Problem 1

- Poisson Arrivals , $\lambda = 100$ Jobs/hour
- Exp. service time, $\mu_0 = \mu = 0{,}05^{-1} = 20$ (3min = 0,05hours)
- 5 servers
- No queue

$$\begin{cases} \mu_1 = 20 \\ \mu_2 = 20\sqrt{2} \\ \mu_3 = 20\sqrt{3} \\ \mu_4 = 40 \\ \mu_5 = 20\sqrt{5} \end{cases}$$

- State Diagram : S_i : i jobs in the system

$$\begin{tikzcd}
S_0 \arrow[r, bend right, "100"'] & S_1 \arrow[l, bend right, "20"'] \arrow[r, bend right, "100"'] & S_2 \arrow[l, bend right, "40\sqrt{2}"'] \arrow[r, bend right, "100"'] & S_3 \arrow[l, bend right, "60\sqrt{3}"'] \arrow[r, bend right, "100"'] & S_4 \arrow[l, bend right, "160"'] \arrow[r, bend right, "100"'] & S_5 \arrow[l, bend right, "100\sqrt{5}"']
\end{tikzcd}$$

- Balance Equations

- $100 P_0 = 20 P_1 \longrightarrow P_1 = 5 P_0$ (1)
- $100 P_1 = 40\sqrt{2} P_2 \longrightarrow P_2 = \frac{100}{40\sqrt{2}} P_1 = \frac{100 \cdot 5}{40\sqrt{2}} P_0 = \frac{12{,}5\sqrt{2}}{2} P_0$ (2)
- $100 P_2 = 60\sqrt{3} P_3 \longrightarrow P_3 = \frac{100}{60\sqrt{3}} P_2 = \frac{100 \cdot 12{,}5\sqrt{2}}{120\sqrt{3}} P_0$ (3)
- $100 P_3 = 160 P_4 \longrightarrow P_4 = \frac{100}{160} \cdot \frac{100 \cdot 12{,}5\sqrt{2}}{120\sqrt{3}} P_0$ (4)
- $100 P_4 = 100\sqrt{5} P_5 \longrightarrow P_5 = \frac{\sqrt{5}}{5} P_4$ (5)
- $P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 1$ (6)

$\longrightarrow P_0 = \ldots = \boxed{0{,}0332}$

Time blocking : $P_5 = \boxed{0{,}07914}$

Call (job) blocking : P_5 (Homogeneous system)

b) Blocking Period $\sim \exp(100\sqrt{5})$, $\bar{x}_B = \frac{1}{100\sqrt{5}}$ hours $\boxed{= 0{,}268 \text{min}}$

c) Non-blocking period : $\frac{\bar{x}_B}{\bar{x}_B + \bar{x}_{NB}} = P_5 \Rightarrow \bar{x}_{NB} = \ldots = \boxed{3{,}21 \text{ min.}}$

d) (Utilization)

- Average number of jobs : $\bar{N} = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5 = \ldots = 2{,}6705$
- Effective arrival rate : $\lambda_{eff} = \lambda \cdot (1 - P_5) = 100(1 - P_5) = \ldots \cong 92$
- Little : $\lambda_{eff} \cdot \bar{T}_{SYSTEM} = \bar{N} \Longrightarrow \bar{T}_{SYSTEM} = \ldots \cong 29{,}02$ msec

Utilization : $\lambda_{eff} \cdot \bar{T}_{SYSTEM} \Longrightarrow$ Utilization / server $= \frac{\lambda_{eff} \cdot \bar{T}_{SYSTEM}}{5} = \ldots = \underline{\underline{0{,}5341}}$

e) Pure M/M/5/5 with $\bar{T} = \bar{T}_{SYSTEM} = 29{,}02\,msec$

So, $\frac{1}{\mu_{eq}} = \bar{T} \longrightarrow \mu_{eq} = \dots = 34{,}45$

Blocking Probability

$$\begin{tikzcd}
S_0 \arrow[r, bend right, "100"'] & S_1 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "100"'] & S_2 \arrow[l, bend right, "2\mu"'] \arrow[r, bend right, "100"'] & S_3 \arrow[l, bend right, "3\mu"'] \arrow[r, bend right, "100"'] & S_4 \arrow[l, bend right, "4\mu"'] \arrow[r, bend right, "100"'] & S_5 \arrow[l, bend right, "5\mu"']
\end{tikzcd}$$

$$\longrightarrow P_5 = \frac{(\lambda/\mu)^5/5!}{\sum_{k=0}^{5} \frac{(\lambda/\mu)^k}{k!}} = \dots \cong 0{,}1$$

2

a)

class 1: $S^*(s) = \frac{4}{(s+2)^2} = \left(\frac{2}{s+2}\right)^2 \Rightarrow S_1 \sim \text{Erlang}(2,2)$

$\bar{S_1} = \frac{2}{2} = 1, \quad \overline{S_1^2} = V[S_1] + \bar{S_1}^2 = \frac{2}{2^2} + 1 = \frac{3}{2}$

class 2: $S^*(s) = \frac{2}{s+2} \Rightarrow S_2 \sim \text{Exp}(2)$

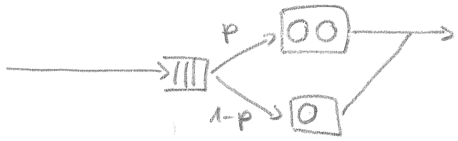
$\bar{S_2} = \frac{1}{2}, \quad \overline{S_2^2} = \frac{2}{2^2} = \frac{1}{2}$

$\bar{S} = p\bar{S_1} + (1-p)\bar{S_2} = \frac{4}{5}\cdot 1 + \frac{1}{5}\cdot\frac{1}{2} = \frac{9}{10} = 0.9$

$\overline{S^2} = p\overline{S_1^2} + (1-p)\overline{S_2^2} = \frac{4}{5}\cdot\frac{3}{2} + \frac{1}{5}\cdot\frac{1}{2} = \frac{13}{10} = 1.3$

$V[S] = \overline{S^2} - \bar{S}^2 = \frac{13}{10} - \left(\frac{9}{10}\right)^2 = \frac{130-81}{100} = \frac{49}{100} = 0.49$

b) $\rho = \lambda\bar{S} < 1 \Rightarrow \lambda < \frac{10}{9}$



c) $\rho = 0.9 \Rightarrow \lambda = \frac{\rho}{\bar{S}} = 1$

$\bar{W} = \frac{\lambda\overline{S^2}}{2(1-\rho)} = \frac{1\cdot\frac{13}{10}}{2\cdot\frac{1}{10}} = \frac{13}{2} = 6.5$ time units

d)

Since $\bar{S_2} < \bar{S_1}$, the class 2 jobs should be given higher priority to reduce the average waiting time of an arbitary job.

class 1: low priority $\lambda_1 = p\lambda = \frac{4}{5}$ $\rho_1 = \lambda_1\bar{S_1} = \frac{4}{5}$

class 2: high priority $\lambda_2 = (1-p)\lambda = \frac{1}{5}$ $\rho_2 = \lambda_2\bar{S_2}$ or $(\rho - \rho_1) = \frac{1}{10}$

$\bar{R} = \frac{1}{2}(\lambda_1\overline{S_1^2} + \lambda_2\overline{S_2^2}) = \frac{1}{2}\left(\frac{4}{5}\cdot\frac{3}{2} + \frac{1}{5}\cdot\frac{1}{2}\right) = \frac{13}{20}$

$\bar{W_2} = \frac{\bar{R}}{(1-\rho_2)} = \frac{\frac{13}{20}}{\frac{9}{10}} = \frac{13}{18}$

$\bar{W_1} = \frac{\bar{R}}{(1-\rho_2)(1-\rho_1-\rho_2)} = \frac{\frac{13}{20}}{\frac{9}{10}\cdot\frac{1}{10}} = \frac{65}{9}$

$\bar{W} = p\bar{W_1} + (1-p)\bar{W_2} = \frac{4}{5}\cdot\frac{65}{9} + \frac{1}{5}\cdot\frac{13}{18} = \frac{533}{90} \approx 5.9$ time units

③

a)

We can assume Poisson arrival Poisson (λ)
Hyperexp. service time ($\alpha_1, \alpha_2, \mu_1, \mu_2$)
One server
Infinite queue

$\Bigg\}$ $\underline{\underline{M/H_2/1}}$

$\lambda = \frac{1}{5}$ cust/min

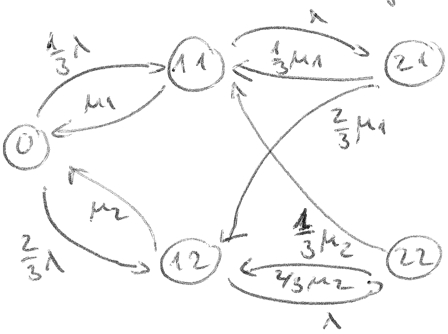
$\mu_1 = \frac{1}{2}$ $\quad \alpha_1 = \frac{1}{3}$

$\mu_2 = \frac{1}{4}$ $\quad \alpha_2 = \frac{2}{3}$

$\Big\}$ $E[x] = \alpha_1 E[x_1] + \alpha_2 E[x_2] = \frac{1}{3} 2 + \frac{2}{3} \cdot 4 = \frac{10}{3}$ min $\Big\}$ $\rho = \lambda E x = \frac{2}{3}$

Post office empty: $\underline{\underline{p_0 = 1 - \rho = \frac{1}{3}}}$

State transition diagram: state: {# customers, type of service}



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b) Average waiting time
- M/G/1 mean value form

$$E[x^2] = \alpha_1 E[x_1^2] + \alpha_2 E[x_2^2] = \alpha_1 \frac{2}{\mu_1^2} + \alpha_2 \frac{2}{\mu_2^2} = \frac{1}{3} \cdot 2 \cdot 4 + \frac{2}{3} \cdot 2 \cdot 16 = \frac{72}{3} = 24$$

$$W = \frac{\lambda E[x^2]}{2(1-\rho)} = \frac{\frac{1}{5} 24}{2 \cdot \frac{1}{3}} = \underline{\underline{\frac{36}{5} = 7.2 \text{ min}}}$$

③ Cont'd

c) Average waiting time of customer arriving when the server is busy but noone is waiting = average remaining service time in "busy" state

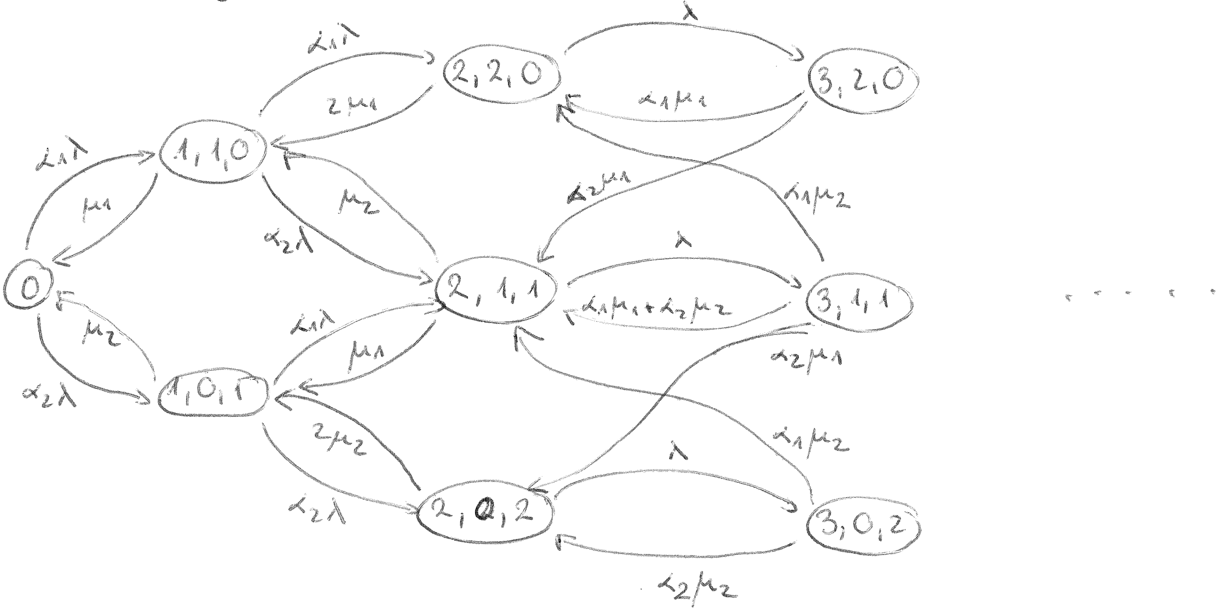
~~As~~ Remaining service time for type 1 cust: $Exp(\mu_1)$
— " — type 2 cust: $Exp(\mu_2)$

$\Rightarrow$ Av. waiting time $W_1 = \alpha_1 \frac{1}{\mu_1} + \alpha_2 \frac{1}{\mu_2} = \underline{\underline{\frac{10}{3} \text{ min}}}$

$P(\text{waiting} > 4 \text{ min}) = 1 - F(t)\Big|_{t=4} = \frac{1}{3} e^{-\mu_1 t} + \frac{2}{3} e^{-\mu_2 t}\Big|_{t=4} = \frac{1}{3} e^{-\frac{1}{2}\cdot 4} + \frac{2}{3} e^{-\frac{4}{4}\cdot 4} \approx \underline{\underline{0.29}}$

d) $M/E_2/2$

State: {# customers, # type 1 under service, # type 2 under service}



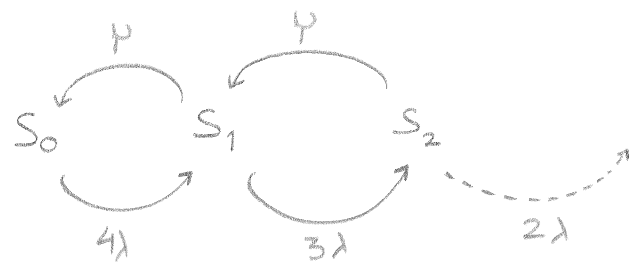
Problem 4

- Finite Population: 4 Students
- 1 server
- 1 waiting position
- Exponential arrivals / departures

- $\lambda = \frac{1}{4\text{hours}}$
- $\mu = \frac{3}{2}\ \text{hour}^{-1}$

M/M/1/2/4

a)



Balance Equations

$$\begin{cases} 4\lambda P_0 = \mu P_1 \longrightarrow & P_1 = \frac{4\cdot 1/4}{3/2} P_0 = \frac{2}{3} P_0 \\ 3\lambda P_1 = \mu P_2 \longrightarrow & P_2 = \frac{3\cdot 1/4}{3/2} P_1 = \frac{1}{2} P_1 = \frac{1}{3} P_0 \\ P_0 + P_1 + P_2 = 1 \end{cases}$$

$$\longrightarrow P_0 \left(1 + \tfrac{1}{3} + \tfrac{2}{3}\right) = 1 \longrightarrow \boxed{P_0 = 0{,}5} \quad \boxed{P_1 = \tfrac{1}{3}} \quad \boxed{P_2 = \tfrac{1}{6}}$$

)

Blocking:

$$P_b = \frac{2\lambda P_2}{4\lambda P_0 + 3\lambda P_1 + 2\lambda P_2} = \frac{\frac{1}{2}\cdot\frac{1}{6}}{\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{6}} = \frac{1/12}{\frac{1}{2} + \frac{1}{4} + \frac{1}{12}} = \frac{1}{6+3+1} = \frac{1}{10}$$

) $P_{busy} = 1 - P_0 = 0{,}5 \longrightarrow$ in 24hours $\longrightarrow$ 12 hours busy.

$$\frac{12\text{hours}}{40\text{min}} = \frac{12}{\frac{2}{3}} = \frac{36}{2} = 18 \text{ cooking rounds}$$

$18/4 = 4{,}5$ "cookings"/student per day d)

$$\begin{tikzcd} S_0 \arrow[r, bend right, "4\lambda"'] & S_1 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "3\lambda\cdot\frac{1}{3}=\lambda"'] & S_2 \arrow[l, bend right, "2\mu"'] \arrow[r, dashed, bend right, "2\lambda\cdot\frac{1}{3}=\frac{2\lambda}{3}"'] & {} \end{tikzcd}$$

Balance Equations

$$4\lambda P_0 = \mu P_1 \rightarrow 4\cdot\frac{1}{4}P_0 = \frac{3}{2}P_1 \rightarrow P_1 = \frac{2}{3}P_0$$

$$\lambda P_1 = 2\mu P_2 \rightarrow P_2 = \frac{1/4}{2\cdot 3/2}P_1 \rightarrow P_2 = \frac{1}{12}P_1 = \frac{1}{18}P_0$$

$$P_0 + P_1 + P_2 = 1$$

$$\rightarrow P_0\left(1+\tfrac{2}{3}+\tfrac{1}{18}\right) = 1 \rightarrow P_0 = \frac{18}{18+2\cdot 6+1} = \frac{18}{31}$$

$$P_1 = \frac{2}{3}\cdot\frac{18}{31} = \frac{12}{31}, \quad P_2 = \frac{1}{31}$$

$$P_{blocked} = \frac{2\lambda/3\, P_2}{4\lambda\cdot P_0 + \lambda P_1 + \frac{2\lambda}{3}P_2} = \frac{1/_{2\cdot 3}P_2}{P_0 + 1/4 P_1 + 1/6 P_2} = \frac{1/6\cdot 1/31}{\frac{18}{31}+\frac{3}{31}+\frac{1}{6\cdot 31}} = \frac{1/6}{18+3+1/6}$$

$$= \boxed{7{,}874\, 10^{-3}}$$

e) (more than one solutions may be accepted)

The students make an error if they decide not to go and the sofa position was empty.

The error is made in state 1: $3\lambda\cdot\frac{1}{3}$ make the correct decision; $3\lambda\cdot\frac{2}{3}$ make an error

In state II the action "not to go" is correct $(2\lambda\cdot\frac{2}{3})$

So, after consulting the monitor, the percentage of students that erroneously stay at the room is:

$$\frac{3\lambda\cdot\frac{2}{3}\cdot P_1}{3\lambda\frac{2}{3}P_1 + 2\lambda\cdot\frac{2}{3}P_2} = \frac{3\lambda P_1}{3\lambda P_1 + 2\lambda P_2} \simeq 0{,}94$$

(5)

a) $X = \min\{X_1, \dots X_k\}$ $\quad X_i \sim Exp(\lambda)$

$$\bar{F}_k(x) = P(X > x) = P(X_1, \dots X_k > x) = \prod_{i=1}^{k} P(X_i > x) = \prod_{i=1}^{k} e^{-\lambda x} = e^{-k\lambda x}$$

$$F_k(x) = 1 - \bar{F}_k(x) = 1 - e^{-k\lambda x} \quad \Rightarrow \quad X \sim Exp(k\lambda)$$

We use this property when we define the service intensity in multiserver systems.

b) M/M/1 with vacation $\quad T = ?$

$V \sim$ const (2 min)

$\lambda = 1$

$\left.\begin{matrix} \lambda = 1 \\ \rho = 0.75 \end{matrix}\right\} \Rightarrow \mu = \frac{4}{3}$

$$T = \bar{x} + \frac{\rho}{\mu - \lambda} + \frac{\bar{V^2}}{2\bar{V}} = \frac{3}{4} + \frac{\frac{3}{4}}{\frac{1}{3}} + \frac{4}{2 \cdot 2} = \underline{\underline{4 \text{ min}}}$$

c) M/G/1

$X = \begin{cases} 1 & p = 0.5 \\ 10 & p = 0.5 \end{cases}$ $\quad \left. E[X] = 5.5 \right\} \rho = 0.55$

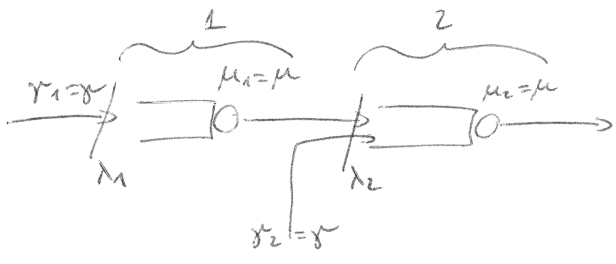
$\lambda = \frac{1}{10}$

$E[\text{idle}] = \frac{1}{\lambda} = 10$

$$\rho = \frac{E[\text{busy}]}{E[\text{idle}] + E[\text{busy}]}$$

$$\Rightarrow E[\text{busy}] = \cancel{12}\ \frac{5.5}{0.45} \approx 12.2$$

d)



$\lambda_1 = \gamma \quad \rho_1 = \lambda_1 \mu$

$\lambda_2 = 2\gamma \quad \rho_2 = \lambda_2 \mu = 2\rho_1$

$$N = N_1 + N_2 = \frac{\rho_1}{1 - \rho_1} + \frac{2\rho_1}{1 - 2\rho_1} = 2.5 \Rightarrow \boxed{\begin{matrix} \rho_1 = \frac{1}{3} \\ \rho_2 = \frac{2}{3} \end{matrix}}$$

Utilization of the queuing systems

Customer entering at queue 1 leaves after queue 2 $\Rightarrow$

$\underline{P(\text{no waiting})} = P(\text{no waiting at queue 1}) P(\text{no waiting at queue 2}) =$

$$P_0^1 \cdot P_0^2 = (1 - \rho_1)(1 - \rho_2) = \underline{\underline{\frac{2}{9}}}$$

~~Utilization = 1 − P(empty) = 1 − $P_0^1 P_0^2$ = $\frac{7}{9}$~~