Queuing Theory, Exam, June 10^{th}, 2011

Problem 1

$\lambda = 4h^{-1}$

$\mu = 1h^{-1}$

servers $= 4$, # queue positions $= 2$, FCFS

a) State space : S_k : k groups in the restaurant

State diagram

$$\begin{tikzcd} S_0 \arrow[r, bend right, "\lambda"'] & S_1 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "\lambda"'] & S_2 \arrow[l, bend right, "2\mu"'] \arrow[r, bend right, "\lambda"'] & S_3 \arrow[l, bend right, "3\mu"'] \arrow[r, bend right, "\lambda"'] & S_4 \arrow[l, bend right, "4\mu"'] \arrow[r, bend right, "\lambda"'] & S_5 \arrow[l, bend right, "4\mu"'] \arrow[r, bend right, "\lambda"'] & S_6 \arrow[l, bend right, "4\mu"'] \end{tikzcd}$$

$\longrightarrow$ HOMOGENEOUS ARRIVALS

Balance Equations:

$4P_0 = P_1$

$4P_1 = 2P_2$

$4P_2 = 3P_3$

$4P_3 = 4P_4$

$4P_4 = 4P_5$

$4P_5 = 4P_6$

$\oplus$ normalization equation $\sum_{k=0}^{6} P_k = 1$ $\Longrightarrow$

Solution

$P_0 = 0{,}018$

$P_1 = 0{,}0719$

$P_2 = 0{,}1437$

$P_3 = 0{,}1916$

$P_4 = P_5 = P_6 = P_3$

$Pr\{\text{immediate service}\} = P_0 + P_1 + P_2 + P_3 = 0{,}4252$

b) $Pr\{\text{leave without eating}\} = P_6 = 0{,}1916$

within an hour $\longrightarrow$ $\lambda \cdot P_6 = 0{,}7664$ groups

c) Blocking Period (T_B) : the time on $S_6 \longrightarrow \exp(4\mu)$

$$\Rightarrow \overline{T}_B = \frac{1}{4\mu} = 15 \text{ min}$$

$$\frac{\overline{T}_B}{\overline{T}_B + \overline{T}_{NB}} = P_6 \longrightarrow \overline{T}_{NB} = \frac{1-P_6}{P_6}\overline{T}_B = 1{,}0548 \text{ hours}$$

Problem 2

4 students, $\lambda = \frac{1}{5}\,\text{min}^{-1}$ per student

$\mu = 1\,\text{min}^{-1}$

2 servers, inf. queue (but only 2 places needed)

- System: M/M/2/4/4

State space: S_k: k papers in the system (printers + queue)

a) State diagram

$$\begin{tikzcd} S_0 \arrow[r, bend right, "0{,}8"'] & S_1 \arrow[l, bend right, "1"'] \arrow[r, bend right, "0{,}6"'] & S_2 \arrow[l, bend right, "2"'] \arrow[r, bend right, "0{,}4"'] & S_3 \arrow[l, bend right, "2"'] \arrow[r, bend right, "0{,}2"'] & S_4 \arrow[l, bend right, "2"'] \arrow[r, dashed, "0"'] & {} \end{tikzcd}$$

$0 \longrightarrow$ NON-HOMOGENEOUS ARRIVAL PROCESS

Balance Equations

$$\left.\begin{aligned} 0{,}8\,P_0 &= P_1 \\ 0{,}6\,P_1 &= 2P_2 \\ 0{,}4\,P_2 &= 2P_3 \\ 0{,}2\,P_3 &= 2P_4 \end{aligned} \quad \oplus \quad \begin{aligned} &\text{normalization eq.} \\ &\sum_{k=0}^{4} P_k = 1 \end{aligned} \right\} \longrightarrow$$

Solution

$P_0 = 0{,}4778$

$P_1 = 0{,}3823$

$P_2 = 0{,}1147$

$P_3 = 0{,}0229$

$P_4 = 0{,}00229$

Printer utilization

$$U = 0 \cdot P_0 + \tfrac{1}{2} P_1 + 1 \cdot (P_2 + P_3 + P_4) = 0{,}33104$$

b) $$P_{WAIT} = \frac{\lambda_2 P_2 + \lambda_3 P_3}{\sum_{k=0}^{4} \lambda_k P_k} = \frac{0{,}4 P_2 + 0{,}2 P_3}{0{,}8 P_0 + 0{,}6 P_1 + 0{,}4 P_2 + 0{,}2 P_3} = \frac{0{,}05046}{0{,}6628} = 0{,}07613$$

c) If a paper observes state 2 $\longrightarrow T_{W_2} \sim \exp(2\,\text{min}^{-1}) \longrightarrow \overline{T}_{W_2} = 0{,}5\,\text{min}$

If it observes $S_3 \longrightarrow T_{W_3} \sim Erl_2(2\,\text{min}^{-1}) \longrightarrow \overline{T}_{W_3} = 1\,\text{min}$

Mean waiting time: $$\overline{W} = \frac{\lambda_2 P_2}{\lambda_2 P_2 + \lambda_3 P_3}\overline{T}_{W_2} + \frac{\lambda_3 P_3}{\lambda_2 P_2 + \lambda_3 P_3}\overline{T}_{W_3} \cong 0{,}5445\,\text{min}$$

Problem 3

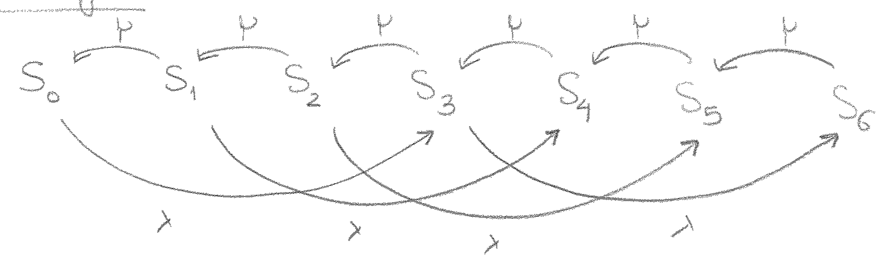
- Poisson arrivals (λ)
- 1 server, 1 queue position
- Erlang-n service time

a) $$c^2 = \frac{Var[x]}{(\bar{x})^2} = \frac{Var[x_1 + \dots + x_n]}{[E[x_1 + \dots + x_n]]^2} \overset{\text{i.i.d.}}{=} \frac{n \cdot Var[x_i]}{[n\, E[x_i]]^2}$$

$$= \frac{n \cdot \frac{1}{\lambda^2}}{n^2 (\frac{1}{\lambda})^2} = \frac{1}{n} < 1 \quad , \text{for } n \geq 1$$

b) State space : S_k : k stages left for service

State diagram



μ: the rate of each exp. stage

Balance Equations

$$\lambda P_0 = \mu P_1$$
$$\lambda P_0 + \lambda P_1 = \mu P_2$$
$$\lambda P_0 + \lambda P_1 + \lambda P_2 = \mu P_3$$
$$\lambda P_1 + \lambda P_2 + \lambda P_3 = \mu P_4$$
$$\lambda P_2 + \lambda P_3 = \mu P_5$$
$$\lambda P_3 = \mu P_6$$

c) Blocking states : S_4, S_5, S_6.

$$P_{block} = P_4 + P_5 + P_6$$

(homogeneous arrivals)

d) Average number of customers:

$$\bar{N} = 1(P_1 + P_2 + P_3) + 2(P_4 + P_5 + P_6)$$

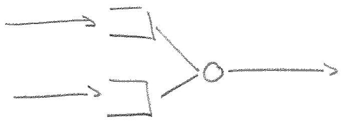
Mean System Time

$$\bar{T}_{SYSTEM} = \frac{\bar{N}}{\lambda_{eff}} \quad \text{(from Little)}$$

$$\lambda_{eff} = \lambda (P_0 + P_1 + P_2 + P_3)$$

4

$\lambda_1 = 300$ pk/s , $\bar{L}_1 = 400$ bits (Exp)



$\lambda_2 = 150$ pk/s

$L_2 = 200$ bits (Det.)

a) $\bar{X}_1 = \frac{\bar{L}_1}{C} = 2 \cdot 10^{-3} s$ $\Big\}$ $R_1 = \frac{1}{2} \lambda_1 \overline{X_1^2} = 1.2 \times 10^{-3}$

$\overline{X_1^2} = 2 \bar{X}_1^2$ $\quad$ $\rho_1 = \lambda_1 \bar{X}_1 = 0.6$

$\bar{X}_2 = \frac{L_2}{C} = 10^{-3} s$ $\Big\}$ $R_2 = \frac{1}{2} \sum_{i=1}^{2} \lambda_i \overline{X_i^2} = 1.275 \cdot 10^{-3}$

$\overline{X_2^2} = \bar{X}_2^2$ $\quad$ $\rho_2 = \lambda_2 \bar{X}_2 = 0.15$

$$T_1 = \frac{R_1}{1-\rho_1} + \bar{X}_1 = 5 \text{ ms}$$

$$T_2 = \frac{(1-\rho_1-\rho_2)\bar{X}_2 + R_2}{(1-\rho_1)(1-\rho_1-\rho_2)} = 15.25 \text{ ms}$$

b) The high priority transmitter can see a simple M/M/1 queue due to the preemptive priority.

$F_W(t) = 1 - \rho_1 e^{-(\mu_1 - \lambda_1)t}$

$P(W > 10^{-3} s) = 1 - F_W(10^{-3}) = 0.49$

c) M/M/1 with vacations

$V = 10$ ms, $\overline{V^2} = \bar{V}^2$

$\bar{W}_{1,save} = \bar{W}_{M/M/1} + \frac{\overline{V^2}}{2V} = 8 \cdot 10^{-3} s$

$\bar{T}_{1,save} = \bar{W}_{1,save} + \bar{x} = 10 \cdot 10^{-3} s$

$N_q = \lambda_1 W_{1,save} = 2.4$

d) $E_{save} = 3600 \cdot (1-\rho_1) \cdot 0.001 = 1.44$ J

Problem 5

a) Hyper-exp distribution

$E[x] = 2$

$C_x^2 = \frac{E[x^2]}{E[x]^2} - 1 = 1.5$

$\underline{E[x^2]} = 2.5 \cdot E[x]^2 = 2.5 \cdot 4 = \underline{10}$

$$E[x] = \underset{\Downarrow}{\overset{\alpha_1}{\frac{1}{2}}} \cdot \frac{1}{\mu_1} + \underset{\Downarrow}{\overset{\alpha_2}{\frac{1}{2}}} \cdot \frac{1}{\mu_2} = 2$$

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} = 4$$

$$\frac{1}{\mu_2} = \left(4 - \frac{1}{\mu_1}\right)$$

$$E[x^2] = \frac{1}{2} \cdot \frac{2}{\mu_1^2} + \frac{1}{2} \cdot \frac{2}{\mu_2^2} = 10$$

$$\frac{1}{\mu_1^2} + \frac{1}{(\mu_2)^2} = 10$$

$$\frac{1}{\mu_1^2} + \left(4 - \frac{1}{\mu_1}\right)^2 = 10 \qquad \frac{1}{\mu_1} = a$$

$$a^2 + 16 + a^2 - 8a = 10$$

$$2a^2 - 8a + 6 = 0$$

$$a^2 - 4a + 3 = 0 \quad \left\{\begin{array}{l} a_1 = 1 \\ a_2 = 3 \end{array}\right\} \Rightarrow \boxed{\mu_1 = 1,\ \mu_2 = \frac{1}{3},\ \alpha_1 = \alpha_2 = 0.5}$$

---

b) M/G/1

$\lambda = 0.5$

$$S^*(s) = \frac{4}{(s+2)^2} = \frac{2}{s+2} \cdot \frac{2}{s+2} \Rightarrow \text{Erlang-2},\ \cancel{\mu_1 = \mu_2 = 2}\ 2\mu = 2 \Rightarrow \mu = 1 \Big\}\ \rho = \underline{0.5}$$

$$\left.\begin{array}{l} E[S] = 1 \\ C_S^2 = \frac{1}{2} \\ \rho = 0.5 \end{array}\right\} \quad \underline{\underline{W}} = \frac{\rho}{1-\rho} \cdot E[S] \cdot \frac{1 + C_S^2}{2} = 1 \cdot 1 \cdot \frac{1 + \frac{1}{2}}{2} = \underline{\underline{\frac{3}{4}}}$$

c) M/G/1 with vacation. Vacation starts with empty buffer $\Rightarrow N_q$ at the end of the vacation period = # arrivals during the vacation period.

$V = 2$

$\lambda = 0.5$

$$\underbrace{P[k] = \frac{(\lambda V)^k}{k!} e^{-\lambda V}}_{\text{Poisson arrival process}}$$

$$\underline{P(\text{more than 2 req. waiting})} = 1 - \left(P(\emptyset \text{ waiting}) + P(1 \text{ waiting}) + P(2 \text{ waiting})\right) =$$

$$1 - \left(e^{-1} + 1 \cdot e^{-1} + \frac{1}{2} e^{-1}\right) = 1 - \frac{5}{2} \cdot e^{-1} = \underline{0.08}$$

d) $\lambda_1 = 1 + 0.25\lambda_1$

$0.75\lambda_1 = 1$

$\boxed{\lambda_1 = \frac{4}{3}} = \lambda_2$

For both queues: $\rho = \frac{\lambda}{\mu} = \frac{4/3}{2} = \frac{2}{3}$

$$N = \frac{\rho}{1-\rho} = \frac{2/3}{1/3} = 2 \Rightarrow \Sigma N = 4$$

$$\underline{\underline{T}} = \frac{N}{\lambda} = \frac{4}{1} = \underline{\underline{4 \text{ time units.}}}$$