

EP2200 Queueing theory and teletraffic systems

Lectures 5-7

Summary of $M/M/*/*$ systems

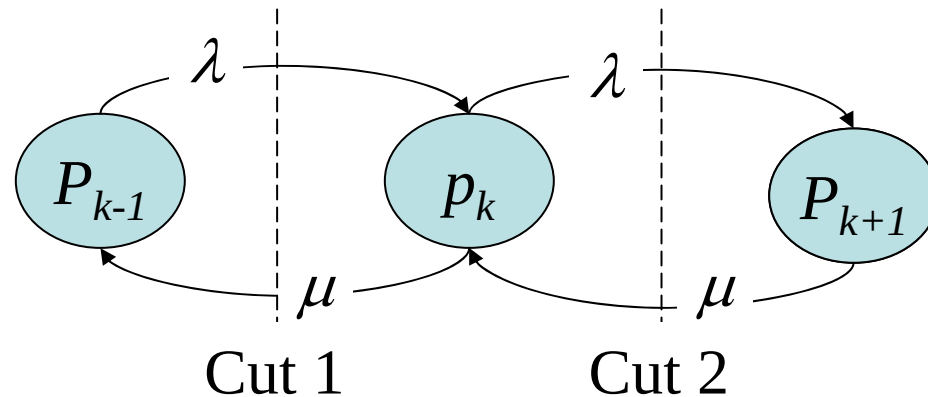
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M/M/1

- Single server, infinite waiting room
- Service times are exponentially distributed
$$P(X < x) = B(x) = 1 - e^{-\mu x}$$
- Arrival process Poisson
 - Models a population of independent customers
 - Each customer arrives to the system at a low rate
 - The total arrival process tends towards a Poisson process for a large population
- The queueing system can be modelled by a homogeneous birth-death process

Local balance equations

- Total transition rates from one part of the chain must balance the transition rates from the other part in stationarity



$$\text{Cut 1: } \lambda p_{k-1} = \mu p_k \Rightarrow p_k = \frac{\lambda}{\mu} p_{k-1}$$

$$\text{Cut 2: } \lambda p_k = \mu p_{k+1} \Rightarrow p_{k+1} = \frac{\lambda}{\mu} p_k = \left(\frac{\lambda}{\mu} \right)^2 p_{k-1}$$

Performance results

- The system is in state k with probability $p_k = (1-\rho)\rho^k$
- An arriving customer finds k customers in the system with probability p_k
 - PASTA: Poisson Arrivals See Time Averages
- Expected number of customers in the system is $N = \rho/(1-\rho)$
 - Time measures by Little's law
- Waiting time distribution
 - In transform domain and in time domain

$$L(f_w(t)) = \sum_{k=0}^{\infty} \left(\frac{\mu}{s + \mu} \right)^k (1-\rho)\rho^k$$

$$P(W < t) = W(t) = 1 - \rho e^{-(\mu-\lambda)t}, t \geq 0$$

M/M/1/K systems

- Poisson arrival, exponential service time, 1 servers, finite buffer capacity
- State transition diagram:
 - K+1 states
 - $\lambda_i = \lambda$, for $i \leq K$
 - $\mu_i = \mu$, for $i > 0$
- State probabilities in equilibrium and blocking probability from the local balance equations

$$p_k = \frac{\rho^k (1 - \rho)}{1 - \rho^{K+1}}, \quad P(\text{block}) = p_K$$

$$\lambda_{\text{eff}} = \lambda(1 - p_K)$$

$$\bar{N} = \sum_{k=0}^K k p_k = \frac{\rho}{1 - \rho} (1 - (K + 1) p_K), \quad \bar{T} = \bar{N} / \lambda_{\text{eff}}$$

M/M/m/m - loss systems (Erlang loss systems)

- Poisson arrival, exponential service time, m identical servers, no buffer,
- Offered load: $a = \lambda/\mu$
- State transition diagram:
 - m+1 states
 - $\lambda_i = \lambda$
 - $\mu_i = i\mu$
- State probabilities and performance measures

$$p_k = \frac{a^k/k!}{\sum_{i=0}^m a^i/i!}, \quad P(\text{block}) = E_m(a) = B(m, a) = p_m \quad (\text{Erlang-B form})$$

$$N = \sum_{k=0}^m k p_k = a(1 - p_m)$$

M/M/m systems

- Poisson arrival, exponential service time, m identical and independent servers, infinite buffer capacity
- Offered load $a = \lambda/\mu$ [Erlang], $a < m$ for stability
- State transition diagram:
 - infinite states
 - $\lambda_i = \lambda$
 - $\mu_i = i\mu$, for $0 < i \leq m$
 - $\mu_i = m\mu$, for $i > m$
- Probability of waiting and waiting time distribution

$$P(\text{wait}) = D_m(a) = C(m, a) = \sum_{k=m}^{\infty} p_k$$

$$D_m(a) = \frac{mE_m(a)}{m - a(1 - E_m(a))} \quad (\text{Erlang-C form})$$

$$P(W < t) = W(t) = 1 - D_m(a)e^{-(m\mu - \lambda)t}, t \geq 0$$

M/M/m/m/C – finite population Engset loss system

- Exponential service time, m identical servers, infinite buffer capacity
- BUT: finite population – can not be modeled with state independent arrivals
- Modeling a single user:
 - thinking time $\text{Exp}(\lambda)$
 - holding time (or service time) $\text{Exp}(\mu)$
 - after blocked call new thinking time
- Markov-chain model:
 - $\lambda_i = (C-i)\lambda$
 - $\mu_i = i\mu$
 - p_i from the balance equations

M/M/m/m/C – finite population

- Time blocking: the proportion of time in blocking state = p_m
- Call blocking: the probability that an arriving call gets blocked = a_m
- Call blocking $\neq$ time blocking

$$a_k = \frac{\lambda_k p_k}{\sum_{i=0}^m \lambda_i p_i}$$

- Effective load and average number of active users

$$\lambda_{eff} = \sum_{i=0}^{m-1} (C-i) \lambda p_i$$

$$\bar{N} = \lambda_{eff} / \mu$$