

EP2200 Queueing theory and teletraffic systems

Lecture 9 M/G/1 systems

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The M/G/1 queue

- Arrival process memoryless (Poisson(λ))
- Service time general, identical, independent, $f(x)$
- Single server
- M/E_r/1 and M/H_r/1 are specific cases, results for M/G/1 can be used
- Rules we can “use” from the Markovian systems
 - $\rho = \lambda E[x] < 1$ for stability (single server, no blocking)
 - Little: $N = \lambda T$
 - PASTA
- Problem: $N(t)$, the number of customers in the system does not give a Markov chain
 - If we consider the system when a new customer arrives, then
 - then the remaining (residual) service time of the customer under service depends on the past of the process (on the elapsed service time)

The M/G/1 queue

- Solution methods
 - Average measures N , T , etc.
 - Mean value analysis
 - Distribution of the number of customers, waiting time, etc.
 - Study the system at time points $t_0, t_1, t_2, \dots$ when a customer departs, and extend for all points of time
 - Embedded Markov chain

The M/G/1 mean value analysis

Pollaczek-Khinchin mean formulas

- To calculate average measures
- We start with the average waiting time:
 - the service of the waiting customers + the remaining (or residual service) of the customer in the service unit
 - Average remaining service time: R_s
 - First conditional average waiting time, then uncondition

$$W_k = R_{s,k} + \sum_{i=1}^{k-1} X_i, \quad k \geq 1$$

$$E[W_k] = E[R_{s,k}] + (k-1)E[X], \quad k \geq 1 \quad (\text{average waiting time for customer arriving at state } k)$$

$$W = E[W] = \sum_{k=0}^{\infty} p_k E[W_k] = p_0 0 + \sum_{k=1}^{\infty} p_k E[R_{s,k}] + \sum_{k=1}^{\infty} p_k (k-1)E[X]$$

$$R_s = \sum_{k=0}^{\infty} p_k E[R_{s,k}], \quad r_{s,0} = 0 \quad (\text{average includes 0 remaining service times at state 0})$$

$$W = R_s + N_q E[X]$$

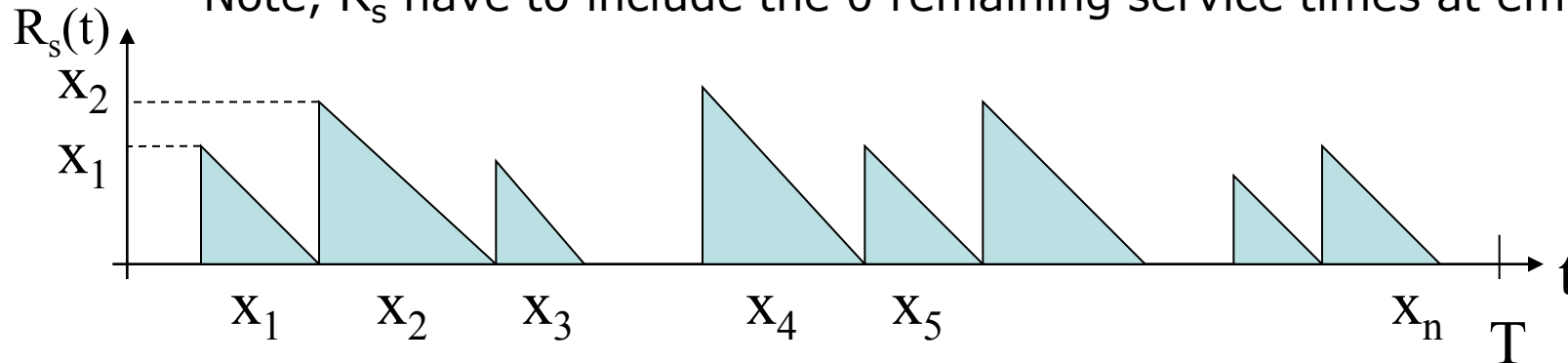
$$W = R_s + W\lambda E[X]$$

$$W = \frac{R_s}{1 - \rho}$$

The M/G/1 mean value analysis

Pollaczek-Khinchin mean formulas

- We have to derive the average remaining service time R_s :
 - n : number of services in a large T = number of Poisson arrivals: $n=\lambda T$ (since the system is stable)
 - $T \rightarrow \infty$ and $n \rightarrow \infty$
 - Note, R_s have to include the 0 remaining service times at empty system.



$$R_s = E[R_s(t)] = \frac{1}{T} \sum_{i=1}^n \frac{1}{2} X_i^2 = \frac{\lambda}{n} \frac{1}{2} \sum_{i=1}^n X_i^2 = \frac{\lambda}{2} E[X^2]$$

$$W = \frac{R_s}{1-\rho} = \frac{\lambda E[X^2]}{2(1-\rho)}$$

Pollaczek-Khinchin mean formula
for the waiting time

The M/G/1 mean value analysis

Pollaczek-Khinchin mean formulas

- From W you can derive T , N , N_q with Little's theorem
- Comments:
 - W depends on the first and the second moment of the service time only
 - Mean values increase with variance (cost of randomness)

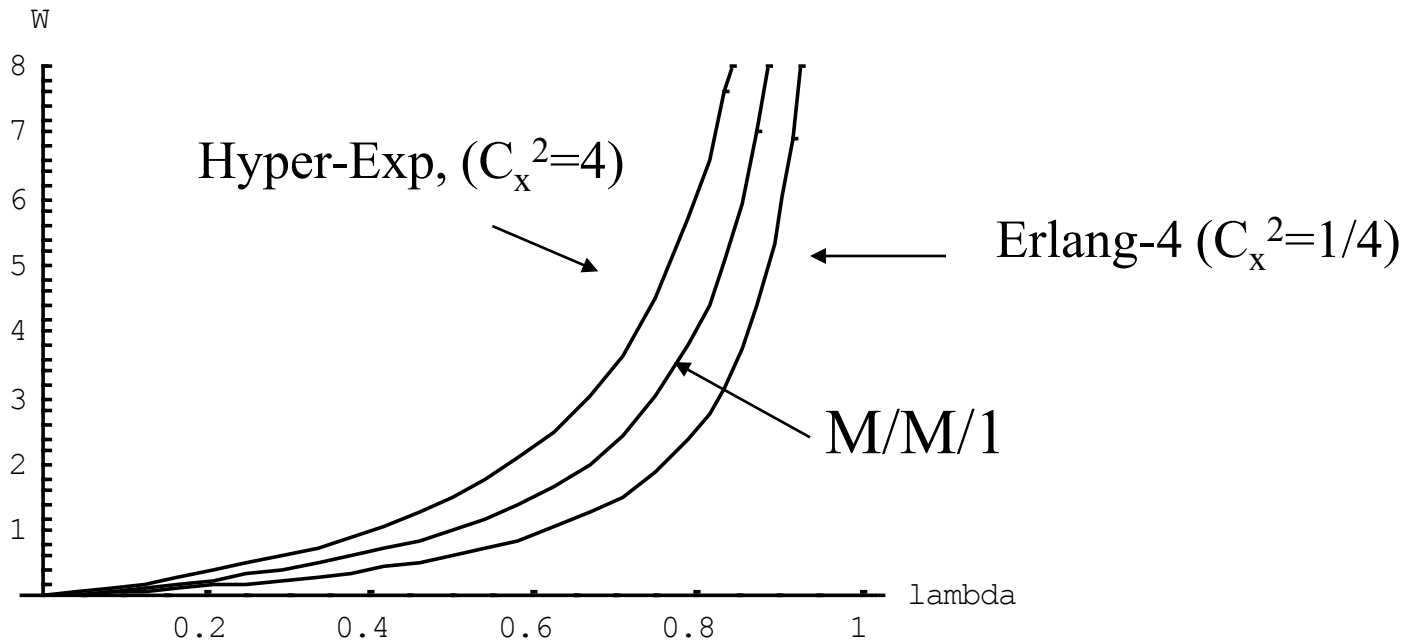
$$W = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\lambda(E[X]^2 + V[X])}{2(1-\rho)} = \frac{\lambda(E[X]^2 + V[X] \frac{E[X]^2}{E[X]^2})}{2(1-\rho)} = \frac{\rho E[X]}{2(1-\rho)} (1 + C_x^2)$$

$$M / M / 1: \quad C_x^2 = 1, \quad W = \frac{\rho E[X]}{(1-\rho)}$$

$$M / D / 1: \quad C_x^2 = 0, \quad W = \frac{\rho E[X]}{2(1-\rho)}$$

M/G/1 waiting time

$$W = \frac{\rho E[X]}{2(1-\rho)} (1 + C_x^2)$$



The M/G/1 mean value analysis

Pollaczek-Khinchin mean formulas

Group work:

- Consider the following system:
 - Single server, infinite buffer
 - Poisson arrival process, 0.1 customer per minute
 - Service process: sum of two exponential steps, with mean times 1 minute and 2 minutes.
 - Calculate the mean waiting time

$$W = \frac{\lambda E[x^2]}{2(1-\rho)} = \frac{\rho E[x]}{2(1-\rho)} (1 + C_x^2)$$

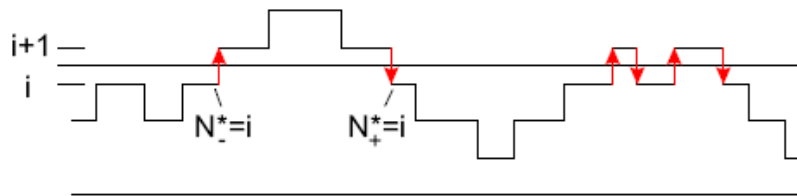
Distribution of number of customers in the system

*** Comment: called as queue-length in the Virtamo notes! ***

- The number of customers, N_t is not a Markov process
 - the residual service time is not memoryless
- We can model the system at departure time and extend the results to all points of time:
 - in the case of Poisson arrival the **distribution of N at departure** times is the same as at arbitrary points of time (PASTA)
 - if we are lucky then N_t follows a **discrete time Markov process** at departure times
 - since this discrete time Markov chain is rather complex, we can express the transform form (z-transform) of the distribution of the number of customers in the system.

Distribution of number of customers in the system

- In the case of Poisson arrival the distribution of N at departure times is the same as at arbitrary points of time (PASTA)
 - PASTA is proved for arrival instants
 - however, departure instants see the same queue length distribution



- Let us follow $N_k, N_{k+1}, N_{k+2} \dots$, that is, the number of customers in the system after departures

N_k : number of customers after the departure of a customer k

V_k : number of arrivals during the service time of customer k ,

$b(x)$ is the service time distribution, then:

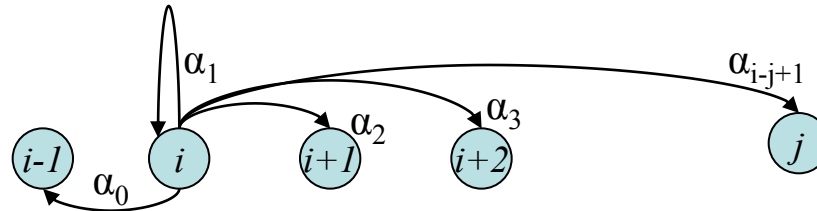
$$N_{k+1} = \begin{cases} N_k - 1 + V_{k+1} & N_k \geq 1 \\ V_{k+1} & N_k = 0 \end{cases}$$

$\Rightarrow N_{k+1}$ depends only on N_k and V ,
 V is independent from k

→ Discrete time Markov Process

$$\alpha_i = P(V = i) = \int \frac{(\lambda x)^i}{i!} e^{-\lambda x} b(x) dx$$

M/G/1 number of customers in the system



- Expressing the steady state of the Markov-chain describing N , we get the z -transform of the distribution of N
- Pollaczek-Khinchin transform form:

$$Q(z) = B^*(\lambda - \lambda z) \frac{(1 - \rho)(1 - z)}{B^*(\lambda - \lambda z) - z}$$

- where: $\rho = \lambda E[X]$ and $B^*(s)$ is the Laplace transform of the service time distribution. ($S^*(s)$ in the Virtamo notes)
- Distribution of N with inverse transform, or moments of the distribution.
- E.g., M/M/1

M/G/1 system time distribution

- Without proof:
- Pollaczek-Khinchin transform form for the system time and waiting time:

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}$$

$$T^*(s) = B^*(s) \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}$$

- where: $\rho = \lambda E[x]$ and $B^*(s)$ is the Laplace transform of the service time distribution. ($S^*(s)$ in the Virtamo notes)
- E.g., M/M/1 system time

The M/G/1 mean value analysis

Pollaczek-Khinchin mean formulas

Group work:

- Consider the following system:
 - Single server, infinite buffer
 - Poisson arrival process, 0.1 customer per minute
 - Service process: sum of two exponential steps, with mean times 1 minute and 2 minutes.
 - Give the Laplace transform of the waiting time, calculate the mean waiting time

$$W^*(s) = \frac{s(1-\rho)}{s - \lambda + \lambda B^*(s)}$$

M/G/1

- Requirements for the exam:
- Derive and use P-K mean value formulas
- Use P-K transform forms
 - typically: for given service time distribution give the transform forms, calculate moments
- Do not forget: M/M/1, M/D/1, M/E_r/1 and M/H_r/1 are specific cases of M/G/1