# EP2200 Queueing theory and teletraffic systems

## Queueing networks

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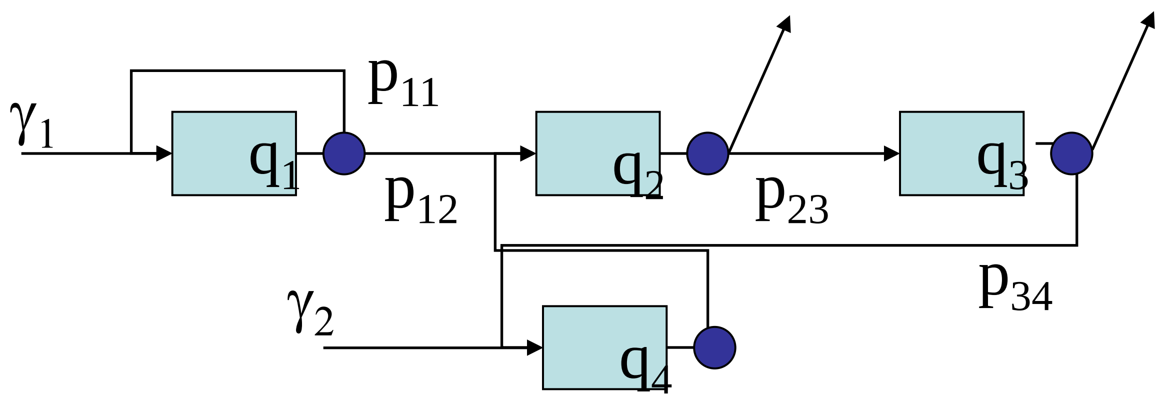
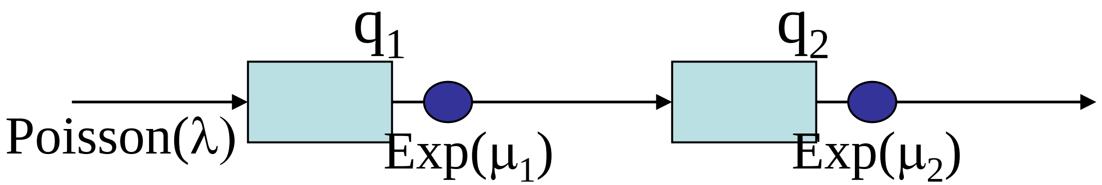
# Open and closed queuing networks

- Queuing network: network of queuing systems
  - E.g., data packets traversing the network from router to router
- Open and closed networks
  - Open queuing network: customers arrive and leave the network (typical application: data communication)
  - Closed queueing networks: in and out flows are missing – constant number of customers circulate in the network (application: computer systems)



# Open queuing networks- A tandem system



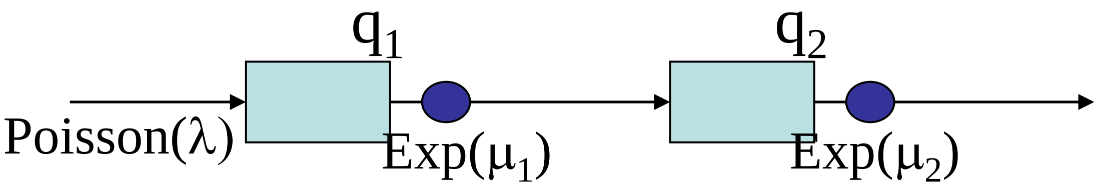

- The most simple open queuing network
- Assume a Poisson arrival process and independent, exponentially distributed service times
- What is the departure process from queue 1?
  - Interdeparture time:
    - Customer leaves queue behind: time of service of next customer
    - Customer leaves empty system behind: time to next arrival + time of service

$$L(f_\tau(t)) = \rho \frac{\mu}{s+\mu} + (1-\rho)\frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu} =$$

$$\frac{\rho\mu(s+\lambda)+\lambda\mu-\rho\lambda\mu}{(s+\lambda)(s+\mu)} = \frac{\lambda s+\lambda^2+\lambda\mu-\lambda^2}{(s+\lambda)(s+\mu)} = \frac{\lambda}{s+\lambda}$$

  - Departure process: Poisson ($\lambda$)!
- Same for M/M/m, but not for systems with losses and not for M/G/m systems

# A tandem system




- The tandem system:
  - Queue 1 is an M/M/1 queue
  - The departure process from queue 1 is a Poisson process with intensity $\lambda$
  - Thus, queue 2 is also a M/M/1-queue, the two queues are independent.

  - State of the tandem queue: $(n_1,n_2)$=(customers in system 1, customers in system 2)
  - Then state probability $p(n_1,n_2)=p_1(n_1)p_2(n_2)$ – product form solution
    - M/M/1 queues: $p(n_1, n_2) = (1-\rho_1){\rho_1}^{n_1} \cdot (1-\rho_2){\rho_2}^{n_2}$
    - prove also with a two dimensional chain (Virtamo notes)
  - Product form valid for M/M/m-system (Burke's theorem )

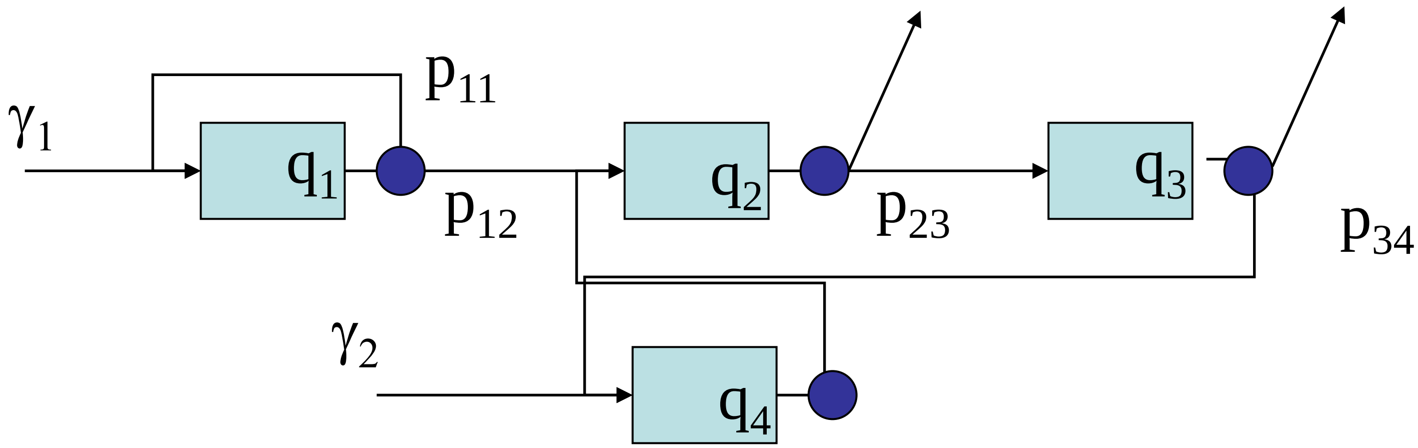
# Modeling communication networks - note on the indepdence assumption

- For two transmission links in series, queue 2 is not a M/M/1-queue
  - Correlation between service times of a customer in the two queues – determined by the packet length and the link transmission rate
  - Correlation between arrival and service times
    - interarrival time of arrivals i, i+1 ≥ service time of customer i
    - There will not be any queuing in queue 2 if the transmission rate at queue 2 is larger
  - Product form solution does not apply

- Kleinrock's assumption on independence
  - Traffic to a queue comes from several upstream queues
    - Superposition of Poisson processes give a Poisson process
  - Traffic from a queue is spread randomly to several downstream queues
    - Partial processes are Poisson with intensity $p_i \lambda$ ($\Sigma p_i$=1)
  - It is assumed to create independence
  - Product form solution applies
  - E.g., network of large routers

# Jackson’s queuing networks

- Open queuing network
  - arrivals to the network
  - from all arrival point a departure point is reachable
- M queues with infinite storage and m exponential servers
  - Even finite storage if “last queue” in the networks
- Customers from outside of the network arrive to node *i* as a Poisson process with intensity $\gamma_i \geq 0$
- The service times are independent of the arrival process (and service times in other queues)
- A customer comes from node *i* to node *j* after service with the probability $p_{ij}$ or leaves the network with the probability $p_{i0}=1-\sum p_{ij}$. Allows feedback.



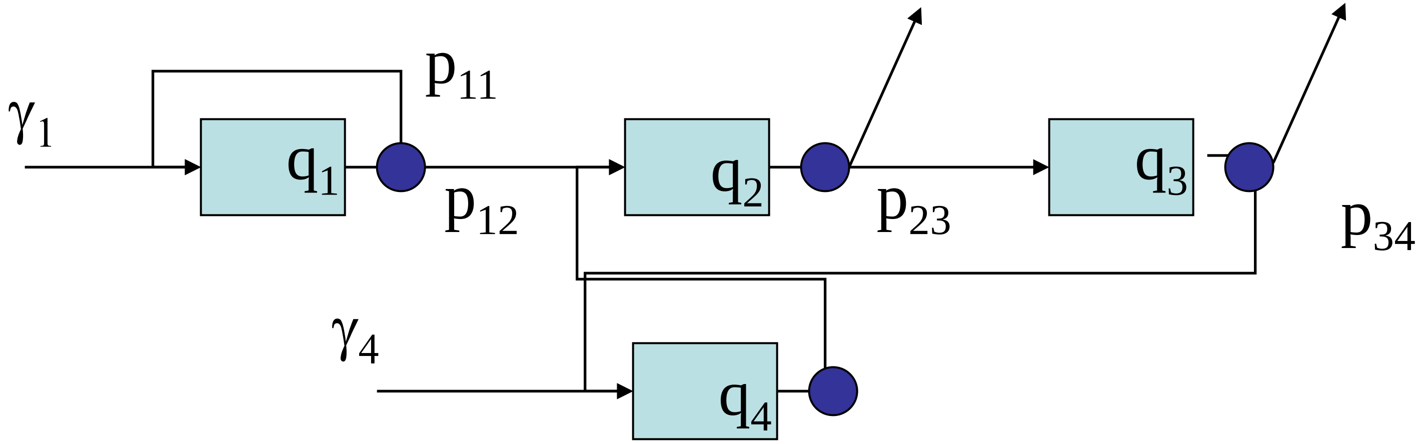
# Jackson’s queuing networks

- Flow conservation: arrival intensity to node *j* is

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

- Jacksons theorem: The distribution of number of customers in the network has *product form* – queues behave as independent M/M/m queues! (we do not prove – see the Virtamo notes)

$$P(n_1, n_2, \ldots, n_M) = P_1(n_1) \cdots P_M(n_M)$$

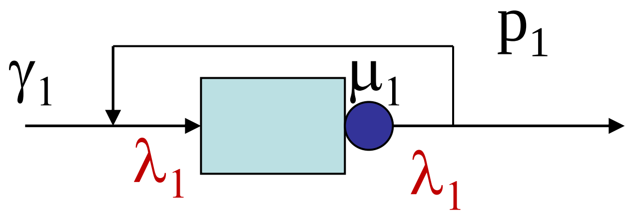


# Jackson's queuing networks

- Flow conservation: arrival intensity to node $j$:

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

- Example 1: single feedback queue




$$\lambda_1 = \gamma_1 + \lambda_1 p_1$$

$$\lambda_1 = \frac{\gamma_1}{1 - p_1}$$

  - Performance measures as if it would be M/M/1
  - Though the arrival process is not Poisson
  - Stability: $\lambda_1 / \mu_1 < 1$

# Jackson's queuing networks

- Arrival intensity and state probability

$$\lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

$$P(n_1, n_2, \ldots, n_M) = P_1(n_1) \cdots P_M(n_M)$$

- For the M/M/1 case:

$$P(n_i) = (1 - \rho_i)\rho_i^{n_i} \text{ and } \rho_i = \lambda_i / \mu_i < 1$$

- Example 2
  - calculate arrival intensities
  - calculate the probability that the network is empty
  - calculate the probability that there is one customer in the network

# Jackson's queuing networks
# Mean performance measures

- Little's theorem applies to the entire network!
- The mean number of customers in the network and the average time spent in the network are (e.g., M/M/1 case)

$$N = \sum_{j=1}^{M} N_j = \sum_{j=1}^{M} \frac{\rho_j}{1-\rho_j}$$

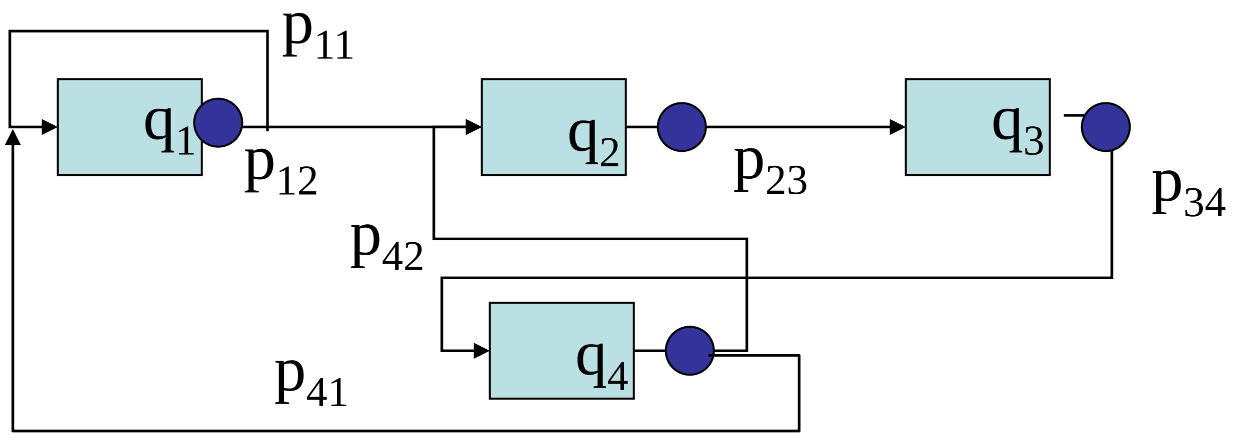
$$T = N / \sum_{j=1}^{M} \gamma_j$$

- The mean number of nodes a customer visits before leaving:
  - {Sum arrival intensity to the queues} / {arrival intensity to the network}

$$V = \sum_{j=1}^{M} \lambda_j / \sum_{j=1}^{M} \gamma_j, \quad \lambda_j = \gamma_j + \sum_{i=1}^{M} \lambda_i p_{ij}$$

# Closed Jackson's queuing networks

- Not exam material this year
- Closed queuing network
- M queues with infinite storage and m exponential servers
- K customers circulating in the network, no arrivals and departures
- The service times are independent of the arrival process (and service times in other queues)
- A customer comes from node *i* to node *j* after service with the probability $p_{ij}$
- Queues can not be independent, since there is a fixed number of customers



# Closed Jackson's queuing networks

- Flow conservation: arrival intensity to node *j:*

$$\lambda_j = \sum_{i=1}^{M} \lambda_i p_{ij} \qquad (*)$$

- Limited set of states, since the sum of the customers is constant K:

$$S = \{(n_1, n_2, \ldots, n_M), \quad n_i \geq 0, \sum_{i=1}^{M} n_i = K\}$$

- MC based solution: state: vector of number of customers per queue - complex
- Algorithmic solution – e.g., M/M/1
  - (*) gives a set of dependent equations, with solution of e.g.:

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots \lambda_M\} = \alpha\{1, e_2, e_3, e_4, \ldots e_M\}$$

  - we have to select the one that gives sum of network state probabilities equal to one
  - Gordon-Newell: state probabilities, without calculating arrival intensities (without proof)

$$P(\underline{n}) = \frac{1}{G_M^K} \prod_{i=1}^{M} \left( \frac{e_i}{\mu_i} \right)^{n_i}, \quad G_M^K = \sum_{\underline{n} \in S} \prod_{i=1}^{M} \left( \frac{e_i}{\mu_i} \right)^{n_i}$$

# Summary

- Queuing networks:
  - set of queuing systems
  - customers move from queue to queue
- Applied to networking problems: independence of queues have to be ensured
- Open queuing networks
  - Burke: Output process of an M/M/m queue is Poissonian
  - Jackson theorem: network state probability has product form if M/M/m queues
- Closed queuing networks – not exam material
  - Number of customers constant
  - State of queues is dependent – Gordon-Newell normalization