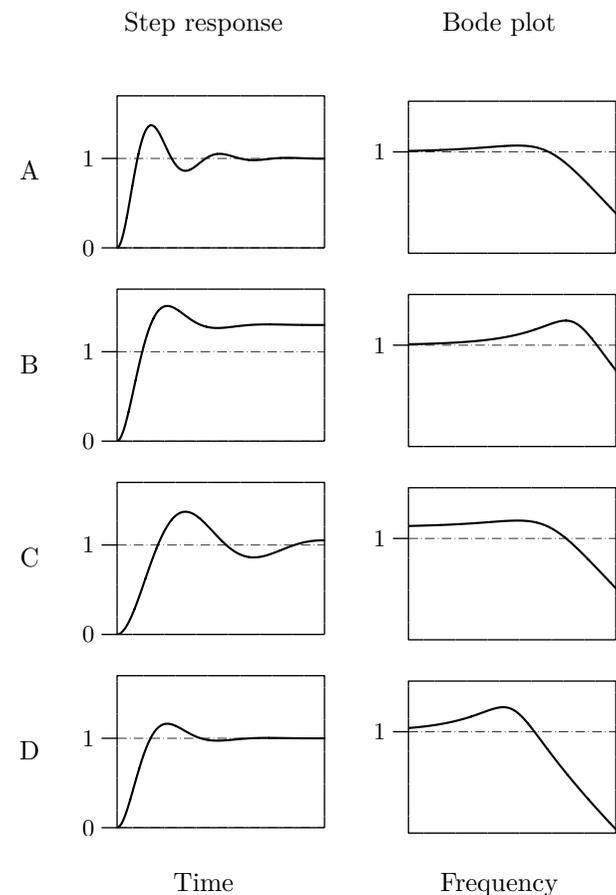


Figure 4.4a. All comparable diagrams have equal scaling.

- 4.5 a) Consider the transfer functions $G_A(s)$, $G_B(s)$, $G_C(s)$, $G_D(s)$, and $G_E(s)$ in Problem 2.6.

$$G_A(s) = \frac{1}{s^2 + 2s + 1}, \quad G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

$$G_C(s) = \frac{1}{s^2 + 5s + 1}, \quad G_D(s) = \frac{1}{s^2 + s + 1}$$

$$G_E(s) = \frac{4}{s^2 + 2s + 4}$$

Study the amplitude curves of the Bode plots for the systems and find the static gain and bandwidth of the systems. In cases when it is relevant find also the resonance frequency and resonance peak.

- b) Describe qualitatively (without formulas) the relationships between T_r (rise time) and ω_B (bandwidth) and between M (overshoot) and M_p (resonance peak) respectively?

- 4.6 A system has the transfer function

$$G(s) = \frac{e^{-2s}}{s(s+1)}$$

What is the output (after transients) when the input is

$$u(t) = 2 \sin(2t - 1/2)$$

- 4.7 For the systems below the input is chosen as $u(t) = \sin(2t)$. Determine the output signal $y(t)$ after transients have faded away, provided that it exists.

- a) $Y(s) = \frac{1}{s+1}U(s)$
 b) $Y(s) = \frac{1}{s-1}U(s)$
 c) $Y(s) = \frac{1}{(s+1)(2s+1)}U(s)$
 d) $Y(s) = \frac{e^{-0.5s}}{s+1}U(s)$

- 4.8 A system is described by $Y(s) = G(s)U(s)$. Figure 4.8a shows $u(t) = \sin(\omega t)$ and the corresponding output $y(t)$ (after all transients have faded away) for the frequencies $\omega = 1, 5, 10,$ and 20 rad/s (from top to bottom).

- a) Determine the gain ($|G(i\omega)|$) and phase ($\arg G(i\omega)$) for the system for each value of ω .
 b) Determine the gain values in dB₂₀ ($20 \log_{10}(|G(i\omega)|)$).
 c) Sketch the Bode plot using the values determined above.

- 4.9 Combine the transfer functions below with the Bode plots in Figure 4.9a.

$$G_1(s) = \frac{1}{s+1}, \quad G_2(s) = \frac{6(s+1)}{(s+2)(s+3)}$$

$$G_3(s) = \frac{1}{s}, \quad G_4(s) = \frac{1}{s(s+1)}$$

$$G_5(s) = \frac{5}{s^2 + 2s + 5} \text{ (poles: } -1 \pm i2)$$

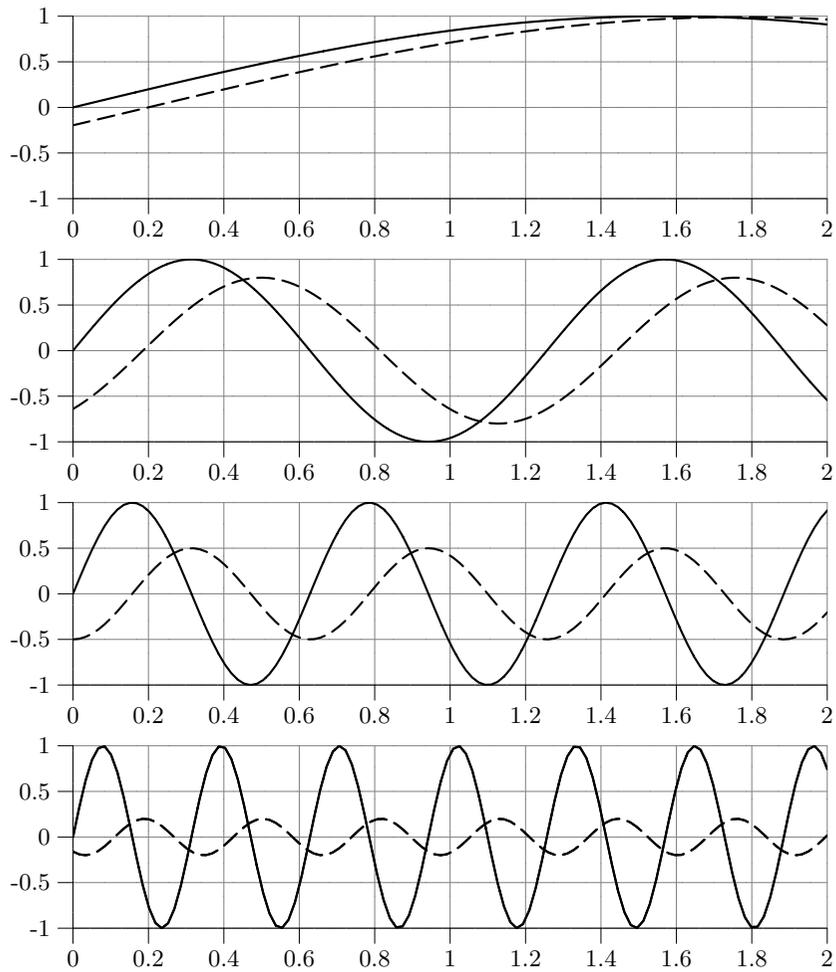


Figure 4.8a. $u(t) = \sin(\omega t)$ (solid) and $y(t)$ (dashed).

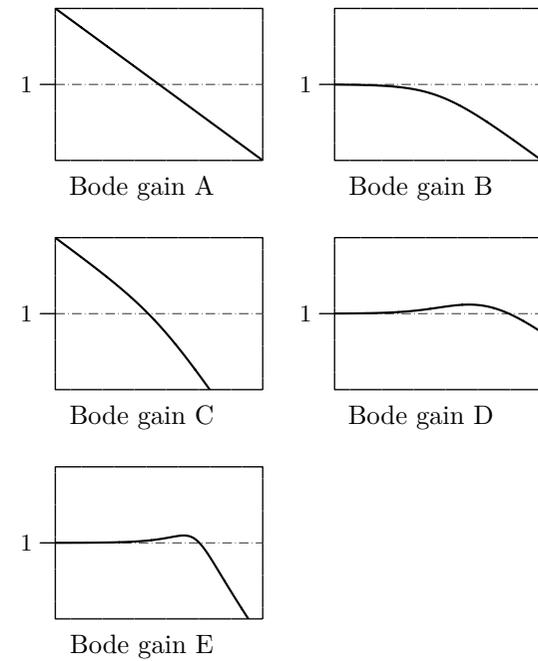


Figure 4.9a. All diagrams have equal scaling.

4.10 Figure 4.10a shows the Bode gain plots and step responses of four different systems, in no particular order. Identify the pair of plots that belongs to each system. That is, for each Bode gain plot (amplitude curve), find the corresponding step response. Motivate your answer by pointing out a set of unique features for each system.

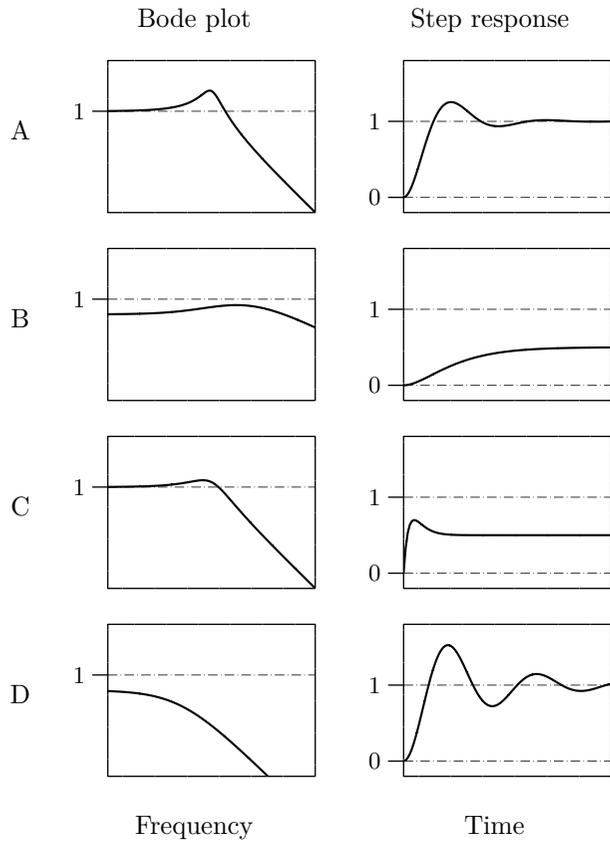


Figure 4.10a. All comparable diagrams have equal scaling.

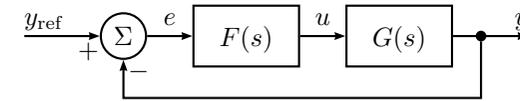


Figure 4.11a

4.11 The pH in a biochemical reactor is controlled by addition of a base. The transfer function $G(s)$ from added base to pH for the open system has been determined by experiments to be

$$G(s) = \frac{1.7}{(s+1)(0.7s+1)(0.5s+1)}$$

In an attempt to control the pH the control structure shown in Figure 4.11a is employed

- a) Make a Bode plot for the transfer function $G(s)$.
- b) Assume that a P controller is used ($F(s) = K$). At what value of K does the pH start to oscillate with constant amplitude?

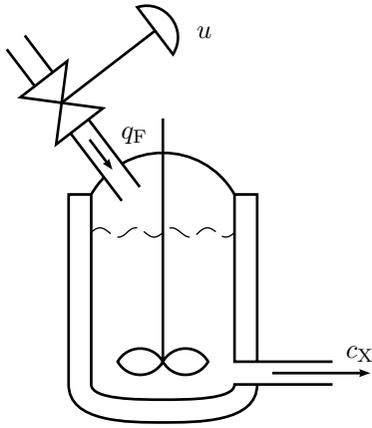


Figure 4.12a

4.12 Consider the biochemical reactor in Figure 4.12a. It is desirable to control the concentration of biochemical material c_X (output y) by manipulating the dilute flow q_F (input u). A model of this system can be described as

$$Y(s) = \frac{2e^{-5s}}{30s + 1}U(s)$$

where the time delay reflects the time it takes to measure the biochemical concentration. The bode digram of the system is shown in Figure 4.12b.

- For which values of K is a P controller going to stabilize the system?
- Construct a controller which has the crossover frequency $\omega_{c,d} = 0.1$ and no steady-state error.

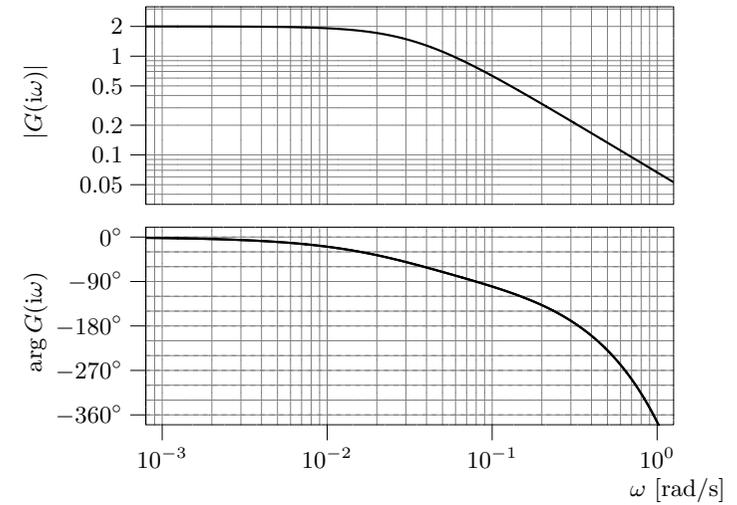


Figure 4.12b

5 Compensation

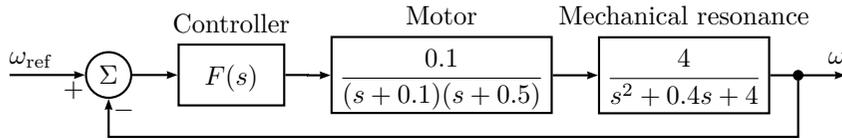


Figure 5.1a

5.1 The block diagram for speed control of a DC-motor is shown in Figure 5.1a. Find a compensator $F(s)$ such that the following specifications are fulfilled.

- The system should be twice as fast as for $F(s) = 1$, but with the same damping as for $F(s) = 1$.
- If ω_{ref} is constant, $|\omega_{\text{ref}} - \omega| / \omega_{\text{ref}}$ should be less than 5%.
- The controller should not be unnecessarily sensitive for high frequency disturbances and the open loop system's low frequency amplification should not be larger than necessary.

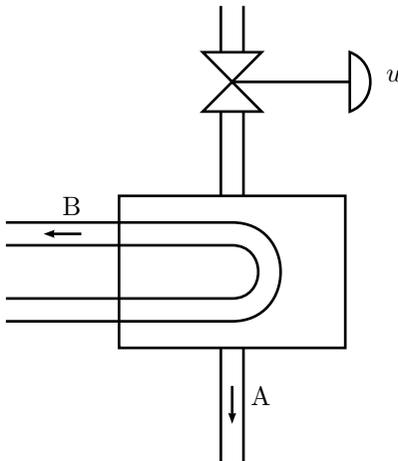


Figure 5.2a

5.2 The outflow temperature θ in the liquid A can be controlled in a heat exchanger by controlling the flow of the liquid B by a valve with the setting denoted u .

See Figure 5.2a. Measurements have been made using a sinusoidal input u and the gain and phase shift have been measured at different frequencies. The results are given in the following table.

Frequency [rad/s]	Gain	Phase shift
0.05	1.37	-67°
0.1	0.80	-106°
0.2	0.34	-153°
0.3	0.18	-185°
0.4	0.11	-210°

- Make a Bode plot for the system.
- What is the largest crossover frequency possible to achieve when using proportional control and wanting a phase margin of at least 50° ?
- Suggest a compensator that doubles the speed compared to b) and still keeps the phase margin.

5.3 A hydraulic system with a valve and a piston is described by the following linearized transfer function

$$G(s) = \frac{k_u/A}{s(\frac{s^2}{\omega_0^2} + 2\zeta\frac{s}{\omega_0} + 1)}$$

where A is the area of the piston and k_u the hydraulic gain.

- Make a Bode plot for the system when $\omega_0 = 150$ rad/s, $\zeta = 0.1$, and $k_u/A = 20$.
- What is the smallest value of the ramp error that can be achieved using proportional control if we want an amplitude margin of 2? What is the crossover frequency in this case?
- Suggest a compensator such that the ramp error decreases 15 times at the same time as the crossover frequency, phase margin and amplitude margin will be the same as in b). Due to physical constraints in the implementation, this regulator has to have finite amplification at all frequencies.

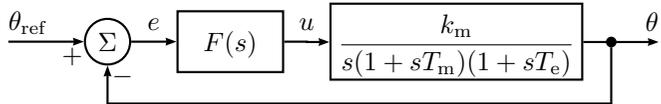


Figure 5.4a

5.4 Figure 5.4a shows a position servo including a DC-motor. The extra time constant T_e is due to the inductance in the winding of the motor, which is usually not taken into account. The parameter values are $k_m = 10$, $T_m = 0.1$ and $T_e = 0.01$. We want the servo to fulfill the following specifications:

- Rise time ≤ 0.1 s.
- Overshoot $\leq 10\%$.
- The steady state error at step in θ_{ref} should be zero.
- The steady state error when θ_{ref} is a ramp with slope $10^\circ/s$ should be less than 0.1° .

Suggest a compensator such that the specifications are fulfilled. (Clue: Suppose that the relationship between rise time, overshoot, and other specifications are the same as for a second order system, that is, according to Figures 5.4b and 5.4c in Solution 5.4 (the figures can also be found in Glad&Ljung).)

5.5 In Figure 5.5a we have arranged step responses and open loop and closed loop Bode plots for five different systems. Identify the three plots that belong to each of the five systems, one open loop and one closed loop Bode plot and one step response. Motivate your answer by pointing out one unique feature for each system.

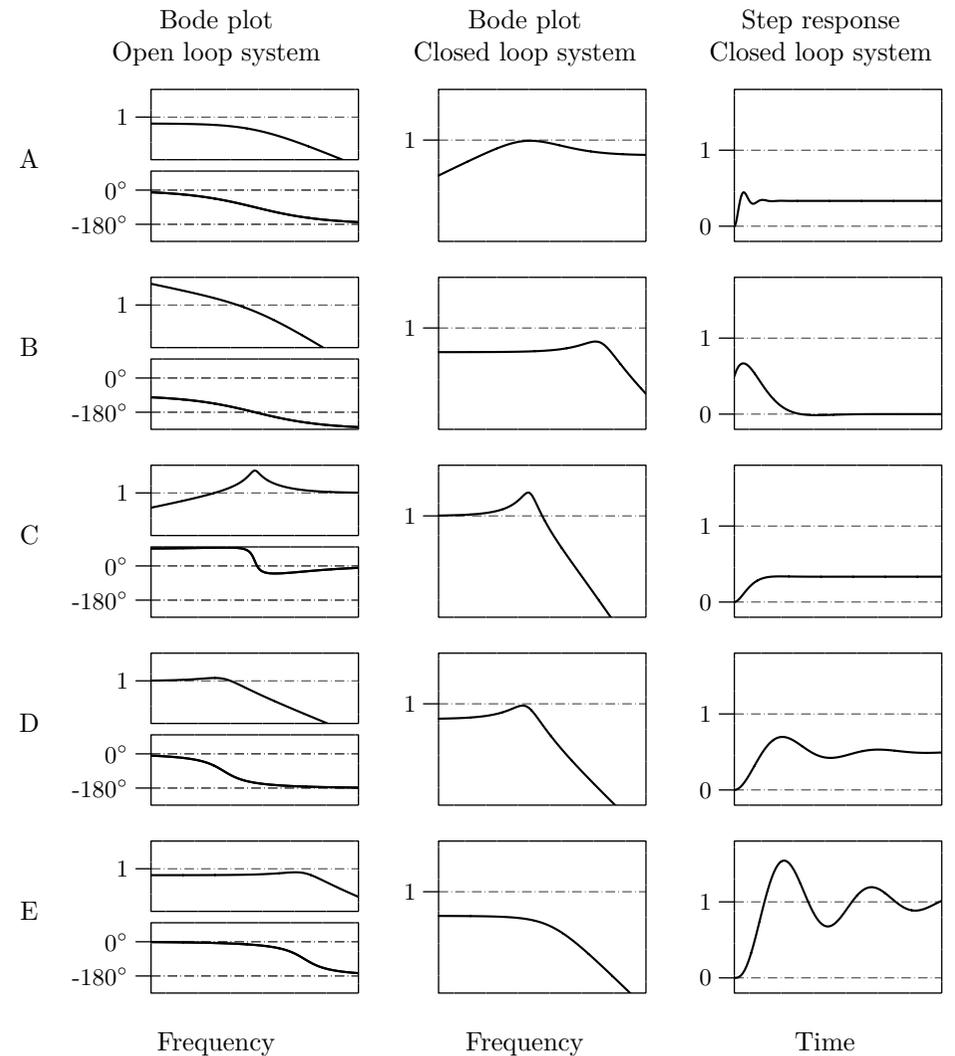


Figure 5.5a. All comparable diagrams have equal scaling.

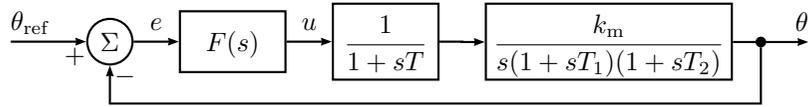


Figure 5.6a

5.6 A DC-servo is described by the block diagram in Figure 5.6a, where $T_1 = 50$ ms is a mechanical time constant, $k_m = 10$ is a proportional constant, $T_2 = 25$ ms is an electrical time constant, and $T = 10$ ms is an amplifier time constant. The system is tested with $F(s) = 1$ and we find that the dynamic properties are satisfactory but that the system is somewhat too slow. Find an $F(s)$ so that the closed loop system is twice as fast as for $F(s) = 1$, without increasing the overshoot. $F(s)$ should also give a closed loop system which fulfills the following accuracy demands:

- $|\theta - \theta_{\text{ref}}| \leq 0.001$ rad in steady state when θ_{ref} is constant.
- When θ_{ref} is a ramp with slope 10 rad/s we should have $|\theta - \theta_{\text{ref}}| \leq 0.01$ rad in steady state.

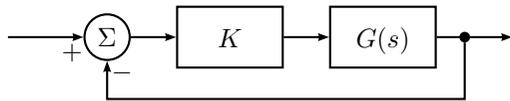


Figure 5.7a

5.7 The amplitude curves and the phase curves in Figure 5.7b have been measured for a system without poles in the right half plane. The system is controlled using feedback according to Figure 5.7a. Use the Nyquist criterion to decide for which values of K the closed loop system is stable ($K > 0$).

5.8 A block diagram for a control system with time delay is shown in Figure 5.8a. The system G_1 has no poles in the right half plane.

- G_1 has a Bode plot according to the plot in Figure 5.8b. Determine for what values of the time delay T the closed loop system is stable.
- The same as in a), but for the plot in Figure 5.8c.

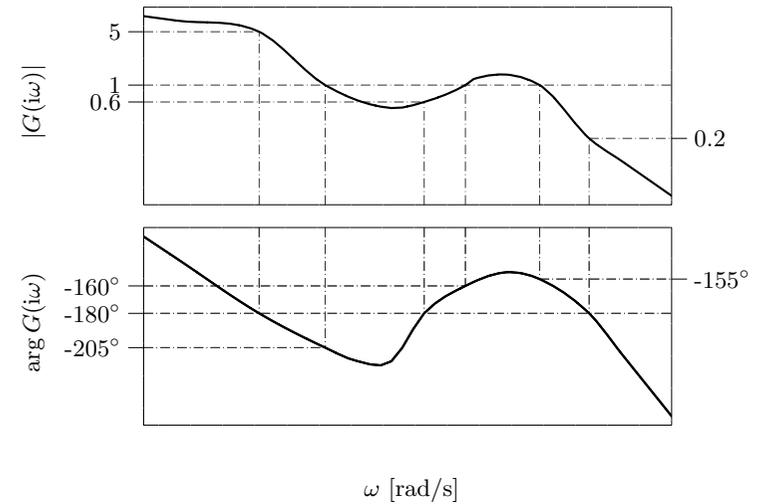


Figure 5.7b

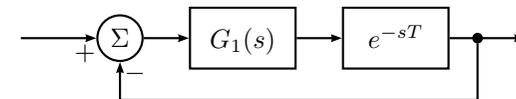


Figure 5.8a

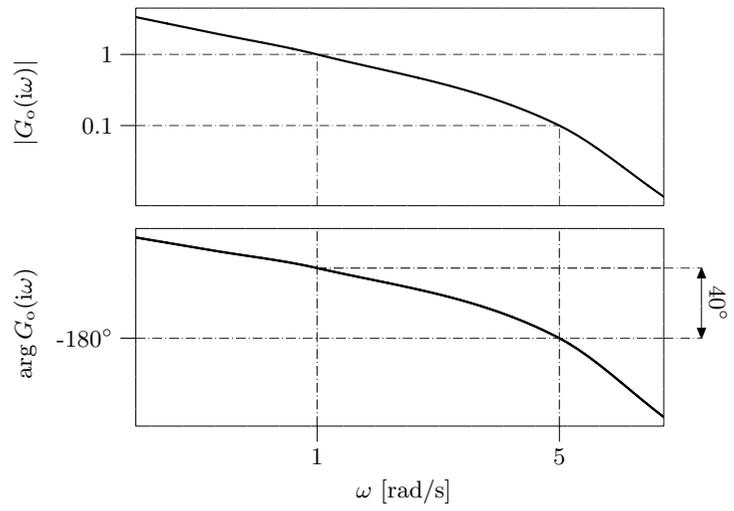


Figure 5.8b

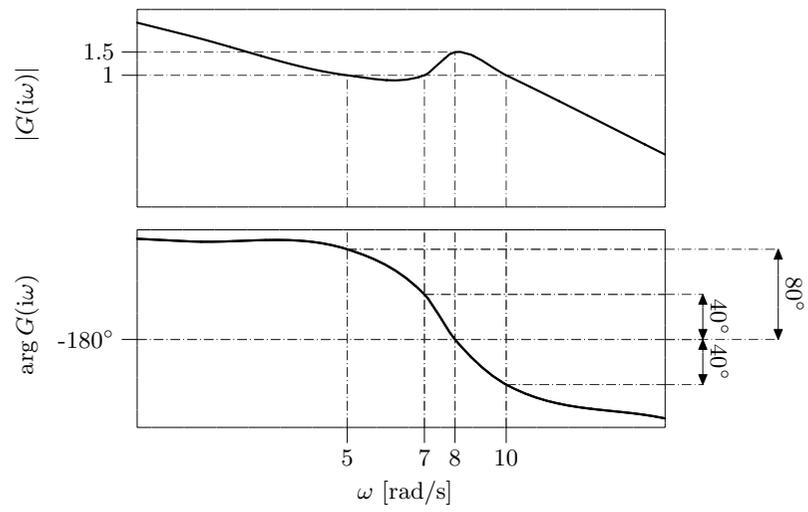


Figure 5.8c

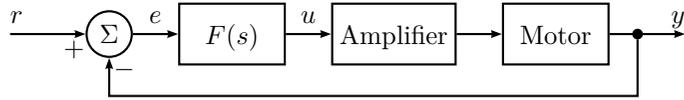


Figure 5.9a

5.9 A servo system based on a DC-motor has to be designed. A block diagram for the system is given in Figure 5.9a. By measuring the phase shift and the gain at different frequencies the Bode plot for the motor, see Figure 5.9b, has been determined. The amplification is a system of the first order, that is, it has the transfer function

$$G_A(s) = \frac{k_A}{s + a}.$$

In order to find the constants k_A and a , a unit step experiment has been carried out on the amplifier, giving the output shown in Figure 5.9c.

- a) Find the constants k_A and a from Figure 5.9c. Also draw the Bode plot for the open loop system, that is, the system from u to y .
- b) Find a compensator $F(s)$, such that the closed loop system fulfills the following demands:
 - ◇ The system has to be 5 times as fast as when using $F(s) = 1$.
 - ◇ The overshoot should not be larger than for $F(s) = 1$.

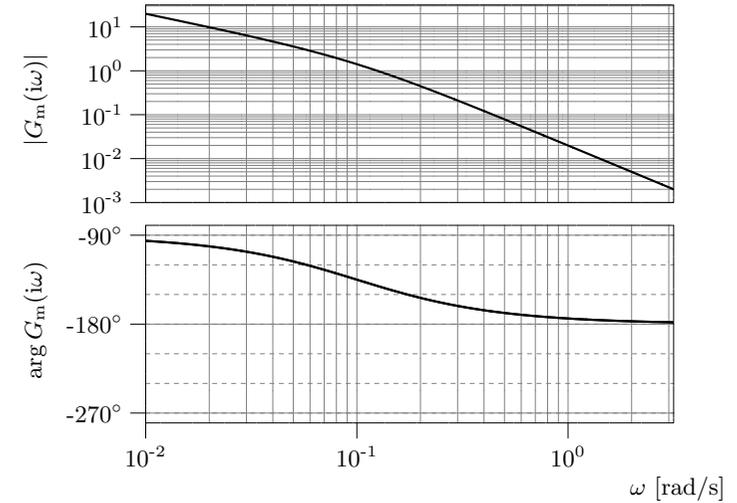


Figure 5.9b

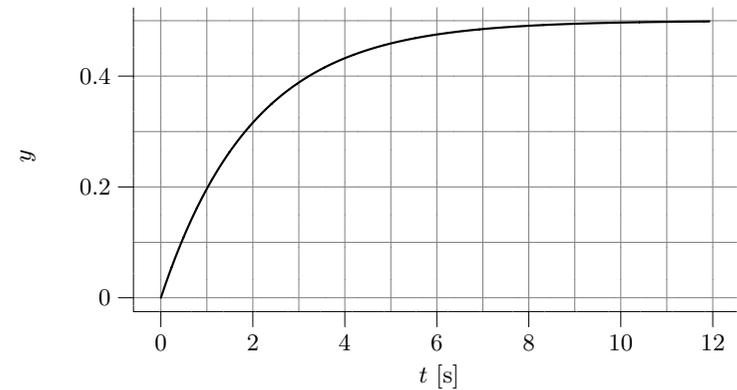


Figure 5.9c

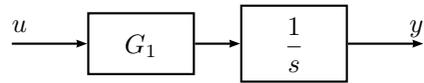


Figure 5.10a

5.10 A system $G(s)$ can be split into two sub-systems

$$G(s) = G_1(s) \frac{1}{s}$$

according to Figure 5.10a. The Bode plot for $G_1(s)$ is given in Figure 5.10b. Find a compensator for the system $G(s)$ such that the following is fulfilled:

- The phase margin for the compensated system is 40° .
- The closed loop system is twice as fast as what is possible to achieve using proportional control with a 40° phase margin.
- The steady state error when the reference signal is a ramp is 1% of the corresponding error with proportional control and 40° phase margin.

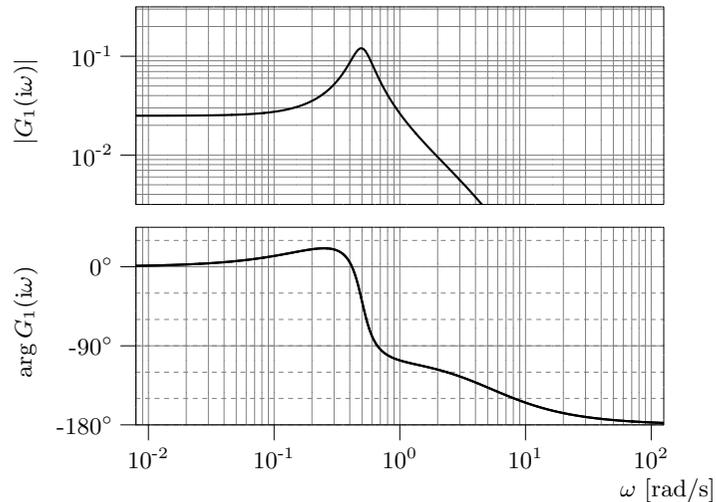


Figure 5.10b

5.11 The Bode plot for a system is given in Figure 5.11a.

- Draw the Nyquist curve of the system.
- Assume that the system is controlled using the proportional feedback

$$U(s) = K(R(s) - Y(s))$$

For which $K > 0$ is the closed loop system asymptotically stable?

- Assume that we choose $K = 2$ in the proportional controller in problem b). What will the steady state error be when $r(t) = 10t$?
- Assume that $y(t)$ is delayed T seconds. How large is T allowed to be in order for the system to still be asymptotically stable with $K = 2$?

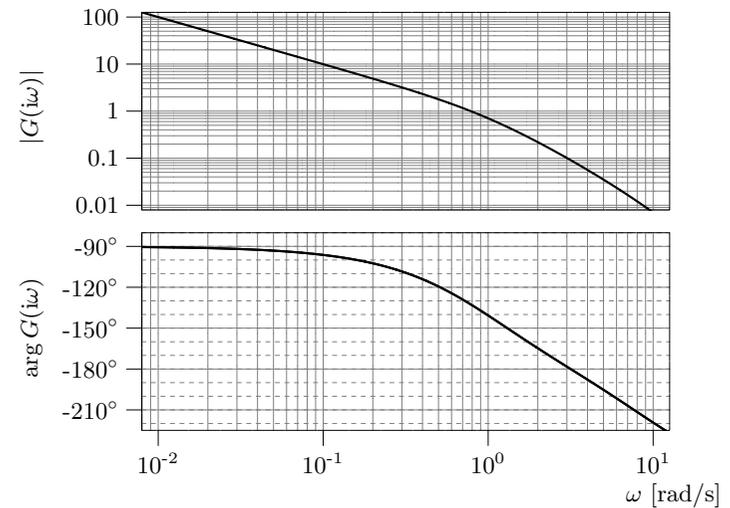


Figure 5.11a

5.12 a) A plot of the amplitude curve of a stable transfer function $G_o(s)$ is given in Figure 5.12a. Choose one of the following alternatives regarding the stability of the closed loop system $\frac{G_o}{1 + G_o}$:

1. It is stable.
2. It is not stable.
3. Impossible to determine given these facts only.

b) Repeat for the transfer function whose amplitude curve is given in Figure 5.12b. Justify your answers carefully.

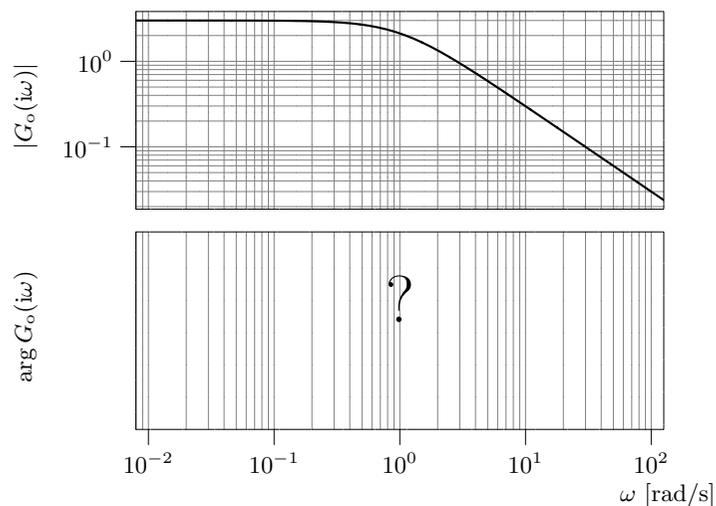


Figure 5.12a

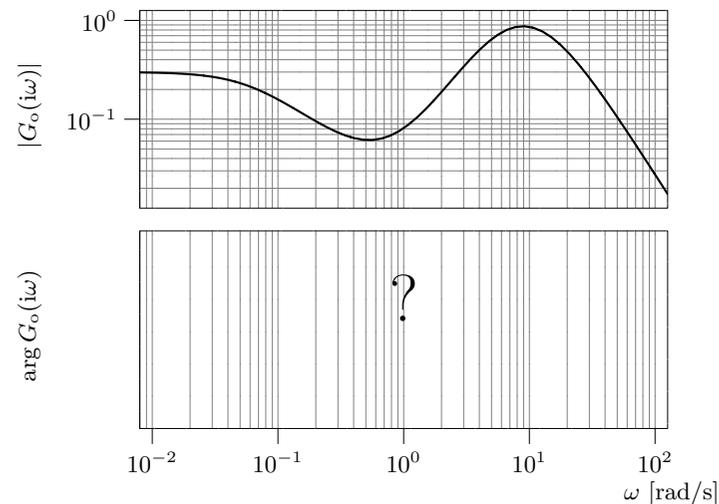


Figure 5.12b

5.13 Consider the relation

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

a) Assume that the system is controlled by

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 1$. Find ω_c , ω_p , φ_m , and A_m for the loop gain.

b) Compute a regulator such that the open loop system fulfills the following requirements:

- (i) $\omega_c = 5$
- (ii) $\varphi_m \geq 60^\circ$

and the closed loop system fulfills:

- (iii) $e_0 = 0$

Draw the Bode plot of the compensated open loop system and check that the requirements are satisfied. Simulate the closed loop system for a step in the reference signal and plot the step response. Check that the requirement on the steady state error is satisfied.

c) Draw the amplitude curve of the Bode plot of the closed loop system with and without the compensator. Describe how the properties of the closed loop system have been changed by the compensation.

d) Simulate the control error when the reference signal is a ramp and the regulator designed in b) is used. Is the stationary error zero?

5.14 When using microorganisms in production it is important to keep the oxygen concentration at a certain level to get maximum productivity. There are many ways to control the amount of dissolved oxygen, in this example we will use the speed of stirring as the controlled signal. The transfer function from the stirrer speed N_Δ to the oxygen measurement $O_{p\Delta}$ becomes (linearized model)

$$G(s) = \frac{b}{s + T_1} \frac{e^{-s\tau}}{1 + sT_2}$$

The parameters $\tau = 2$ s, $T_2 = 20$ s and $b = 0.02$ remain constant with change in stirrer speed while T_1 can vary from 0.02 s⁻¹ to 0.224 s⁻¹ as the stirrer speed increases from 400 r/min to 1200 r/min. A Bode plot for $G(s)$ is given in Figure 5.14a. Construct a controller, for the 1200 r/min case, which has a crossover frequency $\omega_c = 0.2$ rad/s, a phase margin $\varphi_m = 60^\circ$ and no steady-state error.

5.15 Earlier in the history of the basic automatic control courses at LiTH, the lead and lag compensators were parameterized in a different way. This parameterization is used in this problem, which is not only meant to be a crash course in understanding exams with solutions dating back to those days, but should be

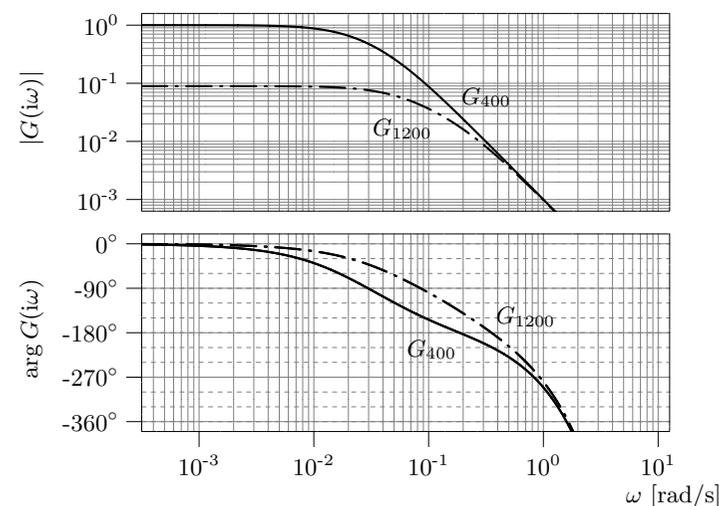


Figure 5.14a. Solid line: 400 r/min. Dash-dotted line: 1200 r/min.

seen as a chance of practicing the *ideas* of loop shaping rather than exercising *recipe* knowledge.

The old parameterization of a lead compensator looked as follows:

$$F_{\text{lead}}(s) = N \frac{s + b}{s + bN}$$

The schematic Bode plot of such a lead compensator is shown in Figure 5.15a, and it can be shown that the maximum phase lead is obtained at the frequency $b\sqrt{N}$, where the gain is \sqrt{N} , and the phase lead is given by

$$\varphi_{\text{max}} = \arctan \frac{\sqrt{N} - 1/\sqrt{N}}{2}$$

(shown in Figure 5.15b).

The old parameterization of a lag compensator looked as follows:

$$F_{\text{lag}}(s) = \frac{s + a}{s + a/M}$$

The schematic Bode plot of such a lag compensator is shown in Figure 5.15c. Just like the parameter τ_1 of the new parameterization, the parameter a used

here shall be determined as a trade-off between the undesirable phase lag and the bandwidth of the desirable low frequency gain. As a rule of thumb, one may use $a = 0.1 \omega_{c,d}$ as an initial guess, and iteratively improve from there. The worst phase lag for this choice of a is about 5.7° .

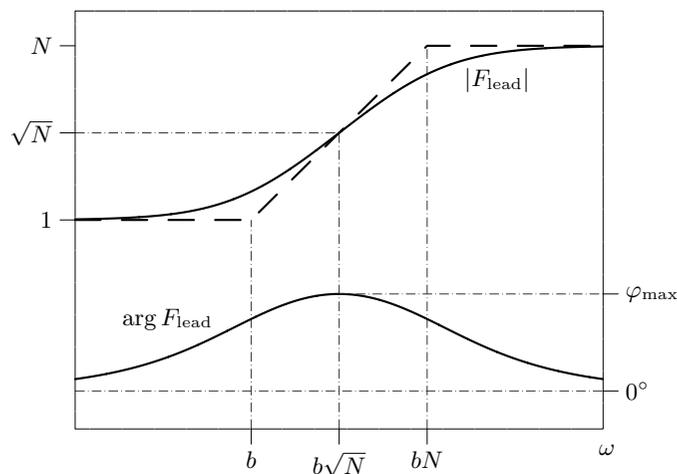


Figure 5.15a. Schematic Bode plot for the old parameterization of a lead compensator.

A read/write head of a hard disk is mounted on a mechanical arm which is moved by a motor. The system from motor input voltage to the angle of the arm is modelled by

$$Y(s) = \frac{5}{\tau_1 s + 1} \cdot \frac{0.05}{s(\tau_2 s + 1)} U(s)$$

where Y and U are the Laplace transforms of the output and input respectively. The numerical values of the constants are $\tau_1 = 10^{-3}$ och $\tau_2 = 0.05$. The Bode plot of the system is given in Figure 5.15d.

- a) To begin with, assume that the arm is controlled using proportional feedback,

$$U(s) = K(R(s) - Y(s))$$

What are the step and ramp error coefficients (often referred to as e_0 and e_1)? For what values of K are they defined?

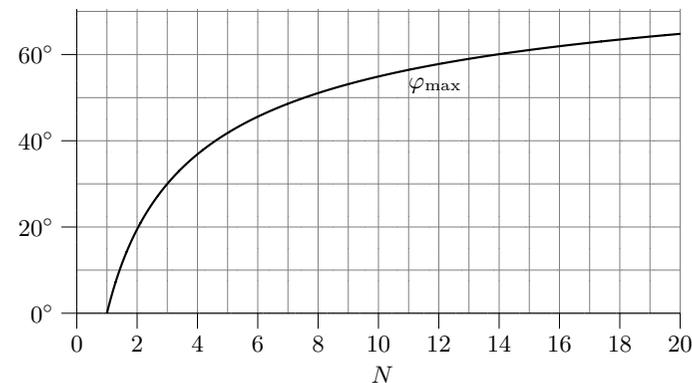


Figure 5.15b. Maximum phase advance as a function of N in a lead compensator parameterized the old way.

- b) Compute a controller,

$$U(s) = F(s)(R(s) - Y(s))$$

for the same system, such that the resulting system fulfills the following requirements:

- ◇ $e_0 = 0$
- ◇ $e_1 \leq 0.001$
- ◇ $\omega_c = 100$ rad/s
- ◇ $\varphi_m \geq 50^\circ$

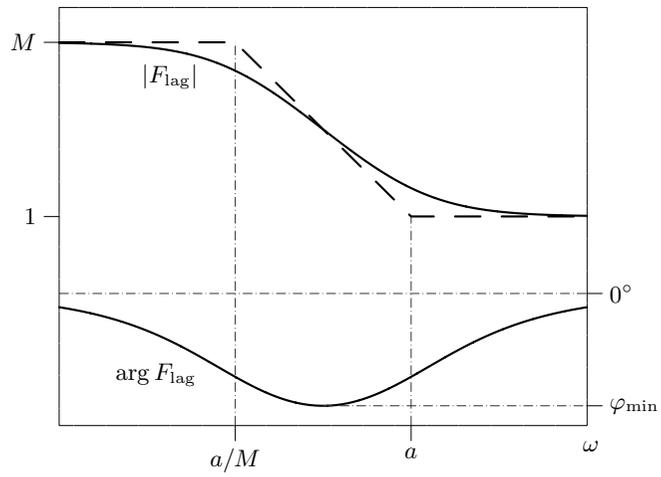


Figure 5.15c. Schematic Bode plot for the old parameterization of a lag compensator.

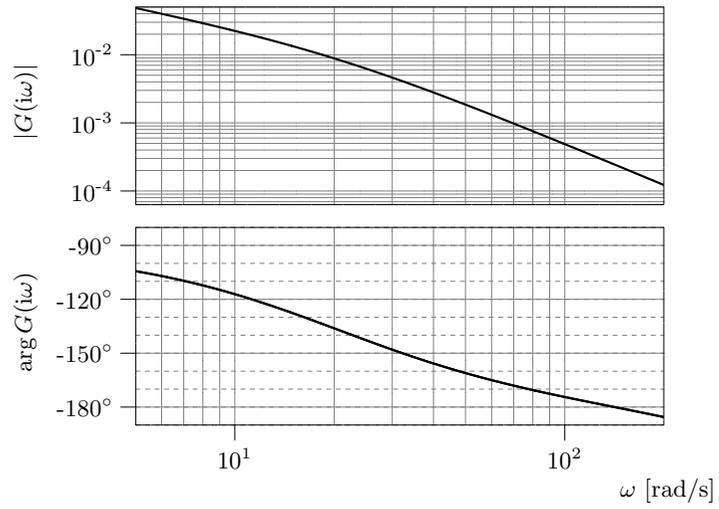


Figure 5.15d

6 Sensitivity and Robustness

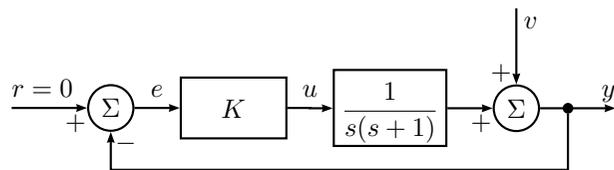


Figure 6.1a

- 6.1 Consider the control system in Figure 6.1a where $v(t)$ is a sinusoidal disturbance, $v(t) = \sin(t)$. Compute the absolute value of the sensitivity function at $\omega = 1$ rad/s as a function of K . How must K be selected if the amplitude of $y(t)$ shall be less than the amplitude of $v(t)$ at this frequency?

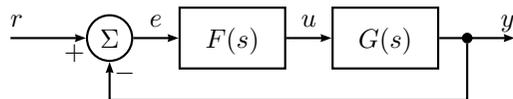


Figure 6.2a

- 6.2 Assume that we have constructed a controller $F(s)$ for the model $G(s)$, see Figure 6.2a, such that there is no steady state error when the reference signal is a step. Let the real system be given by

$$G^0(s) = (s + 1)G(s)$$

and assume that $G^0(s) \rightarrow 0$, $s \rightarrow \infty$. Also assume that the amplitude curve of the closed loop system has no resonance peaks and decreases, at least and asymptotically, with 20 dB₂₀/decade for frequencies over the bandwidth. What is the highest possible bandwidth we can use for the closed loop system in Figure 6.2a, while at the same time guaranteeing stability?

- 6.3 Figure 6.3a shows a Nyquist diagram for the loop gain G_o . Show in a figure for what frequencies (that is, for what part of the Nyquist curve above) additive disturbances on the output are amplified in the sense that the output amplitude of the control system in Figure 6.3b is larger than the disturbance amplitude.

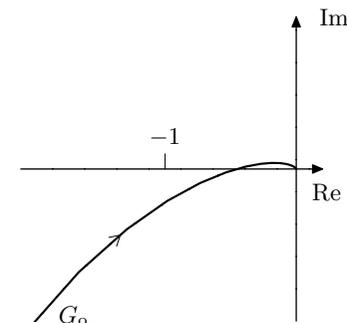


Figure 6.3a

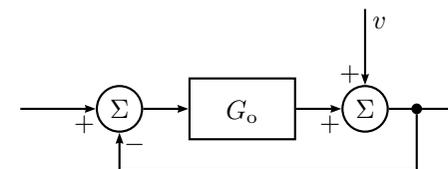


Figure 6.3b

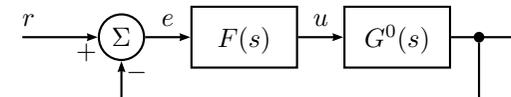


Figure 6.4a

- 6.4 Consider the control system in Figure 6.4a. The true system, denoted $G^0(s)$, is modeled as

$$G(s) = \frac{1}{s + 10}$$

The controller

$$F(s) = \frac{s + 10}{s}$$

gives an asymptotically stable closed loop system with the model $G(s)$. Now

assume that the system is given by

$$G^0(s) = G(s)(1 + G_\Delta(s))$$

where it is known that $G_\Delta(s)$ has no poles in the right half plane, and that

$$|G_\Delta(i\omega)| < \frac{0.9}{\sqrt{1 + \omega^2}}$$

Can we be sure that the closed loop system is asymptotically stable?

6.5 A process is described by the model $G(s)$, while the process in reality has the transfer function

$$G^0(s) = e^{-sT} G(s)$$

a) Draw the absolute value of the inverse of the relative model error, that is,

$$\frac{1}{|G_\Delta(i\omega)|}$$

b) Assume that we design a controller $F(s)$ starting with the model $G(s)$. How large may

$$\left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

be at most, in order to guarantee asymptotic stability of the closed loop system for all values of T , when the controller $F(s)$ is used on the system $G^0(s)$?

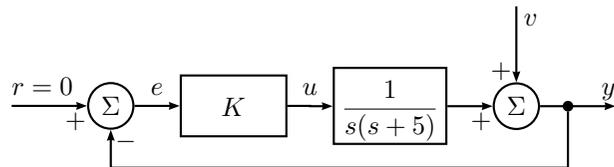


Figure 6.6a

6.6 Consider the control system in Figure 6.6a.

a) Assume that the real system is given by

$$G^0(s) = G(s) + \tilde{G}(s),$$

where

$$G(s) = \frac{1}{s(s+5)}$$

and let $K = 25/2$. Use the robustness criterion to obtain a condition on $|\tilde{G}(i\omega)|$ that guarantees stability in the closed loop system. Does $\tilde{G}(s) = 1$ fulfill the conditions?

b) Now let $\tilde{G}(s) = \alpha$ where α is a scalar. Calculate the characteristic equation for the closed loop system and decide for which α the system is stable. Does this contradict the condition from the robustness criterion?

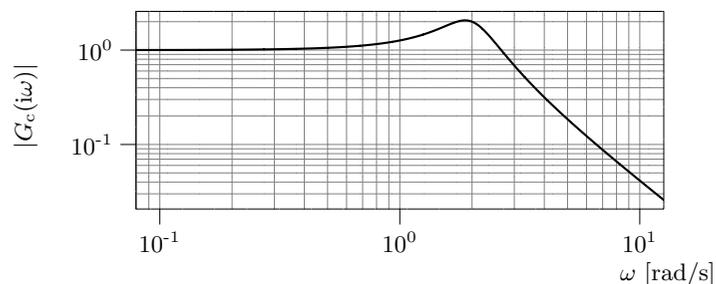


Figure 6.7a

6.7 A DC-motor is assumed to have the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

and it is controlled using proportional feedback,

$$U(s) = F(s)(R(s) - Y(s))$$

where $F(s) = 4$. The amplitude curve of the feedback system

$$|G_c(i\omega)| = \left| \frac{F(i\omega)G(i\omega)}{1 + F(i\omega)G(i\omega)} \right|$$

is given in Figure 6.7a. Assume that the real system is given by

$$G^0(s) = G(s) \frac{\alpha}{s + \alpha}, \quad \alpha > 0$$

and the controller $F(s)$ is used on the system $G^0(s)$.

- Draw a root locus with respect to α for the characteristic equation of the closed loop system and determine for which α the closed loop system is asymptotically stable.
- Use the robustness criterion to decide for which α the closed loop system is asymptotically stable.
- Comment on the possible differences in the demands on α in a) and b).

6.8 A system $G^0(s)$ is controlled using a regulator $F(s)$. In Figure 6.8a the amplitude part the Bode plot of the nominal closed loop system,

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

is shown. It is known that G_c is stable, and it is assumed that G and G^0 have the same number of poles in the right half plane. The model uncertainty $G_\Delta(s)$, defined by

$$G_\Delta = \frac{G^0 - G}{G}$$

is assumed bounded by $|G_\Delta(i\omega)| \leq \gamma\omega$. In what interval must γ lie to guarantee stability of the closed loop system?

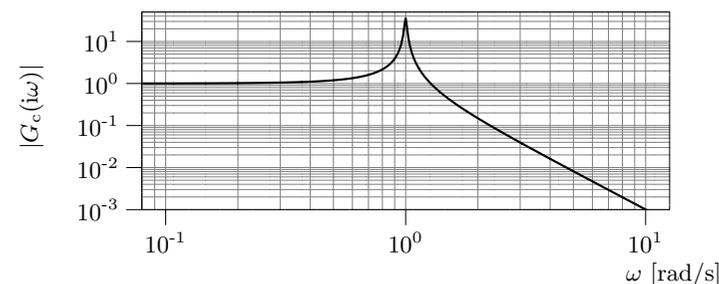


Figure 6.8a

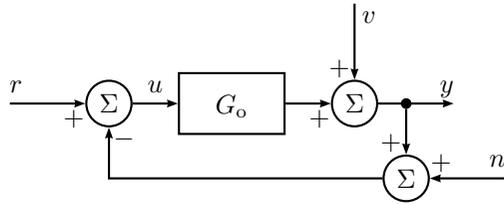


Figure 6.9a

6.9 Consider the system in Figure 6.9a. For $r(t) = 0$, $n(t) = 0$ and $v(t) = \sin t$ the steady-state output is given by

$$y(t) = \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)$$

Determine the steady-state output $y(t)$ when $r(t) = 0$, $v(t) = 0$ and $n(t) = \sin t$.

6.10 Recall the model that was used in the design of a lead-lag controller using MATLAB in Problem 5.13. Assume that the true system contains a time constant that was neglected, and that the transfer function of the system is given by

$$G^0(s) = G(s) \frac{1}{s+1}$$

- Determine the relative model error $G_\Delta(s)$.
- Draw $\frac{1}{|G_\Delta(i\omega)|}$ and $\left| \frac{F(i\omega)G(i\omega)}{1+F(i\omega)G(i\omega)} \right|$ in a Bode plot, when $G(s)$ is given by

$$G(s) = \frac{725}{(s+1)(s+2.5)(s+25)}$$

for the two cases $F(s) = 1$ and $F(s)$ being the controller designed in Problem 5.13. What can be said about the robustness of the closed loop system in these two cases when $F(s)$ is used for control of the “true” system $G^0(s)$?

One possible solution to the design problem in Problem 5.13 was the controller

$$F(s) = 0.46 \cdot \frac{0.43s+1}{0.090s+1} \cdot \frac{2.0s+1}{2.0s}$$

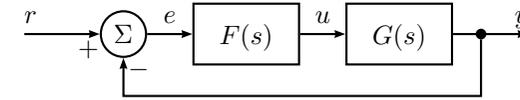


Figure 6.11a

6.11 Consider the control system in Figure 6.11a. The controller $F(s) = 2$ gives the Bode plot of the loop gain $F(i\omega)G(i\omega)$ shown in Figure 6.11b. The Bode plot of the sensitivity function is shown in Figure 6.11c. The reference signal is $r(t) = 2 \sin 0.1t$. Determine the amplitude of the error in steady state.

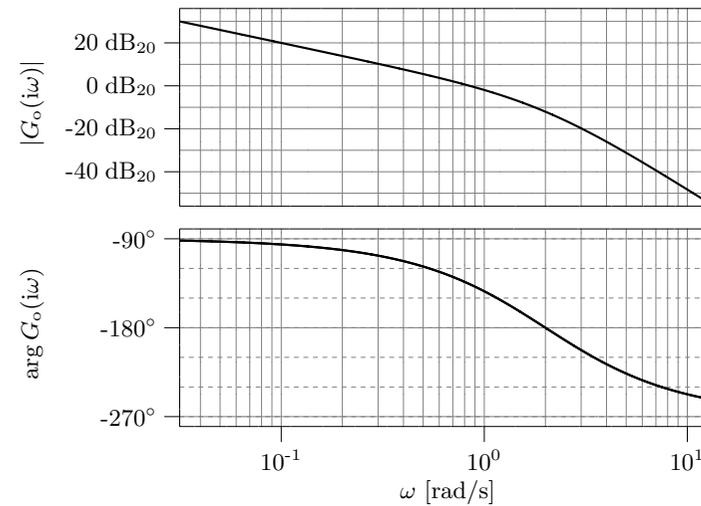


Figure 6.11b

6.12 In Problem 5.14 we saw how the amount of dissolved oxygen depends on the stirring speed. A lead-lag controller was designed for the model linearized around 1200 r/min. Check if this controller also stabilizes the system when the stirring speed is 400 r/min.

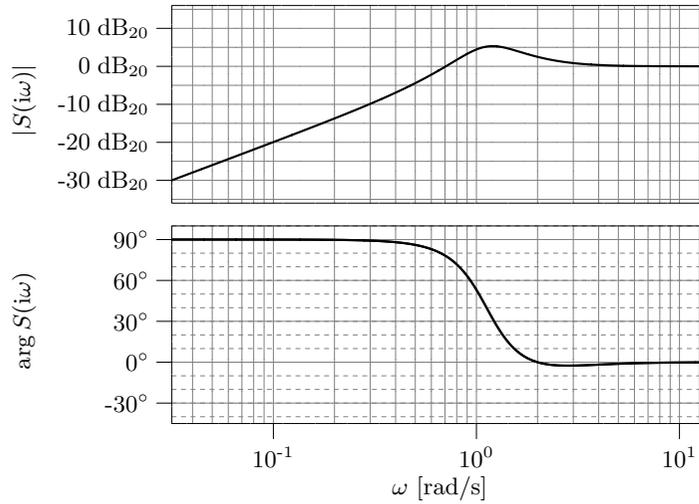


Figure 6.11c

6.13 Consider the connected tank system in Figure 6.13a where u is the inflow to the upper tank and y is the level in the lower tank. The system can approximately be described by the following transfer function

$$G(s) = \frac{1}{(s + 1)^2}$$

The level in the lower tank is controlled by a P controller;

$$U(s) = K(R(s) - Y(s))$$

The goal of the control is to minimize the influence of the disturbance v . This has been formalized as the following demand on the system:

$$|S(i\omega)| \leq 0.1$$

at $\omega = 1$, where $S(s)$ denotes the sensitivity function. How must K be selected if the demand should be satisfied?

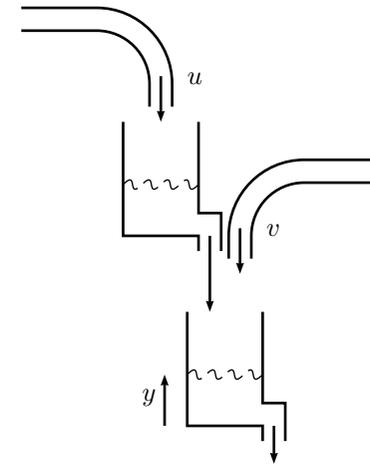


Figure 6.13a

7 Special Controller Structures

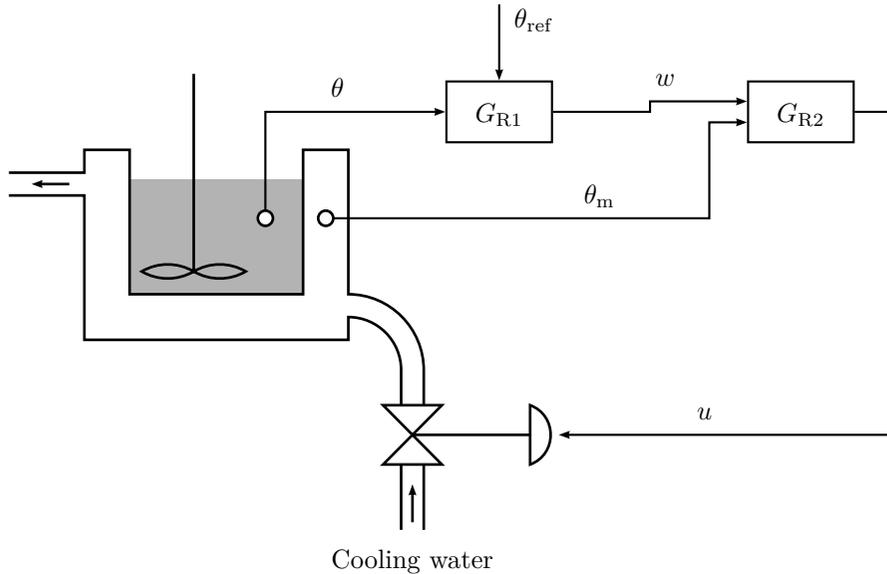


Figure 7.1a

7.1 To control the temperature θ in a chemical reactor, the control system in Figure 7.1a is used, where θ_{ref} is the desired (reference) temperature. The temperatures θ and θ_m in the reactor and the cooler, respectively, are measurable and can be used to control the valve u . The structure of the control system is given by Figure 7.1b. It is here assumed that both G_{R1} and G_{R2} are P controllers.

a) Let $K_2 = 9$ and draw the Bode plot of the transfer function from w to θ . Then choose K_1 so that the gain margin $A_m = 2$. What are the gain crossover frequency ω_c and the steady-state error, if we assume that θ_{ref} is changed stepwise?

b) Suppose that we make a simple feedback loop instead, see Figure 7.1c. How is the Bode plot affected? Again, choose K_1 so that the gain margin $A_m = 2$, and determine the gain crossover frequency ω_c and the steady-state error. Compare with a) with respect to steady-state errors and response times. Conclusions?

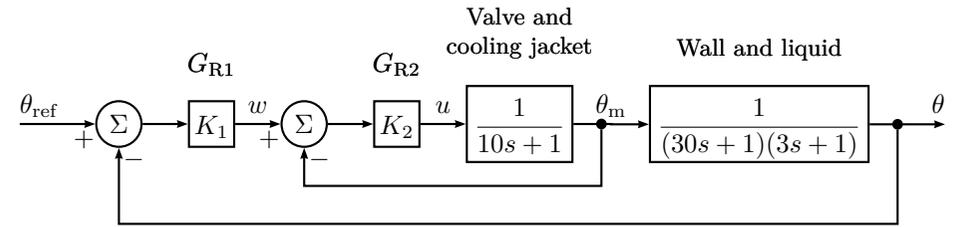


Figure 7.1b

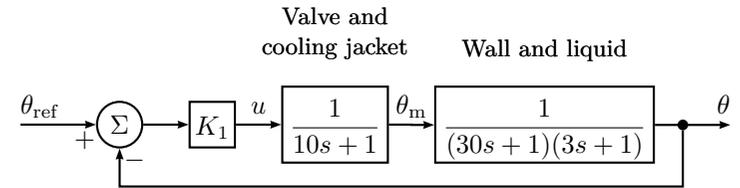


Figure 7.1c

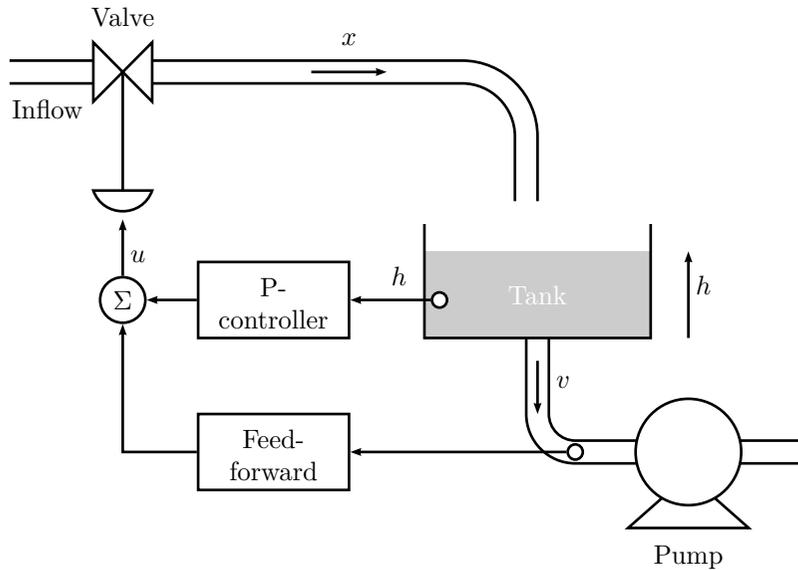


Figure 7.2a

7.2 A level control system for a water tank is shown in Figure 7.2a, where all variables denote offsets from an operation point. The inflow, $x(t)$, to the tank is determined by the valve, and the outflow, $v(t)$, is determined by the pump. Stu Dent has got the assignment to keep the water level in the tank constant, in spite of variations in the outflow $v(t)$. First, Stu determines the transfer function $G_v(s)$ from the valve to $x(t)$. By step response experiments, he obtains the following result:

$$G_v(s) = \frac{1}{0.5s + 1}$$

- a) Because the disturbance $v(t)$ is measurable, Stu first considers a feedforward compensator to completely eliminate it. Stu, who also knows that it is dangerous to differentiate the disturbance, cancels all the derivative terms in the compensator. Compute the feedforward compensator, and determine the response $h(t)$ Stu will get, if the outflow $v(t)$ is changed stepwise with an amplitude of 0.1.

- b) To improve the control system, Stu also introduces a proportional feedback of the water level h . What is the steady-state error in the level h now, if the outflow is changed in the same way as in a)?

7.3 Consider the following system

$$Y(s) = \frac{2}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where u is the control signal, y is the output and v is a disturbance. It is desired that y should be as small as possible despite the disturbance v .

- a) Design a feedforward controller from v to u that eliminates the influence of v on y .
- b) Assume that v is a pure sinusoid with amplitude 2. How large will the control signal be?
- c) The real system is described by

$$Y(s) = \frac{b}{s+3}U(s) + \frac{3}{s+4}V(s)$$

where b value is not exactly known but has its value close to 2. To solve this problem a P controller is added to the feedforward controller that was designed in a). The full controller looks like

$$U(s) = -KY(s) + F_f(s)V(s)$$

where $F_f(s)$ is the feedforward controller. What is the stationary error if $v = 1$?

7.4 The transfer function for a temperature control system is given by

$$Y(s) = \frac{3}{s+1}U(s) + \frac{4}{(s+2)(s+5)}V(s)$$

where y is the controlled temperature, u is the supplied power and v is the temperature of the surroundings. Assume that the desired temperature is zero.

- a) Design a feedforward controller $U(s) = F_f(s)V(s)$ which eliminates the influence of the disturbance v on y .
- b) To simplify implementation $F_f(s)$ is replaced with a constant, $\tilde{F}_f = F_f(0)$. Assume that v is given by $v(t) = -1 - 0.1t$ and that $U(s) = \tilde{F}_f V(s)$ is used. What will $y(t)$ be in steady state?
- c) The previous controller is now extended with a P controller:

$$U(s) = \tilde{F}_f V(s) - KY(s)$$

What will now $y(t)$ be in steady state?

- d) Assume that one only uses the P controller

$$U(s) = -KY(s)$$

What will now $y(t)$ be in steady state?

8 State Space Description

8.1 Define suitable state space variables for the DC motor discussed in Problem 2.1, and write the system in state space form.

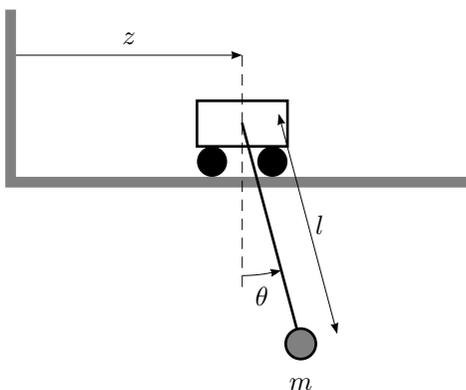


Figure 8.2a

8.2 Consider the system illustrated in Figure 8.2a. It consists of a hinge that can move in the direction marked “z”, and a thereto attached pendulum. The system is described by the equation

$$l\ddot{\theta} + g \sin \theta + \ddot{z} \cos \theta = 0$$

Define state space variables, input, and output as

$$x_1 = \theta \quad x_2 = \dot{\theta} \quad u = \ddot{z}/\ell \quad y = \theta$$

and

$$\omega_0^2 = g/\ell$$

Linearize the system around the equilibrium point given by

$$x_1 = \pi \quad x_2 = 0 \quad u = 0$$

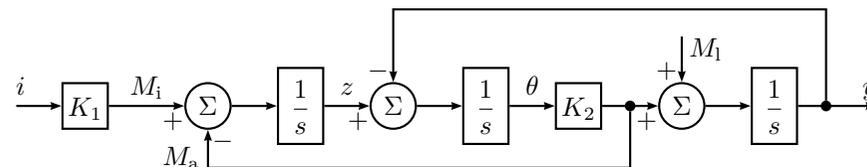


Figure 8.3a

8.3 The block diagram in Figure 8.3a describes an electric motor that drives a load via an elastic axis. Here i is the driving current to the motor, which gives the torque M_i . z is the turning rate of the motor and y is the turning rate of the load. θ is the angle of the transmission axis. $M_a = K_2\theta$ is the torque this angle causes. M_1 is the torque from the load. Give a state space description for the system with M_1 and i as inputs and y as output. (There are at least two different ways to solve this problem.)

8.4 Write the following systems in state space form.

a)

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 6u$$

b)

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 4\frac{d^2u}{dt^2} + \frac{du}{dt} + 2u$$

c)

$$G(s) = \frac{2s + 3}{s^2 + 5s + 6}$$

Use for example controllable or observable canonical form or diagonal form.

8.5 A system has the impulse response (weight function)

$$g(t) = 2e^{-t} + 3e^{-4t}$$

Write the system in state space form.

8.6 Consider the system

$$\dot{x} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (-1 \quad 2) x$$

Compute the transfer function of the system.

8.7 Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

The input is being held constant, $u = u_0$, for the time $t_0 \leq t \leq t_0 + T$.

Give a relation between $x(t_0)$, $x(t_0 + T)$, $y(t_0)$, $y(t_0 + T)$ and u_0 .

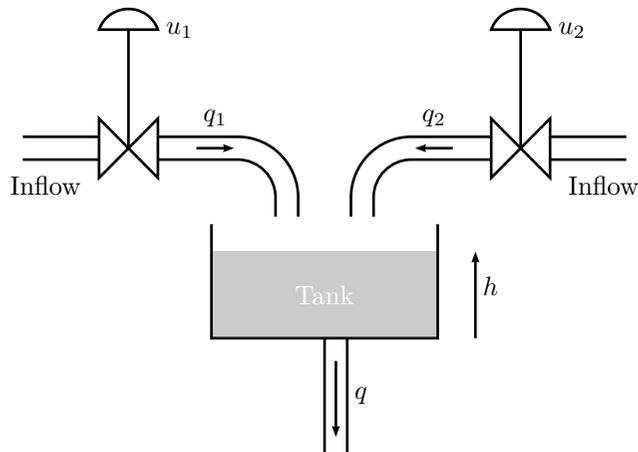


Figure 8.8a

8.8 Consider the tank in Figure 8.8a. The tank can be filled from two different pipes, where the flows q_1 and q_2 are determined by the valve settings u_1 and u_2 . If q_1 , q_2 , u_1 , u_2 , and h denote the deviation from a nominal value, we get the linearized equation

$$\dot{h} + \frac{1}{\tau} h = u_1 + u_2$$

where $\tau = 1$. It is desired that the level should follow a reference value h_{ref} and that q_1 and q_2 should be of approximately the same size. Therefore two PI controllers are used so that

$$u_1 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.1)$$

$$u_2 = (h_{\text{ref}} - h) + \int_0^t (h_{\text{ref}} - h) d\tau \quad (8.2)$$

- Introduce the state variable $x_1 = h$, and let x_2 and x_3 represent the integrals in (8.1) and (8.2) respectively. Derive a state space description of the closed loop system with h_{ref} as input and h as output.
- Verify that the closed loop system is unobservable and that the unobservable subspace is spanned by the vector

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Give a practical interpretation of this phenomenon.

- The level is measured by two different sensors, and due to the poor accuracy in the first sensor it delivers the level signal together with an error. The equation of the first regulator can hence be written

$$u_1 = -(h + n) + \int_0^t -(h + n) d\tau \quad (8.3)$$

where it for simplicity has been assumed that $h_{\text{ref}} = 0$. The second regulator is then given by

$$u_2 = -h + \int_0^t -h d\tau \quad (8.4)$$

Modify the state space model by letting the measurement disturbance be the input to the state space model of the closed loop system.

8.9 Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

Is it possible to control the system from the origin to $x = (1 \ 3)^T$ within 4 seconds?

8.10 Give the dimensions of the controllable and unobservable subspaces to the systems below. Give also the controllable and unobservable subspaces.

a)

$$\dot{x} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \ 3 \ 1.5) x$$

b)

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} u$$

$$y = (0 \ 3 \ 0) x$$

8.11 A state space representation of

$$G(s) = \frac{1}{s+1}$$

is given by

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0) x$$

a) Compute $x_1(t)$, $x_2(t)$ and $y(t)$ if $x(0) = 0$ and

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

b) Is the system asymptotically stable? Input-output-stable?

c) Examine the controllability and observability for the system.

d) Explain why the realization is not suitable for simulating a system whose transfer function is $G(s)$.

8.12 Compute the poles and zeros of the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

$$y = (1 \ 1) x$$

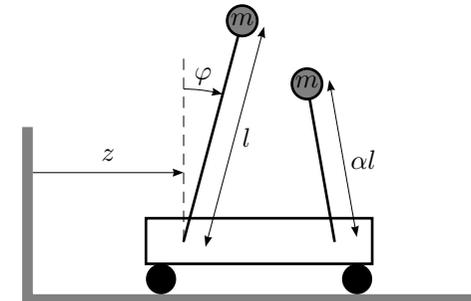


Figure 8.13a

8.13 Two mathematical pendulums are mounted on a trolley. They are mounted so that they can move without friction in a plane coinciding with the direction of movement for the trolley. The lengths of the pendulums are l and αl and their masses are m . For one pendulum we have

$$\ddot{z} \cos \varphi + \ddot{\varphi} l = g \sin \varphi$$

a) Linearize the system to the left in Figure 8.13a around $\varphi = 0$ and put the constants l , m , and g to 1 and write the equations in the form $\dot{x} = Ax + Bu$.

b) Give the values on α for which the system is controllable. Give a practical motivation.

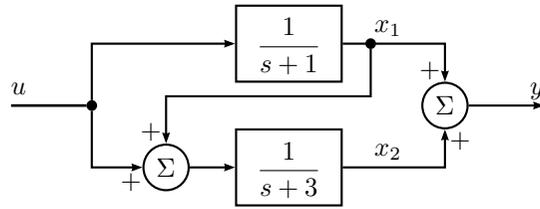


Figure 8.14a

- 8.14 A system is given by the block diagram in Figure 8.14a. Derive a state space model of the system, with the state space variables given in the figure.
- 8.15 The substances A and B react according to, $3A \rightarrow B$, in a tank. The reaction speed is given by $r_A = -k_1 c_A^3$. The inflow, q to the tank has concentration $c_{A,in}$. The tank volume V and the in- and outflow can be considered constant.
- Determine the dynamical mass balance for the components A and B in the form of differential equations.
 - Linearize the differential equations around a stationary point, c_A^* , c_B^* , $c_{A,in}^*$, and use the state space representation

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

where the state x consists of the deviations $c_{A,\Delta}$ and $c_{B,\Delta}$ of the concentrations. The input signal u is the deviation $c_{A,in,\Delta}$ in the inflow concentration and the output signal y is the deviation $B c_{B,\Delta}$ in concentration of component.

- 8.16 Figure 8.16a shows a protein that changes between the conformations x_1 and x_2 . The kinetic parameters k_{ij} determine the rate of the conformational changes. The input signal, u , together with the constant, K , represent introduction of x_2 to the system. We assume that we can measure x_2 , i.e. $y = x_2$. All reaction rates are assumed to be given by expressions proportional to the concentration of the states they emanate from. That is, for a state i the differential equation will be $\dot{x}_i = \sum_p k_{ip} x_p - \sum_q k_{qi} x_i$, where p is the number of incoming and q the number of outgoing flows from the state i . This corresponds to mass action kinetics for a regular biochemical reaction network.

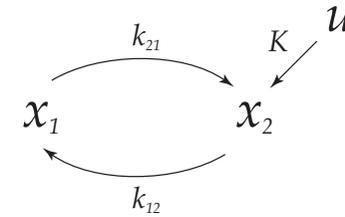


Figure 8.16a. Protein in two possible conformations.

- Write down the state space equations that describe the system.
- Compute the poles of the system. Is the system stable?
- What are the conditions on the kinetic parameters k_{ij} for the system to be observable? Give an interpretation of this result.
- Assume that the conformation change from x_2 to x_1 is catalyzed by an enzyme. To describe this, include Michaelis-Menten saturation in the flow from x_2 to x_1 , i.e. replace the expression $k_{12}x_2$ with $\frac{V_{max}x_2}{K_M + x_2}$, where V_{max} is the maximal rate and K_M the saturation constant. Write down the differential equations for the states x_1 and x_2 .
- Sketch the relation between x_2 and the saturated reaction rate in d), i.e. \dot{x}_2 as a function of x_2 .
- Linearize the system around the stationary point $x_1 = \frac{V_{max}}{2k_{21}}$, $x_2 = K_M$, $u = 0$.

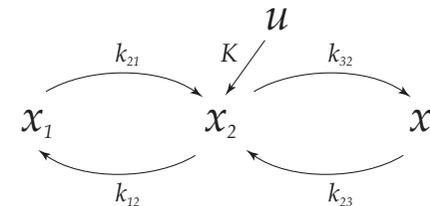


Figure 8.16b. Protein in three possible conformations.

Now, assume that the protein can exist in a third conformation x_3 , according to Figure 8.16b. We measure $y = x_2$ as before.

- g) Write down the state space equations that describe the system.
- h) What are the conditions on the kinetic parameters k_{ij} for the system to be observable? Give an interpretation of this result.

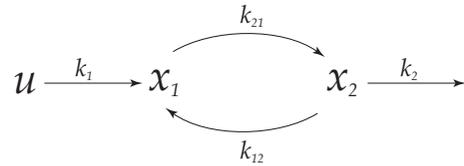


Figure 8.17a

8.17 The model in Figure 8.17a describes the interaction between two proteins with concentrations x_1 and x_2 , where the kinetic parameters k_1, k_{21}, k_{12} and k_2 determine the reaction rate.

The law of mass actions for biochemical reaction networks leads to the following state space model for the system.

$$\dot{x} = \begin{pmatrix} -k_{21} & k_{12} \\ k_{21} & -k_{12} - k_2 \end{pmatrix} x + \begin{pmatrix} k_1 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

The input u is a unit step.

- a) What are the steady state values of x_1 and x_2 , when the input u is a unit step?
- b) Compute the transfer function of the system, from u to y .
- c) Use the final value theorem to find the steady state gain of y . Compare with the result in a). What conclusions can be drawn?

9 State Feedback

9.1 Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (1 \quad 0) x \end{aligned}$$

- Calculate a state feedback that places the poles in I) $\{-3, -5\}$, II) $\{-10, -15\}$. What limits the possibility to achieve arbitrary dynamics of the closed loop system?
- Suppose only the output is measured. Calculate an observer that makes the transfer function from the reference signal to the output the same as in a). Discuss the influence of the poles of the observer.

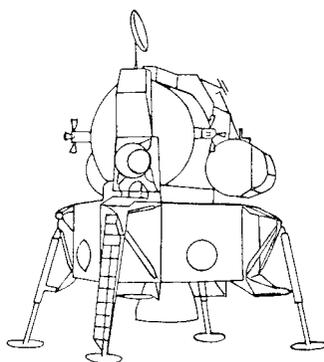


Figure 9.2a

9.2 Figure 9.2a shows the Lunar Excursion Module from the Apollo project. Consider the module hovering a short distance above the surface of the moon using its main engine. If the pitch angle of the module (angle between the vertical line and the direction of movement) differs from zero, a horizontal component of the force is obtained and the module is accelerating along the surface.

We will study a block diagram which shows the connection between the input u (the control signal to the attitude thrusters), the pitch angle θ and the position coordinate z . See Figures 9.2b and 9.2c.

The module is both in the θ -direction and in the z -direction obeying Newton's law of motion without any kind of damping. The transfer function from the control signal of the astronaut (y_{ref}) to velocity \dot{z} is

$$\frac{K_1 K_2}{s^3}$$

which is very difficult to control by hand.

- Write the system in state space form.

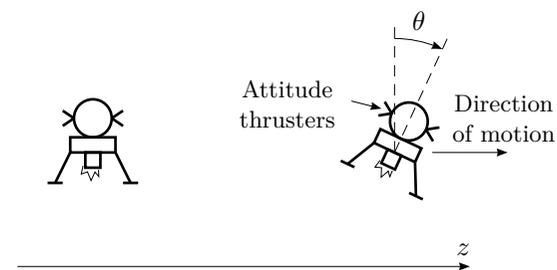


Figure 9.2b

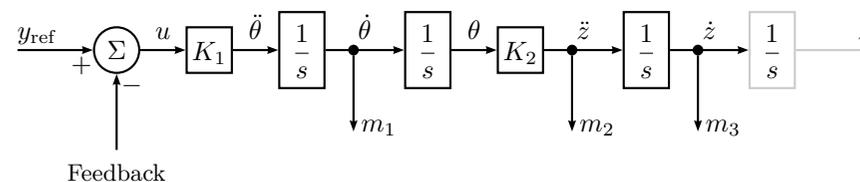


Figure 9.2c

b) In order to make the control duty of the astronaut easier we change the dynamics of the module by making internal feedback. The following signals are measurable:

m_1 , the attitude angular velocity measured using rate gyro.

m_2 , the acceleration in z -direction measured using accelerometers positioned on gyro-stabilized platforms.

m_3 , the velocity in z -direction measured using doppler-radar.

Calculate a state-feedback using these signals such that the closed loop system obtains its poles in $s = -\frac{1}{2}$ and the control signal of the astronaut becomes the reference signal of the velocity in z -direction.

c) Suppose we by safety reasons are interested in the possibility of controlling the module even if the sensors measuring m_1 and m_2 are not working. Design a controller that can handle this and has approximately the same behavior as in a).

9.3 A DC motor with an external load, T , is described by

$$\begin{aligned}\omega &= \dot{\theta} \\ \dot{\omega} &= -\frac{1}{\tau}\omega + c_1 u + c_2 T\end{aligned}$$

where θ is the angle, ω the angular velocity, u the control signal, T the torque of the load, and c_1 , c_2 , and τ are constants.

a) Introduce a controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega$$

such that the poles of the closed loop system becomes $\frac{1}{\tau}(-1 \pm i)$ and $\theta = \theta_{\text{ref}}$ in steady-state if $T = 0$ and θ_{ref} is constant.

b) Introduce a modified controller

$$u = l_0 \theta_{\text{ref}} - l_1 \theta - l_2 \omega + u'$$

such that $\theta = \theta_{\text{ref}}$ in steady-state even for constant non-zero T and constant θ_{ref} .

9.4 A system can be described in state space form as

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (1 \quad -1) x\end{aligned}$$

We want to place the poles in $\{-2, -3\}$. Suggest an observer, and use a linear state feedback controller. Which are the poles of the closed loop system?

9.5 Is it possible to design an observer with poles in $\{-5, -6, -7, -8\}$ for the system below? Motivate your answer.

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 10 \\ -3 \\ 2 \end{pmatrix} u \\ y &= (1 \quad 0 \quad 0 \quad 0) x\end{aligned}$$

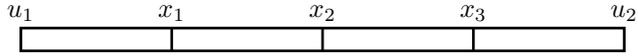


Figure 9.6a

9.6 We want to control the temperature in a long copper rod by heating or cooling its endpoints. Principally, this problem is described by a partial differential equation. To simplify the problem we assume that the temperature profile in the rod can be approximated by the temperatures x_1 , x_2 , and x_3 at three points. The temperatures in the end points are the inputs, u_1 and u_2 . All temperatures are relative to the temperature of the surroundings.

We get the following ordinary differential equations:

$$\begin{aligned}\dot{x}_1 &= \alpha(u_1 - x_1) + \alpha(x_2 - x_1) \\ \dot{x}_2 &= \alpha(x_1 - x_2) + \alpha(x_3 - x_2) \\ \dot{x}_3 &= \alpha(x_2 - x_3) + \alpha(u_2 - x_3)\end{aligned}$$

where α is a constant that depends on the coefficient of thermal conductivity and the specific heat of the rod. For simplicity, let $\alpha = 1$. Consider the problem of controlling the temperature in x_1 , x_2 , and x_3 with u_1 only, assuming $u_2 = 0$.

- Assume that we want to have an arbitrary temperature profile, that is, arbitrary values of x_1 , x_2 , and x_3 . Is this possible? Why/why not?
- Assume that all the temperatures x_1 , x_2 and x_3 are measurable. Find a state feedback that brings any initial state to zero as e^{-3t} .
- Assume that only one of the temperatures x_1 , x_2 , or x_3 is measurable, and that we still want a controller which damps a disturbance as e^{-3t} by using an observer. The sensor can be placed so that any of the three values x_1 , x_2 , or x_3 is measured. Which choices of measure point make it possible to control the system as desired? Give a motivation. Choose one of the points making the wanted design possible and design a controller, that is, an observer and a state feedback, giving the desired error damping.

9.7 Consider the lunar excursion module in Problem 9.2. Suppose that there are no rate-gyro measurements available but that the sensors measuring m_2 and m_3 are still working.

Show how m_1 can be reconstructed from u and measurements of m_2 such that the reconstruction error decreases arbitrarily fast without differentiation of any of the measured signals.

Propose a filter and describe the resulting controller when the feedback consists of both measured and reconstructed states as in Problem 9.2.

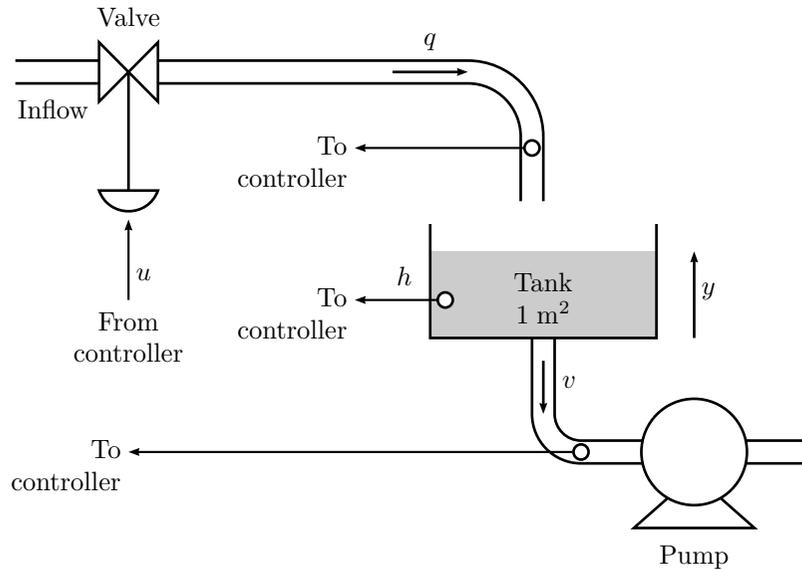


Figure 9.8a

9.8 In Figure 9.8a a level control system for a tank is shown. The objective is to keep the level at a desired value. Let u , h , q , and v denote small variations around the desired working point. The inflow, q , to the tank is determined by the valve, u , calculated by the controller. The outflow v is determined by the pump and deviations from zero is considered as process noise. The valve has some dynamics, which is modeled with the transfer function

$$Q(s) = \frac{k_1}{1 + Ts} U(s)$$

where $k_1 = 1$ and $T = 0.5$. The level is given by

$$A\dot{h} = q - v$$

where the tank cross-section area is $A = 1\text{m}^2$.

- Let q and h be state variables and give a corresponding state space model of the process. Compute a state feedback $u = -l_1q - l_2h + r$, such that the closed loop system poles both are at -2 .
- How large is the steady-state level error for a constant disturbance $v = 0.1$ if $r = 0$?

- Consider the closed loop system in a) and compute a feedforward control law from v to r such that the influence from v is completely eliminated. Exclude all terms in the control law in which v is differentiated to make it implementable. How does this modified feedforward control law work? Steady-state level error?
- Suppose that k_1 differs slightly from 1, but that the same control law as in c) is used (the control law derived under the assumption $k_1 = 1$). What happens with the steady-state level error?
- Propose a modified control law such that the stationary level error is zero for constant disturbances regardless of small deviations from the nominal value of k_1 .

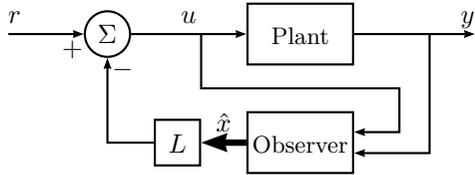


Figure 9.9a

9.9 We want to control the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

with a state feedback. (This can be interpreted as a moving vehicle in one dimension, where x_1 is the position, x_2 is the velocity, and the acceleration is the control signal.) We introduce the control law

$$u(t) = -L\hat{x}(t) + r(t)$$

where \hat{x} is constructed by an observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

We choose the vectors K and L as $L = \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $K^T = \begin{pmatrix} 4 & 4 \end{pmatrix}$. These choices put the eigenvalues of $A - BL$ in -1 and the eigenvalues of $A - KC$ in -2 . A block diagram of the closed loop system is shown in Figure 9.9a. Due to a time delay, the real input is given by the equation

$$u(t) = -L\hat{x}(t - T) + r$$

What is the largest possible time delay T without the closed loop system getting unstable?

9.10 One wants to construct an observer for the system

$$\dot{x}(t) = \begin{pmatrix} -1 & 1 \\ a & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t)$$

- Suppose $a = 1$. Construct an observer with the poles in $\{-5, -10\}$. For which values of a is this possible?
- Suppose that the measured signal y is given by

$$y(t) = \begin{pmatrix} 2 & 1 \end{pmatrix} x(t) + v(t)$$

Here $v(t)$ is the measurement noise. Compute the transfer function from v to $\tilde{x}_1(t)$, that is, the first element in the state vector for the observer error $\tilde{x}(t) = x(t) - \hat{x}(t)$.

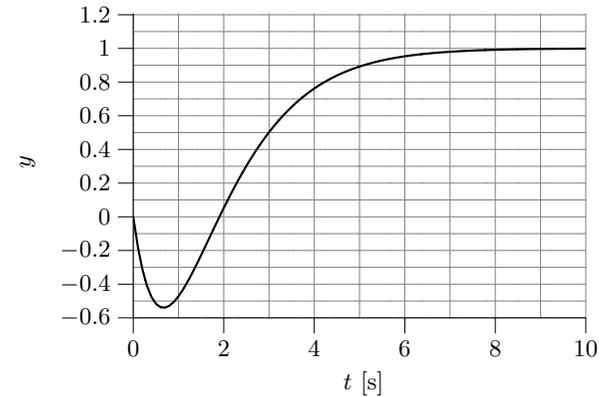


Figure 9.11a

9.11 Consider the system

$$G(s) = \frac{1 - s/\alpha}{(1 + s/\beta)^2}, \quad \alpha > 0, \beta > 0$$

Systems of this kind, that is, with a zero in the RHP have the property that the step response goes in the “wrong direction” initially, see Figure 9.11a.

- Show that the derivative of the step response at $t = 0$, that is, $\dot{y}(0)$, decreases as the zero of the system approaches the origin.
- Is it possible to use state feedback to eliminate the problem that the step response goes in the wrong direction initially? Justify your answer.

9.12 Theoretically one can place the poles of a controllable system arbitrarily. Which practical difficulties limit the performance that one can actually achieve?

9.13 Consider the system

$$\dot{x} = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} -1 \\ 2 \end{pmatrix} u$$

$$y = (1 \quad 1) x$$

- Determine L of a state feedback $u = r - Lx$, that places the poles at $-2 \pm i$.
- The state feedback of a) is used. It is observed that the output $y(t) = 0$ for all t is obtained for a reference signal of the form $r(t) = e^{\alpha t}$. For what value(s) of α does this occur?

9.14 Consider the model of a DC-motor

$$Y(s) = G(s)U(s)$$

where

$$G(s) = \frac{1}{s(s+1)}$$

- Generate a state space representation using MATLAB. Which physical signals are represented by the states?
- Suppose that the system is going to be controlled using state feedback

$$u(t) = -Lx(t) + l_0 r(t)$$

Compute the gain vector L and simulate the closed loop system for the following two choices of closed loop poles:

- ◊ Real poles at $\{-2.2, -2.1\}$
- ◊ Poles at $-1 \pm i$

Also compute l_0 such that the closed loop system gets static gain one. In particular look at the properties of the step response and the magnitude of the control signal in the two cases. Which pole locations give the best trade off between response speed and control signal magnitude?

- Now let L be computed using linear quadratic optimization (LQ) in order to minimize the cost function

$$\int_0^{\infty} x(t)^T Q x(t) + u(t)^2 dt$$

for the three choices of weight matrices given below. Compute the closed loop poles and the step responses of the closed loop system for the three cases. Describe how the properties of the step responses in the different cases.

$$(i) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(ii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$$

$$(iii) \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0.1 \end{pmatrix}$$

- Start from case (ii) and increase the weight on the control signal gradually until the cost function becomes

$$\int_0^{\infty} x(t)^T Q x(t) + 10 u(t)^2 dt$$

Compare the result with the result obtained for case (i).

- Start from case (i) and introduce a weight on the velocity $\dot{y}(t)$. Increase the weight gradually and study how the poles and the step response of the closed loop system change.

9.15 The ingestion and metabolism of a drug in a human body can be described by the following equations:

$$\begin{aligned}\frac{dq(t)}{dt} &= -k_1q(t) + u(t) \\ \frac{dy(t)}{dt} &= k_1q(t) - k_2y(t)\end{aligned}$$

where the input signal $u(t)$ is the ingestion rate of the drug, the output $y(t)$ is the mass of the drug in the blood, and $q(t)$ is the mass of the drug in the gastrointestinal tract. The constants k_1 and k_2 are metabolism rates, satisfying $k_1 > k_2 > 0$. k_2 characterizes the excretory process of the individual. In this example, $k_1 = 0.05$ and $k_2 = 0.02$.

- Is the system controllable?
- Design a state feedback that places the closed loop poles in -0.1 .

$q(t)$ (the mass of the drug in the gastrointestinal tract) cannot be measured, so to be able to use the state feedback in b) we need an observer.

- How should the poles of the observer be selected?
- Design an observer with poles in -0.2 .

9.16 A system is described by the state space equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t - \tau) \\ y(t) &= \kappa Cx(t) + e(t)\end{aligned}\tag{9.1}$$

where it is known that

$$\begin{aligned}\tau &< 0.3 \\ 0.9 &\leq \kappa \leq 1.1\end{aligned}$$

It is also known that e is a sinusoidal disturbance with angular frequency 10 rad/s. The control design is based on the simplified model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{9.2}$$

The specifications for the control system are given by:

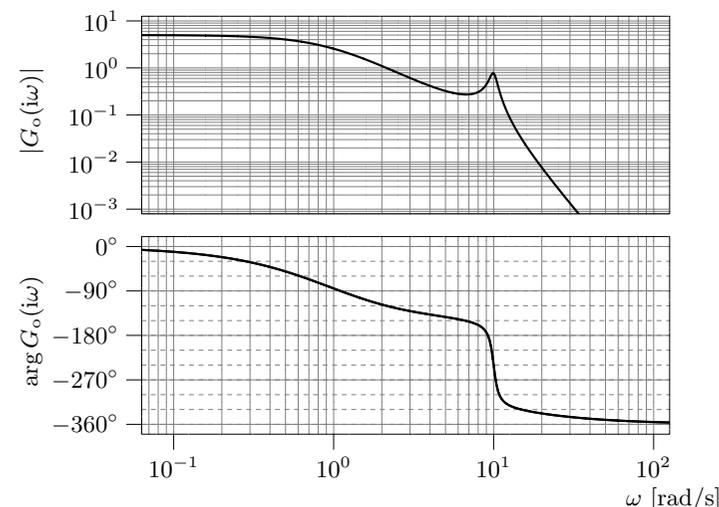


Figure 9.16a. Bode plot of the loop gain.

- The bandwidth of the closed loop system must fulfill $\omega_B \geq 5$ rad/s.
- The closed loop system shall be stable despite the disturbance and the uncertainties in the parameters τ and κ .
- The static gain of the closed loop system shall be 1 despite the disturbance and the uncertainties in the parameters τ and κ .
- The closed loop system shall handle the measurement disturbance sufficiently well.

The regulator design is carried out using state space methods. The poles of the closed loop system, that is, eigenvalues of $A - BL$, are placed in $\{-4, -2 \pm 2i\}$ and the poles of the observer, that is, the eigenvalues of $A - KC$, are placed in $\{-17, -0.2 \pm 10i\}$.

The figures below show different aspects of the control system. It is important to note that the diagrams are obtained using the model (9.2) and the designed regulator. Determine if the requirements 1–4 are fulfilled when controlling the system given by (9.1).

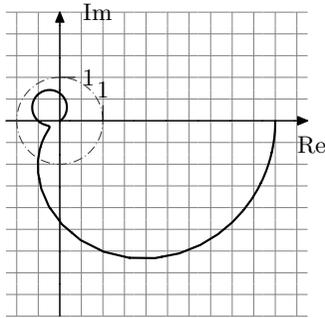


Figure 9.16b. Nyquist curve of the loop gain.

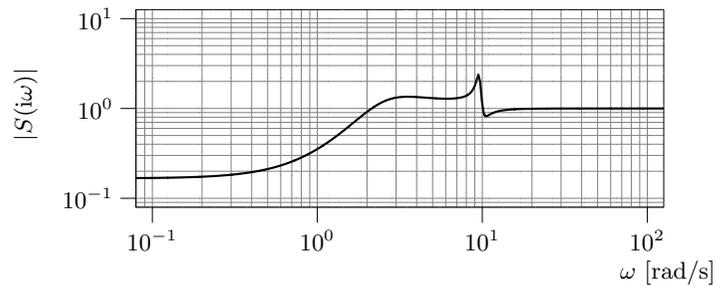


Figure 9.16c. Gain curve of the sensitivity function.

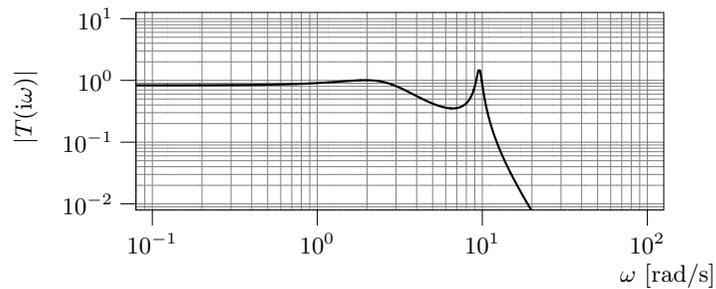


Figure 9.16d. Gain curve of the complementary sensitivity function.

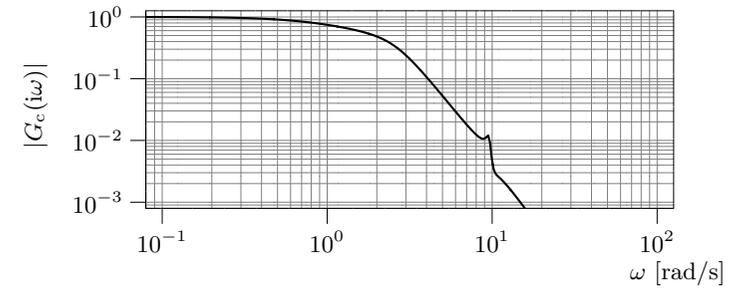


Figure 9.16e. Gain curve of the transfer function of the closed loop system.

9.17 In purification processes sometimes bacteria are used to consume the unwanted substance (possibly converting it to something more useful). Let ξ denote the amount of bacteria, η the amount of substance to be removed, and q the input flow (that contains the substance to be removed). The system is then described by a set of nonlinear differential equations

$$\begin{aligned}\dot{\xi} &= f_1(\xi, \eta, q) \\ \dot{\eta} &= f_2(\xi, \eta, q)\end{aligned}$$

When considering small deviations from an equilibrium the equations can be approximated by a linear system. In this example, the numeric values of the linearization are given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$

where x_1 , x_2 , and u denote the deviations from the equilibrium values of ξ , η , and q respectively.

- Assume that both x_1 and x_2 can be measured. Determine a state feedback placing the closed loop poles in $\{-2, -4\}$.
- Assume that only x_2 is measured. Is it possible to calculate the amount of bacteria x_1 from this measurement if u is known? If the answer is yes: Why is it important that u is known?
- Assume that the value of u is unknown, but let it be known that it is constant. Is it then possible to calculate the amount of bacteria, x_1 from a measurement of x_2 ? In case it is, show some way of doing the computation.

9.18 A certain species of bacteria grows by consuming glucose, whose inflow is controlled. The following model is used

$$\dot{m} = (f - 1)m \quad \dot{f} = -m + q$$

where m is the amount of bacteria, f the amount of glucose and q the inflow of glucose. One wants the system to operate in the neighborhood of the operating point $m = 1, f = 1, q = 1$. Using the notation $x_1 = m - 1, x_2 = f - 1, u = q - 1$, an approximate model ($x_1 x_2$ neglected) is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

Sometimes there are disturbances that are modeled as a constant, unknown external signal w :

$$\begin{aligned}\dot{x}_1 &= x_2 + w \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

One wants to drive x_1 to a reference value r using u as control variable.

- a) x_1 is measured. What performance (stationary error, speed of response) can be obtained using a P controller?
- b) x_2 is measured. What performance (stationary error, speed of response) can be obtained by using a control law where u depends linearly on r and x_2 if $w = 0$? What happens when $w \neq 0$?
- c) x_2 is measured. Determine a controller that does not differentiate x_2 , gives an asymptotically stable closed loop system and makes x_1 converge to r asymptotically for an arbitrary constant but unknown w .

11 Implementation

11.1 If you “translate” the compensator

$$U(s) = KN\left(\frac{s+b}{s+bN}\right)E(s)$$

with Tustin’s formula you get a controller of the form

$$u(t) = \beta_1 u(t-T) + \alpha_1 e(t) + \alpha_2 e(t-T)$$

What are the values of α_1 , α_2 , and β_2 , if $T = 0.1$, $N = 10$, $b = 0.1$, and $K = 2$?

11.2 Consider the system

$$\dot{y}(t) = u(t)$$

Suppose it is controlled with a computer, so that the control signal is constant over the sampling interval, that is,

$$u(t) = u_k, \quad kT \leq t < (k+1)T$$

- Introduce the notation $y_k = y(kT)$ and derive a relation between y_{k+1} , y_k , and u_k .
- Suppose we use the proportional feedback

$$u_k = -Ky_k$$

and that $y(0) = y_0$. What are the values of K , for which the closed loop system is stable?

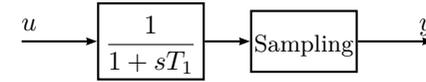


Figure 11.3a

11.3 Consider the system in Figure 11.3a, which illustrates sampling with prefiltering. Suppose we are sampling with the sampling period T and that $u = u_0 + u_1$, where u_0 is an “interesting” low frequency signal in the frequency interval $0 < \omega < \pi/T$ and that u_1 is a sinusoidal control signal

$$u_1(t) = \sin \omega_2 t, \quad \frac{\pi}{T} < \omega_2 < \frac{2\pi}{T}$$

Since the sampling causes aliasing, the output will be

$$y(t) = y_0 + y_1$$

where y_0 is interesting and y_1 is a disturbance signal

$$y_1(kT) = A \sin(\omega_1 kT + \varphi), \quad \omega_1 < \pi/T$$

- What are A , ω_1 and φ ?
- It is clear from a) that the choice of T affects the amplitude of the disturbance signal y_1 . What is the smallest amplitude you can get if you do not want to damp any frequencies in u_0 more than $\sqrt{2}$ times?

Introduktion till Control System Toolbox 5.0

This version: January 2011

1 Inledning

Denna skrift är en kort inledning till hur MATLAB och Control System Toolbox (CST) används i kurserna i Reglerteknik.

2 System

I Control System Toolbox finns datastrukturer för att hantera så kallade *LTI-objects*, dvs linjära tidsinvarianta system, på ett bekvämt sätt. Vi kommer inledningsvis främst att arbeta med system på överföringsfunktionsform, men senare även med system på tillståndsform. Ett objekt som representerar ett system på överföringsfunktionsform skapas med funktionen `tf`. Detta kan göras på två olika sätt, och det första alternativet visas i exemplet nedan.

Betrakta överföringsfunktionen

$$G(s) = \frac{4}{s(s^2 + 2s + 4)}$$

Mata in systemet och ge objektet namnet `G`. Argumenten till funktionen `tf` utgörs av radvektorer innehållande täljarens respektive nämnarens koefficienter.

```
>> G = tf( 4, [ 1 2 4 0 ] )  
  
Transfer function:  
4  
-----  
s^3 + 2 s^2 + 4 s
```

Med det andra alternativet kan man mata in överföringsfunktionen på symbolisk form genom att först skapa ett objekt bestående av symbolen `s`. Därefter kan man `tf` addera och multiplicera med denna symbol på samma sätt som görs med Laplace-variabeln s vid handräkning.

Skapa ett objekt bestående av symbolen `s`. Bilda överföringsfunktionen genom att använda vanliga räkneoperationer.

```
>> s = tf( 's' );  
>> G = 4 / ( s * ( s^2 + 2*s + 4 ) )  
  
Transfer function:  
4  
-----  
s^3 + 2 s^2 + 4 s
```

En finess med överföringsfunktioner representerade som LTI-objekt är att man kan multiplicera och addera överföringsfunktioner på ett rättfram sätt.

Skapa en ny överföringsfunktion `G2` genom att seriekoppla `G(s)` och överföringsfunktionen

```
>> G2 = G * 1 / ( s + 1 )  
  
Transfer function:  
4  
-----  
s^4 + 3 s^3 + 6 s^2 + 4 s
```

$$\frac{1}{s + 1}$$

3 Poler och nollställen

Poler och nollställen till överföringsfunktioner beräknas med funktionerna `pole` respektive `tzero`. Poler och nollställen kan även ritas med funktionen `pzmap`.

Beräkna polerna till $G(s)$. Systemet har en reell pol i origo och två komplexa poler.

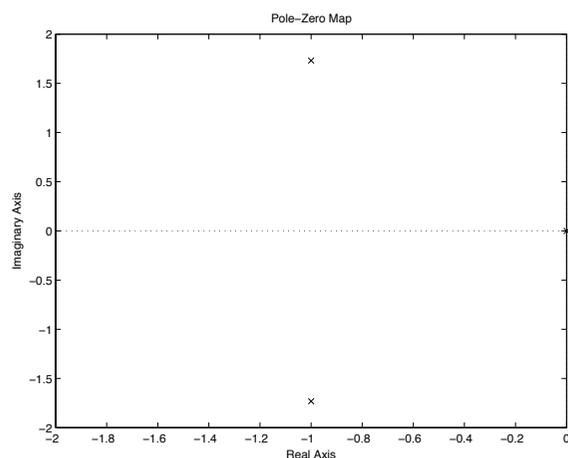
```
>> pole( G )  
  
ans =  
  
0  
-1.0000 + 1.7321i  
-1.0000 - 1.7321i
```

Beräkna nollställena till $G(s)$. Eftersom täljaren i överföringsfunktionen är konstant saknar systemet nollställen.

```
>> tzero( G )  
  
ans =  
  
Empty matrix: 0-by-1
```

Rita in systemets poler och nollställen i det komplexa talplanet. Poler markeras med kryss och nollställen, i de fall de förekommer, markeras med ringar.

```
>> pzmap( G )
>> axis([ -2 0 -2 2 ])
```



4 Återkoppling

I kursen behandlas återkopplade reglersystem enligt figur 1.

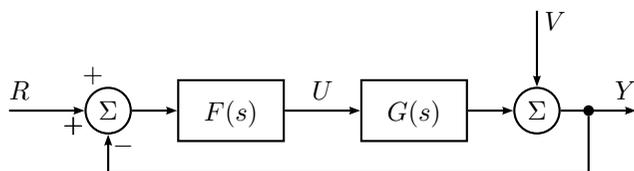


Figure 1. Reglersystem

Med systembeskrivningen

$$Y(s) = G(s)U(s) + V(s)$$

och återkopplingen

$$U(s) = F(s)(R(s) - Y(s))$$

ges det återkopplade systemet av

$$Y(s) = G_c(s)R(s) + S(s)V(s)$$

där

$$G_c(s) = \frac{F(s)G(s)}{1 + F(s)G(s)}$$

och

$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Överföringsfunktionerna för det återkopplade systemet kan beräknas med funktionen **feedback**.

Generera överföringsfunktionen för en proportionell regulator med förstärkning $K_p = 0.7$.

```
>> F = tf( 0.7 )
```

Transfer function:
0.7

Beräkna överföringsfunktionen för det återkopplade systemet.

```
>> Gc = feedback( F * G, 1 )
```

Transfer function:
2.8

$$s^3 + 2 s^2 + 4 s + 2.8$$

Beräkna känslighetsfunktionen.

```
>> S = 1 / ( 1 + F * G )
```

Transfer function:
 $s^3 + 2 s^2 + 4 s$

$$s^3 + 2 s^2 + 4 s + 2.8$$

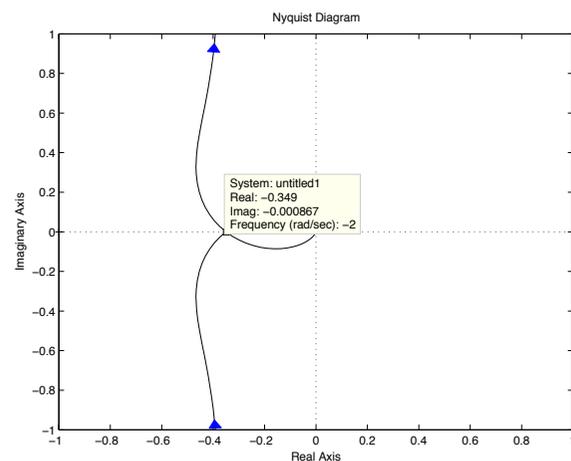
I exemplet ovan hade vi kunnat beräkna G_c på motsvarande sätt som S beräknades, dvs $G_c = F * G / (1 + F * G)$. Med denna metod får dock täljaren och nämnaren i G_c ett antal gemensamma faktorer som kan förkortas bort. Genom att använda funktionen **feedback** undviks detta. De gemensamma faktorerna i det första alternativet kan elimineras genom att använda funktionen **minreal(Gc)** Testa själv och jämför.

5 Nyquistdiagram

Nyquistkurvor för en eller flera överföringsfunktioner ritas med funktionen `nyquist`. Eftersom funktionen `nyquist` graderar axlarna automatiskt kan diagrammet ibland bli svårläst. Läsbarheten kan förbättras genom att man själv väljer axlarnas gradering med funktionen `axis`. Man kan få ut mycket information ur figuren genom att använda vänster respektive höger musknapp. Med vänster musknapp kan man t ex markera en punkt på kurvan och få ut motsvarande värde på ω samt nyquistkurvas värde i denna frekvens. Med höger musknapp får man en meny med olika operationer som kan göras med figuren.

Rita nyquistkurvan för det öppna systemet då systemet $G(s)$ styrs med en proportionell återkoppling med förstärkning $K_P = 0.7$. Justera axlarnas gradering och markera punkten där nyquistkurvan passerar negativa delen av reella axeln.

```
>> nyquist( F * G )
>> axis([ -1 1 -1 1 ])
```



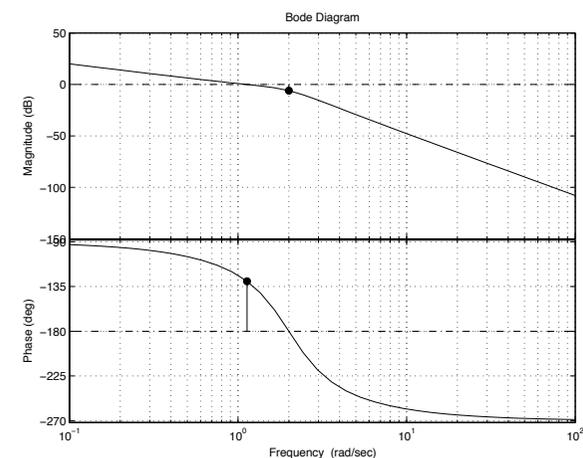
6 Bodediagram

Bodediagram för en eller flera överföringsfunktioner ritas med funktionen `bode`. Även i detta fall kan man läsa av punkter i figuren genom att markera med vänster

musknapp. Med höger knapp får man en meny där man t ex kan välja att markera frekvenserna där stabilitetsmarginalerna läses av.

Beräkna frekvensfunktionen för systemet G och rita upp den i ett bodediagram. Notera att amplitudkurvan graderas i decibel. Använd höger musknapp och lägg in rutnät i figuren samt markera var fas- och amplitudskärfrekvenserna ligger.

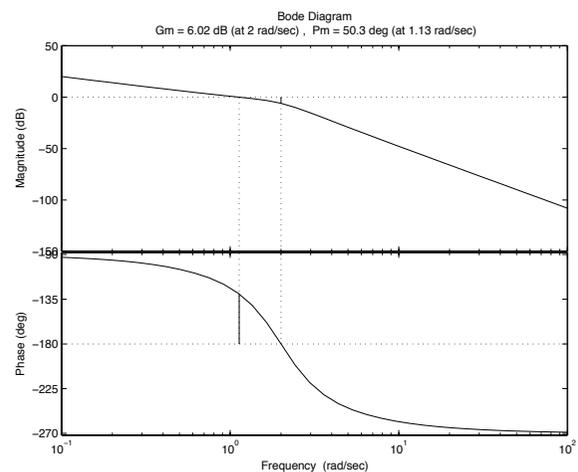
```
>> bode( G )
```



För att bestämma skärfrekvenser samt fas- och amplitudmarginal kan man även använda funktionen `margin`, vilken förutom att rita upp amplitud- och faskurvorna även skriver ut dessa värden. G_m och P_m betecknar amplitud- respektive fasmarginal.

Beräkna frekvensfunktionen för systemet G och rita upp den i ett bodediagram.

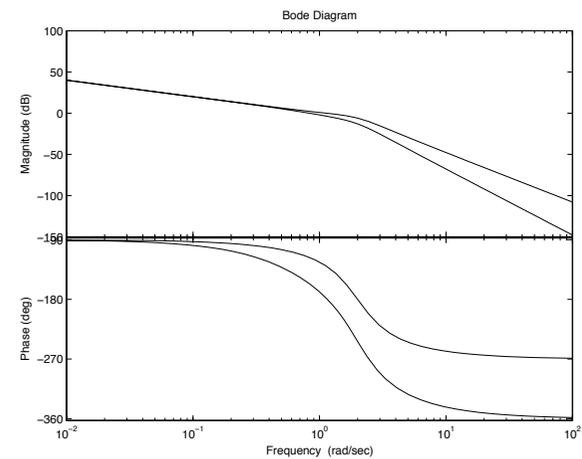
```
>> margin( G )
```



För att t ex kunna göra jämförelser mellan två frekvensfunktioner kan dessa ritas i samma diagram.

Beräkna frekvensfunktionerna för systemen G och $G2$.

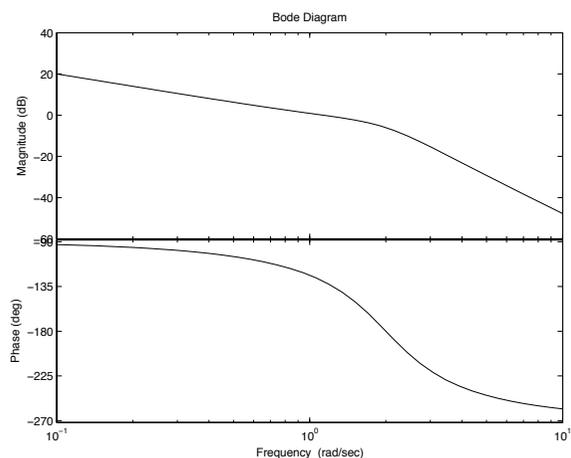
```
>> bode( G, G2 )
```



Skalan på frekvensaxeln kan väljas genom att som sista argument i funktionsanropet ange största och minsta frekvensvärdet mellan krullparenteser .

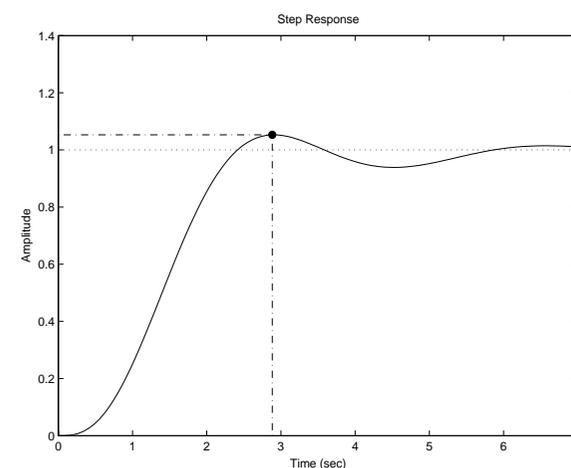
Beräkna frekvensfunktionen för systemet G från 0.1 till 10 rad/s och rita upp den i ett bodediagram.

```
>> bode( G, { 0.1, 10 } )
```



Beräkna och rita upp det återkopplade systemets stegsvar. Markera stegsvarets översläng.

```
>> step( Gc )
```



7 Simulering

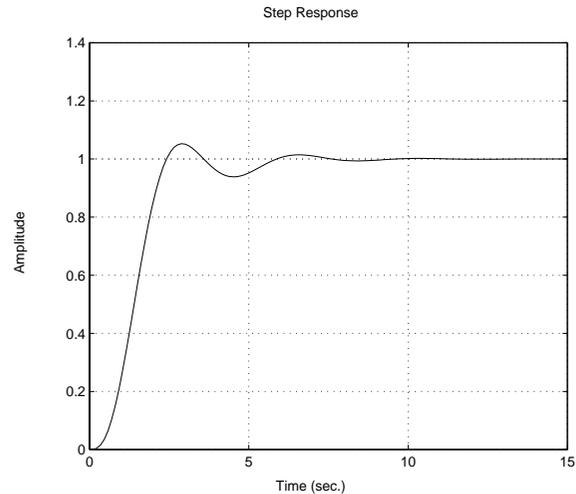
7.1 Stegsvvar

Den vanligaste typen av simulering är att beräkna ett systems stegsvar. Detta kan utföras med funktionen `step`, med vilken man både simulerar systemet och ritar dess stegsvar. I likhet med tidigare kan man läsa av enskilda värden i figuren med vänster musknapp och få en meny med olika val med höger knapp. Genom att t ex välja **Peak Response** från **Characteristics** markeras tidpunkt och värde för överslängen. Placera markören över punkten i diagrammet visas tillhörande numeriska värden.

Antag att systemet G styrs med proportionell återkoppling med förstärkning $K_p = 0.7$.

Beräkna det återkopplade systemets stegsvar under femton sekunder och rita upp resultatet.

```
>> step( Gc, 15 )
```



7.2 Allmän insignal

För att simulera linjära system med allmänna insignaler kan man använda funktionen `lsim(G,u,t)`. Indata till denna funktion är ett (eller flera) system G , en insignalvektor u och en tidsvektor t .

Antag exempelvis att vi vill studera reglerfelet för det återkopplade systemet ovan då referenssignalen är en ramp. Vi vet att sambandet mellan referenssignal och reglerfel ges av känslighetsfunktionen

$$E(s) = S(s)R(s)$$

där

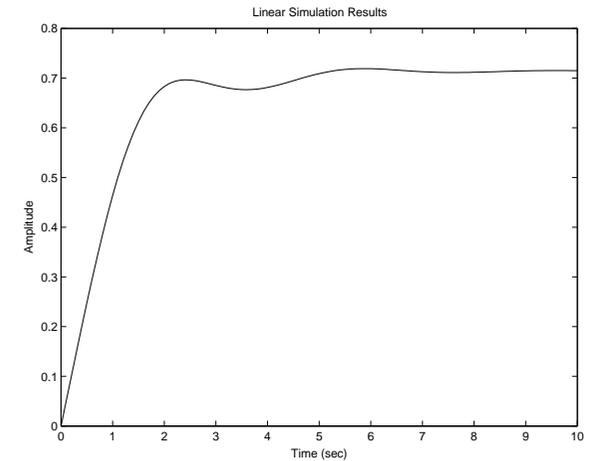
$$S(s) = \frac{1}{1 + F(s)G(s)}$$

Skapa en tidsvektor mellan 0 och 10 med steget 0.1.

```
>> t = ( 0 : 0.1 : 10 ).';
```

Simulera det återkopplade systemet då referenssignalen är en ramp med lutning 0.5. Reglerfelet går i detta fall mot 0.71. Funktionen ritas även insignalen, men den kan väljas bort på menyn som nås via höger musknapp.

```
>> lsim( S, 0.5*t, t )
```



För att skapa sinus- och fyrkantsignaler kan funktionen `gensig` användas.

8 Rotort

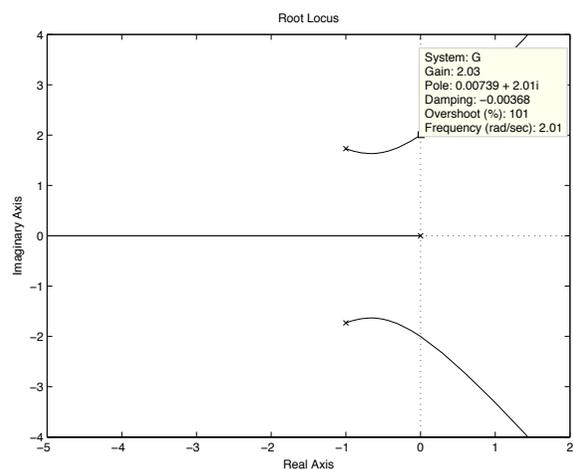
För att avgöra hur rötterna till ekvationen

$$P(s) + KQ(s) = 0$$

rör sig i komplexa talplanet då K går från noll och mot oändligheten kan man rita ekvationens rotort med funktionen `rlocus`. Indata till funktionen är en överföringsfunktion med polynomet $Q(s)$ som täljare och polynomet $P(s)$ som nämnare. Med höger musknapp kan man markera relevanta punkter i figuren, såsom t ex då rotorten passerar imaginäraxeln.

Rita upp rotorten för det återkopplade systemets karakteristiska funktion då systemet G styrs med en proportionell återkoppling. Markera där en av rötterna passerar imaginäraxeln.

```
>> rlocus( G )
```



För att t ex kontrollera för vilken förstärkning polerna har viss dämpning kan man med höger musknapp lägga in ett nät vilket markerar polplaceringar med samma avstånd till origo respektive samma dämpning.

9 SISO Design Tool

Ett ytterligare användbart verktyg är *SISO Design Tool*, vilket är ett användargränssnitt med vilket man enkelt kan studera ett system ur olika aspekter såsom stegsvar, bodediagram, poler och nollställen, etc. Verket *SISO Design Tool* startas genom att skriva `sisotool`. Automatiskt kommer de skapade LTI-objekten att finnas tillgängliga för analys. I figuren nedan visas ett exempel på vilka figurer som kan visas samtidigt. Testa dig fram!

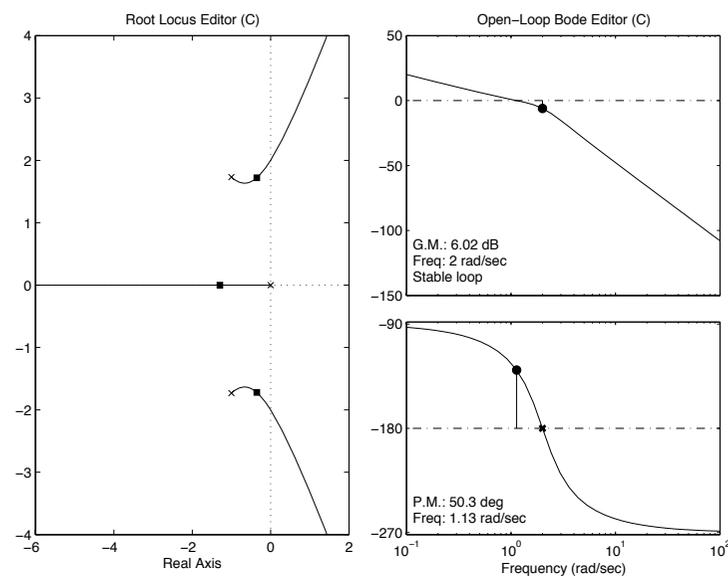


Figure 1. *SISO Design Tool*.

10 Tillståndsbeskrivning

I Control System Toolbox finns även en datastruktur för att hantera system på tillståndsform

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

För att skapa ett system på denna form används funktionen `ss`, med vilken man kan skapa ett system på tillståndsform från början eller konvertera ett system från överföringsfunktionsform.

Överför systemet `G` till tillståndsform. `>> G = ss(G)`

```
a =
      x1  x2  x3
x1  -2  -2  0
x2   2   0  0
x3   0   1  0
```

```
b =
      u1
x1   2
x2   0
x3   0
```

```
c =
      x1  x2  x3
y1   0   0   1
```

```
d =
      u1
y1   0
```

Continuous-time model.

Matriserna A, B, C och D i tillståndsbeskrivningen ingår nu i datastrukturen `G`. För att komma åt matriserna kan man referera till dem direkt genom att skriva `G.a`, `G.b` etc.

```
Beräkna egenvärdena till ma- >> eig( G.a )
trisen A i tillståndsmodellen
ans =
      0
 -1.0000 + 1.7321i
 -1.0000 - 1.7321i
```

Denna möjlighet är användbar t ex när man skall beräkna polplacering tillståndsåterkoppling på formen

$$u(t) = -Lx(t) + r(t)$$

vilket kan göras med funktionen `place` (och i special-fall med `acker`).

```
Bestäm en tillståndsåterkop- >> L = place( G.a, G.b, ...
pling som placerar det återkop-      -2 * ( 1 + [ -0.01 0 0.01 ] ) )
plade systemets poler i
närheten av -2. Läger man
alla polerna exakt i -2 kan
återkopplingen inte beräknas
med hjälp av place, så polerna
sprids ut en aning. För att
få alla polerna exakt i -2
kan acker användas, men
den funktionen har så dåliga
numeriska egenskaper att den
alltid bör undvikas till förmån
för place.
```

```
L =
      2.0000      1.9999      1.9998
```

Det återkopplade systemet

$$\dot{x}(t) = (A - BL)x(t) + Br(t)$$

$$y(t) = Cx(t)$$

kan nu skapas t ex med funktionen `ss`.

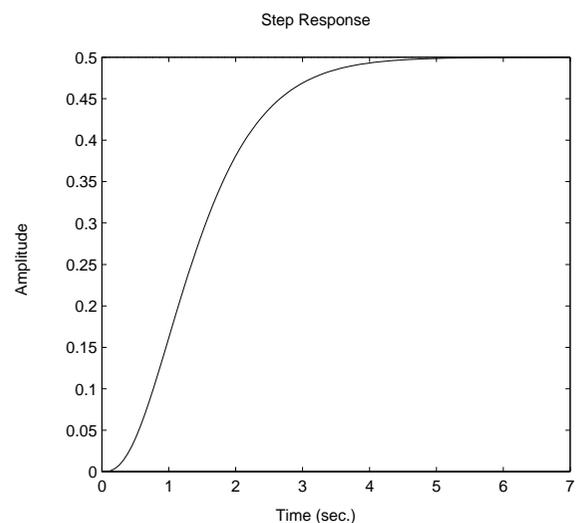
Generera tillståndsbeskrivningen för det återkopplade systemet. Kontrollera att polerna placerats på önskat sätt.

```
>> Gc = ss( G.a - G.b * L, G.b, G.c, 0 );  
>> eig( Gc.a )  
  
ans =  
  
-2.0200  
-2.0000  
-1.9800
```

Det återkopplade systemets stegsvar kan nu beräknas och ritas upp med funktionen `step`.

Beräkna och rita upp det återkopplade systemets stegsvar.

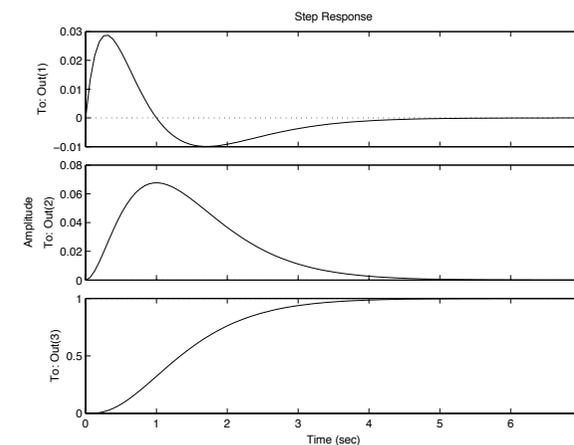
```
>> step( Gc )
```



På detta sätt ser vi endast den utsignal som definieras av vektorn C . Vill vi studera samtliga tillstånd kan detta göras genom att låta C vara en enhetsmatris med dimension lika med systemets ordningstal.

Skapa det återkopplade systemet på nytt, men med samtliga tre tillstånd som utsignaler.

```
>> Gc = ss( G.a - G.b * L, G.b, eye(3), 0 );  
>> step( Gc )
```



För att beräkna linjärkvadratisk tillståndsåterkoppling kan funktionen `lqr` användas.

11 Sammanfattning av kommandon

11.1 Användbara kommandon i Control System Toolbox

<code>tf</code>	System på överföringsfunktionsform
<code>ss</code>	System på tillståndsform
<code>pole</code>	Poler
<code>step</code>	Stegsvar
<code>tzero</code>	Nollställen
<code>feedback</code>	Återkoppling
<code>nyquist</code>	Nyquistdiagram
<code>bode</code>	Bodediagram
<code>bodemag</code>	Bodediagrammets amplitudkurva
<code>sigma</code>	Generalisering av bodemag
<code>margin</code>	Bodediagram och stabilitetsmarginaler
<code>rlocus</code>	Rotort
<code>lsim</code>	Simulering med godtycklig insignal
<code>place</code>	Polplacerande tillståndsåterkoppling
<code>lqr</code>	Linjärvadratisk tillståndsåterkoppling
<code>ctrb</code>	Styrbarhetsmatris
<code>obsv</code>	Observerbarhetsmatris
<code>ltiview</code>	Startar LTI Viewer
<code>pzmap</code>	Pol-nollställediagram
<code>minreal</code>	Förkortning av gemensamma faktorer
<code>sisotool</code>	Grafiskt gränssnitt

11.2 Användbara MATLAB-kommandon

<code>abs</code>	Absolutbelopp
<code>eig</code>	Egenvärden
<code>conv</code>	Polynommultiplikation
<code>det</code>	Determinant
<code>diag</code>	Diagonalmatris
<code>imag</code>	Imaginär del
<code>inv</code>	Matrisinvers
<code>real</code>	Realdel
<code>roots</code>	Rötter till polynom
<code>grid</code>	Nät i figurer
<code>hold</code>	Frysning av figur
<code>loglog</code>	Diagram i log-log skala
<code>plot</code>	Diagram i linjär skala
<code>cd</code>	Byte av bibliotek
<code>dir</code>	Listning av bibliotek
<code>clear</code>	Radering av variabler och funktioner i arbetsminnet
<code>load</code>	Inläsning av variabler från fil
<code>save</code>	Lagring av variabler på fil
<code>who</code>	Listning av variabler i arbetsminnet
<code>helpdesk</code>	Startar HTML-baserad hjälpfunktion