261318 Exam solutions 2004.12.16.

1,

a,

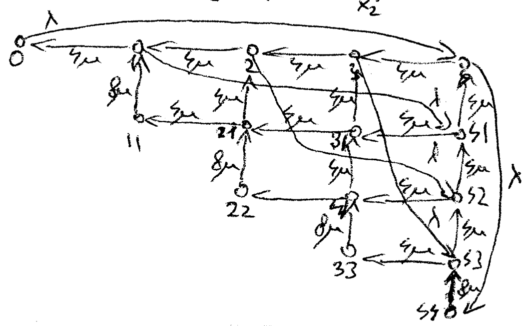
Poisson: Large number of independent sources generate arrivals

Durations:

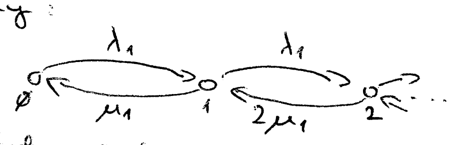
$\bar{x}_1 = 40$ min $\quad c_1 = 1 \quad$ Exponential, intensity $\mu_1 = \frac{1}{\bar{x}_1}$

$\bar{x}_2 = 360$ min $\quad c_2 = \frac{1}{2} \quad$ Erlang-4, intensity $\mu_2 = \frac{1}{\bar{x}_2}$

Night:



Day:



States: # of cars

b,

Day: $M/M/\infty$

$\lambda_1 = \frac{3}{2}$/min, $\mu_1 = \frac{1}{40}$/min $\Rightarrow \rho_1 = \frac{\lambda_1}{\mu_1} = 60 \qquad \bar{N}_1 = \rho_1 = 60$

Night: $M/G/\infty$

$\lambda_2 = \frac{1}{5}$/min, $\mu_2 = \frac{1}{360}$/min $\Rightarrow \rho_2 = \frac{\lambda_2}{\mu_2} = 72 \qquad \bar{N}_2 = \rho_2 = 72$

(due to insensitivity)

c, Day: $M/M/m/m$

$\rho_1 = 60 \qquad E_m(60) < 0.02 \Rightarrow m = 71$

Night: $M/G/m/m$

$\rho_2 = 72 \qquad E_m(72) < 0.1 \Rightarrow m = 71$

$\}\ m = 71$ they are the same

d,

Daytime: $M/M/m$

$$D_m(\rho) = \frac{m \cdot E_m(\rho)}{m - \rho(1 - E_m(\rho))}\Bigg|_{\substack{m=71\\ \rho=60}} = \frac{71 \cdot 0.019671}{71 - 60(1-0.019671)} = 0.1147$$

$D_m(60) < 0.1 \Rightarrow m = ?$

$m = 72$: $\quad D_{72}(60) = \frac{72 \cdot 0.016125}{72 - 60(1 - 0.016128)} = 0.0895$

$m = 72$ is enough

$N_q = \frac{\lambda_1}{m\mu_1 - \lambda_1} \cdot P(\text{wait}) = 5 \cdot 0.0895 = 0.4475$ cars

$P(W>5) = D_m(\rho) \cdot e^{-\mu(m-\rho)5} = 0.0895 \cdot e^{-1.5} = 0.02$

e,

Departure: day: in case (d) Poisson(λ_1) due to Burke's theorem (M)

(c) general (G)

night: general (G)

2)

$\lambda = 1000/s$

a) $\mu = 1250/s \quad \Rightarrow \quad E[B] = \frac{10^6}{1250} = 800 \text{ ops}$

$$\sigma_B^2 = C_B^2 \cdot E[B]^2 = 760000 \text{ ops}^2$$

$$E[B^2] = \sigma_B^2 + E[B]^2 = 1400000 \text{ ops}^2$$

$$\left.\begin{array}{l} 0.4 \cdot \frac{1}{\mu_1} + 0.6 \cdot \frac{1}{\mu_2} = 800 \\ 0.4 \cdot \frac{2}{\mu_1^2} + 0.6 \cdot \frac{2}{\mu_2^2} = 1400000 \end{array}\right\} \Rightarrow \begin{array}{l} \frac{1}{\mu_1} = 500 \text{ ops} \\ \frac{1}{\mu_2} = 1000 \text{ ops} \end{array}$$

$$\bar{x} = 0.4 \cdot \frac{500}{10^6} + 0.6 \cdot \frac{1000}{10^6} = 8 \cdot 10^{-4} s$$

$$\overline{x^2} = 0.4 \cdot 2 \left(\frac{500}{10^6}\right)^2 + 0.6 \cdot 2 \cdot \left(\frac{1000}{10^6}\right)^2 = 1.4 \cdot 10^{-6} s^2$$

$$\rho = \lambda \cdot \bar{x} = 10^3 \cdot 8 \cdot 10^{-4} = 0.8 \Rightarrow W = \frac{\lambda \overline{x^2}}{2(1-\rho)} = \frac{10^3 \cdot 1.4 \cdot 10^{-6}}{2(1-0.8)} = 3.5 \cdot 10^{-3} s$$

b)

$$R = \frac{1000}{2} \left\{ 0.4 \cdot 2 \cdot \left(\frac{500}{10^6}\right)^2 + 0.6 \cdot 2 \cdot \left(\frac{1000}{10^6}\right)^2 \right\} = 0.7 \cdot 10^{-3}$$

$$W_1 = \frac{0.7 \cdot 10^{-3}}{1-\rho_1} = 0.875 \cdot 10^{-3} s \qquad \left(\rho_1 = 0.4 \cdot 10^3 \cdot \frac{500}{10^6} = 0.2\right)$$

$$W_2 = \frac{0.7 \cdot 10^{-3}}{(1-\rho_1)(1-\rho_1-\rho_2)} = 4.375 \cdot 10^{-3} s \qquad W = 0.4 \cdot W_1 + 0.6 \cdot W_2 = 2.97 \cdot 10^{-3} s$$

c)

$$W_1 = \frac{R}{1-\rho_1} + \frac{\overline{V^2}}{2\bar{V}} = 1.69 \cdot 10^{-3} s \qquad \bar{V} = 0.1 + 0.9 \text{ ms} = 10^{-3} s$$

$$W_2 = \frac{R}{(1-\rho_1)(1-\rho_1-\rho_2)} + \frac{\overline{V^2}}{2\bar{V}} = 5.19 \cdot 10^{-3} s \qquad \overline{V^2} = 0.1^2 + 2 \cdot 0.9^2 \cdot 10^{-6} s^2 = 1.63 \cdot 10^{-6}$$

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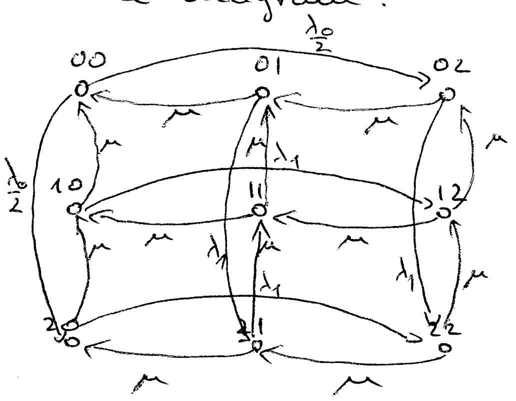
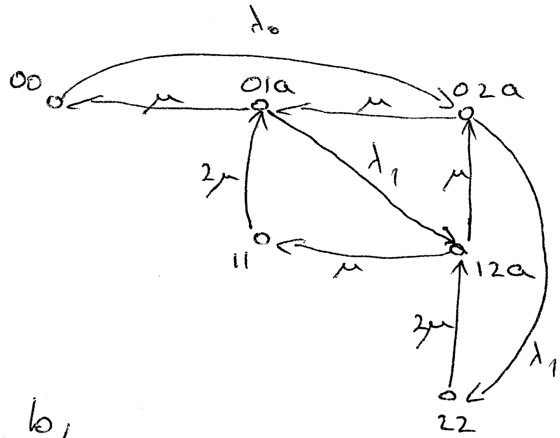
3,

a, $M/E_2/2/2/3$ $\lambda_0 = 1$, $\lambda_1 = \frac{2}{3}$, $\lambda_2 = \frac{1}{3}$, $\mu = 1$

State diagram:





$$p_{00}\left(\frac{\lambda_0}{2}+\frac{\lambda_0}{2}\right) = p_{01}\mu + p_{10}\mu$$

$$\left.\begin{array}{l} p_{01}(\lambda_1+\mu) = p_{02}\mu + p_{11}\mu \\ p_{10}(\lambda_1+\mu) = p_{20}\mu + p_{11}\mu \end{array}\right\} + \quad p_{01a} = p_{01}+p_{10}$$

$$\left.\begin{array}{l} p_{02}(\lambda_1+\mu) = p_{12}\mu + p_{00}\frac{\lambda_0}{2} \\ p_{20}(\lambda_1+\mu) = p_{21}\mu + p_{00}\frac{\lambda_0}{2} \end{array}\right\} + \quad p_{02a} = p_{02}+p_{20}$$

$$\left.\begin{array}{l} p_{12}(\mu+\mu) = p_{22}\mu + p_{10}\lambda_1 \\ p_{21}(\mu+\mu) = p_{22}\mu + p_{01}\lambda_1 \end{array}\right\} + \quad p_{12a} = p_{12}+p_{21}$$

$$p_{11}(\mu+\mu) = p_{21}\mu + p_{12}\mu$$

$$p_{22}(\mu+\mu) = p_{20}\lambda_1 + p_{02}\lambda_1$$

b,

I $p_{00}\lambda_0 = p_{01a}\mu \Rightarrow p_{01a} = p_{00}\frac{\lambda_0}{\mu}$

II. $p_{01a}(\lambda_1+\mu) = p_{02a}\mu + p_{11}2\mu$

III. $p_{02a}(\lambda_1+\mu) = p_{00}\lambda_0 + p_{12a}\mu$

IV $p_{11}2\mu = p_{12a}\mu$

V $p_{12a}2\mu = p_{01a}\lambda_1 + p_{22}2\mu$

VI $p_{22}2\mu = p_{02a}\lambda_1$

I, II, IV:

$$\left.p_{00}\left(\frac{\lambda_0}{\mu}\lambda_1 + \frac{\lambda_0}{\mu}\mu\right) = p_{02a}\mu + p_{12a}\mu\right\} - \quad \text{III}$$

$$p_{00}\left(\frac{\lambda_0\lambda_1}{\mu}+\lambda_0\right) - p_{02a}(\lambda_1+\mu) = p_{02a}\mu - p_{00}\lambda_0$$

$$p_{02a} = \frac{\frac{\lambda_0\lambda_1}{\mu}+\lambda_0}{\lambda_1+2\mu}p_{00} = p_{00}\frac{\lambda_0}{\mu}$$

V:

$$p_{12a}2\mu = p_{00}\frac{\lambda_0}{\mu}\lambda_1 + p_{00}\frac{\lambda_0}{\mu}\lambda_1$$

$$p_{12a} = p_{00}\frac{\lambda_0\lambda_1}{\mu^2}$$

IV: $p_{11} = p_{00}\frac{\lambda_0\lambda_1}{2\mu^2}$ VI: $p_{22} = p_{00}\frac{\lambda_0\lambda_1}{2\mu^2}$

$$p_{00} = \frac{1}{1+\frac{\lambda_0}{\mu}+\frac{\lambda_0}{\mu}+\frac{\lambda_0\lambda_1}{\mu^2}+\frac{\lambda_0\lambda_1}{2\mu^2}+\frac{\lambda_0\lambda_1}{2\mu^2}} = \frac{3}{13}$$

$p_{01a} = \frac{3}{13}$, $p_{02a} = \frac{3}{13}$, $p_{12a} = \frac{2}{13}$, $p_{11} = \frac{1}{13}$, $p_{22} = \frac{1}{13}$

c, $P\{\text{time blocking}\} = p_{11}+p_{12a}+p_{22} = \frac{4}{13}$

$$P\{\text{call blocking}\} = \frac{(p_{11}+p_{12a}+p_{22})\lambda_2}{p_{00}\lambda_0 + (p_{01a}+p_{02a})\lambda_1 + (p_{11}+p_{12a}+p_{22})\lambda_2} = \frac{4}{25}$$

d, Erlang-2 distributed:

$P(T>2) = P\{\emptyset \text{ service with intensity } \mu\} + P\{1 \text{ service with intensity } \mu\} =$

$= e^{-\mu 2} + \mu 2\, e^{-\mu 2} = e^{-2} + 2e^{-2} = 3 \cdot e^{-2} \approx 0.4$

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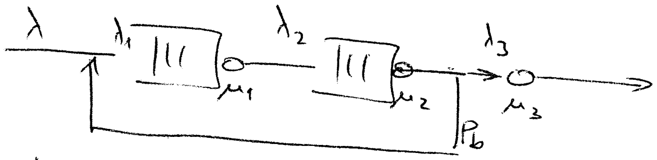
4,



$\lambda = 1, \quad \mu_1 = 2, \quad \mu_2 = 4, \quad \mu_3 = 3$

a,

$\lambda_1 = \lambda_2 \qquad \lambda_3 = \lambda$ in steady state

$\lambda = \lambda_2 (1 - P_b)$ where $P_b = 1 - \frac{1}{1 + \frac{\lambda_2}{\mu_3}} = \frac{\lambda_2}{\lambda_2 + \mu_3}$

$\lambda = \lambda_2 \frac{\mu_3}{\lambda_2 + \mu_3} \Rightarrow \lambda_2 = \frac{\lambda \mu_3}{\mu_3 - \lambda} = 1.5$

$p_0 = \frac{1}{1 + \frac{\lambda_2}{\mu_3}} = \frac{\mu_3}{\lambda_2 + \mu_3}$

$p_1 = 1 - p_0 = \frac{\lambda_2}{\lambda_2 + \mu_3}$

$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{1.5}{2} = 0.75, \quad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{1.5}{4} = \frac{3}{8}, \quad \rho_3 = \frac{\lambda_2}{\mu_3} = \frac{1.5}{3} = 0.5$

b, Two customers in the system (we assume product form solution)

$$P\{\text{two customers}\} = P_1(2) \cdot P_2(0) \cdot P_3(0) + P_1(1) \cdot P_2(1) \cdot P_3(0) + P_1(1) \cdot P_2(0) \cdot P_3(1) + P_1(0) \cdot P_2(2) \cdot P_3(0) + P_1(0) \cdot P_2(1) \cdot P_3(1) =$$

$$= 0.25 \cdot 0.75^2 \cdot \frac{5}{8} \cdot \frac{2}{3} + 0.25 \cdot 0.75 \cdot \frac{5}{8} \cdot \frac{3}{8} \cdot \frac{2}{3} + 0.25 \cdot 0.75 \cdot \frac{5}{8} \cdot \frac{1}{3} + 0.25 \cdot \frac{5}{8} \left(\frac{3}{8}\right)^2 \cdot \frac{2}{3} + 0.25 \cdot \frac{5}{8} \cdot \frac{3}{8} \cdot \frac{1}{3} = \frac{165}{1024}$$

c,

$$\left. \begin{array}{ll} N_1 = \frac{\rho_1}{1 - \rho_1} = 3 & N_3 = p_1 = \frac{1}{3} \\ N_2 = \frac{\rho_2}{1 - \rho_2} = \frac{3}{5} & \end{array} \right\} N = \frac{59}{15} \Rightarrow T = \frac{N}{\lambda} = \frac{59}{15}$$

d,

$$\left. \begin{array}{ll} \text{system } 1: & \lambda_1 < \mu_1 = 2 \\ 2: & \lambda_1 < \mu_2 = 4 \end{array} \right\} \lambda_1 = \lambda_2 < 2 \qquad \frac{\lambda \mu_3}{\mu_3 - \lambda} < 2 \Rightarrow \lambda < \frac{2 \mu_3}{2 + \mu_3} = \frac{6}{5}$$

e,

stability: $\lambda_1 < 1.25$

$\lambda_1 = \frac{\lambda}{1 - P_b} \Rightarrow P_b < \frac{1.25 - \lambda}{1.25} = 0.2$

$m = 2$: $\rho_{max} = \frac{5}{12} \approx 0.42$

$E_2(\rho_{max}) = 0.0585 < 0.2$

$m = 2$ is good

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5)

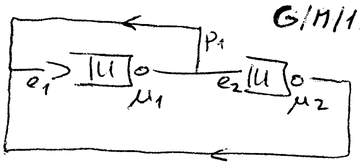
a) $M/G/2$

M : large population of independent users

$\lambda = 230/\text{min} = \frac{23}{6}/s$

$T = 13\,s$

$N = \lambda \cdot T = \frac{23}{6}/s \cdot 13s = \frac{299}{6} \approx 50$

b) 

$G/M/1//3$

$p_1 = 0.1$

$\mu_1 = 1$

$\mu_2 = \frac{1}{2}$

$e_2 = 0.9 e_1$

$e_1 = 10$

$e_2 = 9$

Possible states: | $G_2(3) \cdot P\{state\}$ | $P\{state\}$

3 0	$\left(\frac{10}{\mu_1}\right)^3$	0.0842
2 1	$\left(\frac{10}{\mu_1}\right)^2 \frac{9}{\mu_2}$	
1 2	$\frac{10}{\mu_1} \cdot \left(\frac{9}{\mu_2}\right)^2$	
0 3	$\left(\frac{9}{\mu_2}\right)^3$	

$$G_2(3) = \left(\frac{10}{\mu_1}\right)^3 + \left(\frac{10}{\mu_1}\right)^2 \cdot \frac{9}{\mu_2} + \frac{10}{\mu_1}\left(\frac{9}{\mu_2}\right)^2 + \left(\frac{9}{\mu_2}\right)^3 = 11872$$

c) $\frac{1}{\mu} = 20 \qquad \lambda = \frac{1}{5} \Rightarrow \rho = \frac{\lambda}{\mu} = 4 \qquad M/M/\infty$

Bandwith needed to download in 15 secs: $\frac{10 \cdot 2 \cdot 8\ \text{Mbit}}{15s} = 5.55\ \text{Mbps}$

There can be at most 2 persons in the shop (1 or 2):

$$P\{success\} = \frac{p_1 + p_2}{1 - p_0} = \frac{\frac{\rho}{1} \cdot e^{-\rho} + \frac{\rho^2}{2} \cdot e^{-\rho}}{1 - e^{-\rho}} = \frac{12 \cdot e^{-4}}{1 - e^{-4}} = 0.2239$$

d) $D/D/1$

random observer: $p_0 = \frac{1}{7} \qquad p_1 = \frac{6}{7}$

arriving chassis: $p_0 = 1 \qquad p_1 = 0$