EP2200

Solutions [Exam: 15 Dec 2011]

Problem 1

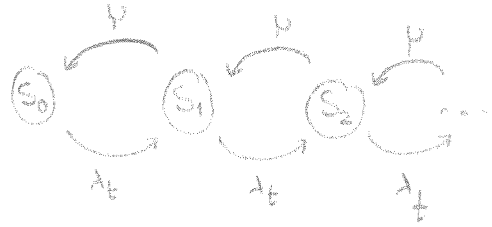
a) $\lambda_1 = 25 s^{-1}$, $\lambda_2 = 75 s^{-1}$ $\Longrightarrow \lambda_t = \lambda_1 + \lambda_2 = 100 s^{-1}$

$\mu = \frac{1}{\frac{1}{120}} = 120 s^{-1}$

$\Big\}\ \rho = \frac{\lambda}{\mu} = \frac{100}{120} = \frac{5}{6}$ Erl

Kendall: M/M/1

Diagram:



$\bar{N} = \frac{\rho}{1-\rho} = \frac{5/6}{1-5/6} = 5$

Little: $T_{SYSTEM} = \frac{\bar{N}}{\lambda_t} = 0{,}05 s$

b) Preemptive resume for Type I > Type II

Type I arrivals see M/M/1 with $\rho_1 = \frac{\lambda_1}{\mu} = \frac{25}{120} = \frac{5}{24}$ Erl.

$\bar{N}_1 = \frac{\rho_1}{1-\rho_1} = \frac{5}{19} \xrightarrow{\text{Little}} \bar{T}_{SYSTEM} = \frac{\bar{N}_1}{\lambda_1} = \frac{5/19}{25} = \frac{1}{95}$ sec.

$\bar{W}_1 = \bar{T}_{SYSTEM} - \bar{T}_{SERVICE} = \frac{1}{95} - \frac{1}{120} = \frac{1}{456}$ sec

$P\{W_1 > 1\} = \rho_1 e^{-(\mu-\lambda_1)\cdot 1} = \frac{5}{19} e^{-95}$

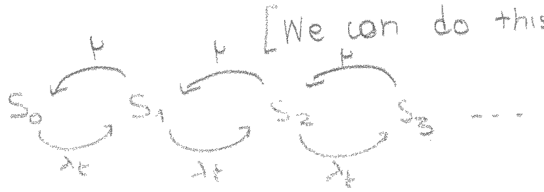
c) Interrupted, if a Type I arrives before service is completed.

$$P_{Interrupt} = \int_0^\infty \underbrace{(1-e^{-\lambda_1 t})}_{T_I \text{ arrive before } t} \underbrace{\mu \cdot e^{-\mu t}}_{\text{service of } T_{II} \text{ is } t} dt = 1 - \mu \int_0^\infty e^{-(\lambda_1+\mu)t} dt = 1 - \frac{\mu}{\mu+\lambda_1} = \frac{5}{29}$$

d) [Many solutions can be accepted]

However: if S_k: k jobs in the system, then nothing changes!

[We can do this because of the exp. service distribution]



e) $\bar{N} = 5$ [Same as in a)]

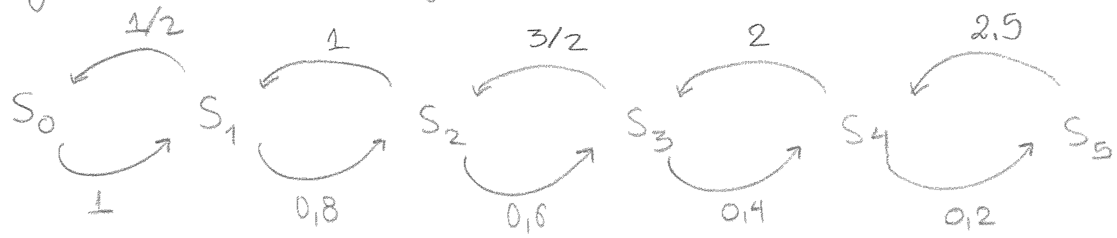
$\bar{N}_1 = \frac{5}{19}$ [Same as in b)]

$\longrightarrow \bar{N}_2 = \bar{N} - \bar{N}_1 \cong 4{,}7368$

Problem 2

- $\lambda = 60$ groups/hour $= 1$ /min
- $\mu_k = \frac{k}{2} \text{ min}^{-1}$ [State - Dependent service times]
- discouraged arrivals : $P_k = 1 - \frac{k}{5}$, $k < 5$

a) Diagram : S_k : k groups waiting in the queue



→ Balance Equations

$$\begin{cases} P_0 = 0{,}5 P_1 \\ 0{,}8 P_1 = P_2 \\ 0{,}6 P_2 = \frac{3}{2} P_3 \\ 0{,}4 P_3 = 2 P_4 \\ 0{,}2 P_4 = 2{,}5 P_5 \end{cases} \xrightarrow{\sum_{k=0}^{5} P_k = 1} \begin{cases} P_0 = 0{,}1859 \\ P_1 = 0{,}3719 \\ P_2 = 0{,}2975 \\ P_3 = 0{,}1190 \\ P_4 = 0{,}0238 \\ P_5 = 0{,}001904 \end{cases} \Rightarrow \bar{N} = 1{,}4286$$

HOMOGENEOUS [time invariant] arrivals

$$\Rightarrow \Pr\{\text{observe } S_5\} = P_5 = 0{,}001904$$

b) $\Pr\{\text{decide to wait}\} = \sum_{k=0}^{5} \underbrace{\left(1 - \frac{k}{5}\right)}_{P_k} \cdot \Pr\{\text{observe } S_k\} =$

$= P_0 + 0{,}8 P_1 + 0{,}6 P_2 + 0{,}4 P_3 + 0{,}2 P_4 = 0{,}7143$.

c) $\lambda_{eff} = \lambda \cdot \Pr\{\text{decide to wait}\} = 0{,}7143 \text{ min}^{-1}$

Little : $\bar{T}_s = \frac{\bar{N}}{\lambda_{eff}} = 2 \text{ min}$

Problem 2 (cont)

d) "Impatient" customers

Groups depart from the chain, if
- they enter the club (μ_k)
- they leave, while waiting ($\mu' = 0{,}1$)

State Diagram
(same state space)

$$
\begin{tikzcd}
S_0 \arrow[r, bend right, "1"'] & S_1 \arrow[l, bend right, "1/2+0{,}1"'] \arrow[r, bend right, "0{,}8"'] & S_2 \arrow[l, bend right, "1+0{,}2"'] \arrow[r, bend right, "0{,}6"'] & S_3 \arrow[l, bend right, "3/2+0{,}3"'] \arrow[r, bend right, "0{,}4"'] & S_4 \arrow[l, bend right, "2+0{,}4"'] \arrow[r, bend right, "0{,}2"'] & S_5 \arrow[l, bend right, "2{,}5+0{,}5"']
\end{tikzcd}
$$

- Balance Equations

$$
\left\{
\begin{aligned}
P_0 &= 0{,}6 P_1 \\
0{,}8 P_1 &= 1{,}2 P_2 \\
0{,}6 P_2 &= 1{,}8 P_3 \\
0{,}4 P_3 &= 2{,}4 P_4 \\
0{,}2 P_4 &= 3 P_5
\end{aligned}
\right\}
\xrightarrow{\sum_{k=0}^{5} P_k = 1}
\left\{
\begin{aligned}
P_0 &= 0{,}2373 \\
P_1 &= 0{,}3955 \\
P_2 &= 0{,}2637 \\
P_3 &= 0{,}0879 \\
P_4 &= 0{,}0146 \\
P_5 &= 0{,}000976
\end{aligned}
\right.
$$

We recalculate $\bar{N} = 1{,}2499$

We recalculate $\lambda_{eff} = \lambda(P_0 + 0{,}8P_1 + 0{,}6P_2 + 0{,}4P_3 + 0{,}2P_4) = 0{,}75$

We recalculate $\bar{T}_{SYSTEM} = \dfrac{\bar{N}}{\lambda_{eff}} = \dfrac{1{,}2499}{0{,}75} = 1{,}6665$

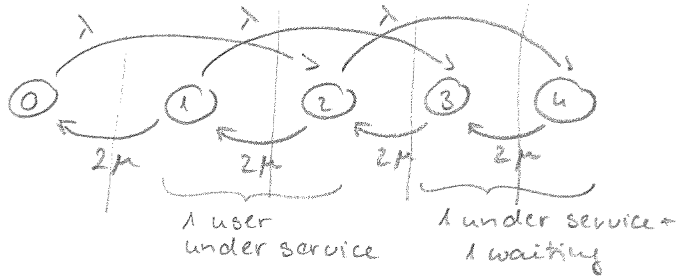
Problem 3

M/E_2/1/2 : Erlang-2 service, 1 server, 1 buffer position

$\lambda = 0.5$ arriv./sec.

$\bar{x} = 2$ sec. $\mu = \frac{1}{2}$

a)



$$\left.\begin{aligned} \lambda p_0 &= 2\mu p_1 \\ \lambda p_0 + \lambda p_1 &= 2\mu p_2 \\ \lambda p_1 + \lambda p_2 &= 2\mu p_3 \\ \lambda p_2 &= 2\mu p_4 \\ \textstyle\sum_i p_i &= 1 \end{aligned}\right\} \{p_0, p_1, p_2, p_3, p_4\} = \left\{\frac{4}{13}, \frac{4}{26}, \frac{6}{26}, \frac{5}{26}, \frac{3}{26}\right\}$$

$$P(\text{block}) = p_3 + p_4 = \frac{4}{13}$$

b) $N = p_0 0 + (p_1 + p_2)1 + (p_3 + p_4)2 = \ldots = 1$

$$T = \frac{N}{\lambda_{eff}} = \frac{N}{(1 - P(\text{block})\lambda} = \ldots = \frac{26}{9}$$

$$W = T - \bar{x} = \frac{26}{9} - 2 = \frac{8}{9}$$

c) Waiting time of accepted, but waiting customers → customers arriving when the system is in state 1 or state 2.

i) System in state 1: 1 Exp(2μ) step is left from service $\bar{W}_1 = 1$ s

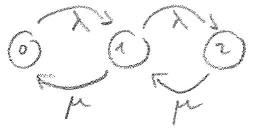
ii) 2: 2 Exp(2μ) steps are left. $\bar{W}_2 = 2$ s

$$\Rightarrow \bar{W} = \frac{p_1}{p_1 + p_2}\bar{W}_1 + \frac{p_2}{p_1 + p_2}\bar{W}_2 = \ldots = 1.6 \text{ sec.}$$

$$\mathcal{L}(f(w)) = \frac{p_1}{p_1 + p_2}\,\frac{2\mu}{s + 2\mu} + \frac{p_2}{p_1 + p_2}\left(\frac{2\mu}{s + 2\mu}\right)^2$$

$$f_W(t) = \frac{p_1}{p_1 + p_2} \cdot 2\mu e^{-2\mu t} + \frac{p_2}{p_1 + p_2}(2\mu)^2 t\, e^{-2\mu t}$$

d) M/M/1/2

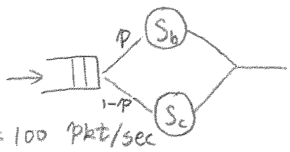


$$\Rightarrow \{p_0, p_1, p_2\} = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\} \quad P(\text{block}) = p_2 = \frac{1}{3} > \frac{4}{13}$$

⇒ We overestimate the blocking, because the variance of the service time is now higher.

In an M/D/1/2 system the services are deterministic, which decreases the variability of the service, and the blocking probability decreases compared to M/E_2/1/2.

Problem 4.



$\lambda = \lambda_b + \lambda_c = 100$ pkt/sec

$p = \frac{\lambda_b}{\lambda} = 0.6$

$S_b \sim Const_b + Exp(u_b)$ $\quad const_b = \frac{L_h}{C_b} = \frac{20}{10K} = 2 \times 10^{-3}$ sec

$\frac{1}{u_b} = \frac{\bar{L_p}}{C_b} = \frac{100}{10K} = 10 \times 10^{-3}$ sec

$S_c \sim Const_c + Exp(u_c)$ $\quad Const_c = \frac{L_h}{C_c} = \frac{20}{20K} = 1 \times 10^{-3}$ sec

$\frac{1}{u_c} = \frac{\bar{L_p}}{C_c} = \frac{100}{20K} = 5 \times 10^{-3}$ sec

a) $\bar{S_b} = Const_b + \frac{1}{u_b} = 12 \times 10^{-3}$ sec

$\bar{S_c} = const_c + \frac{1}{u_c} = 6 \times 10^{-3}$ sec

$\bar{S} = p\bar{S_b} + (1-p)\bar{S_c} = (0.6 \times 12 + 0.4 \times 6) \times 10^{-3} = 9.6 \times 10^{-3}$ sec $= 9.6$ ms

$P_{idle} = 1 - \rho = 1 - \lambda\bar{S} = 1 - 0.96 = 0.04$

b) $\bar{N_q} = \lambda\bar{w}$, $\bar{T} = \bar{w} + \bar{S}$, $\bar{w} = \frac{\lambda\bar{S^2}}{2(1-\rho)}$

$V[S_b] = V[const_b] + V[Exp(u_b)] = 0 + \frac{1}{u_b^2} = 100 \times 10^{-6}$

$V[S_c] = V[const_c] + V[Exp(u_c)] = 0 + \frac{1}{u_c^2} = 25 \times 10^{-6}$

$\bar{S_b^2} = V[S_b] + \bar{S_b}^2 = (100 + 144) \times 10^{-6} = 244 \times 10^{-6}$

$\bar{S_c^2} = V[S_c] + \bar{S_c}^2 = (25 + 36) \times 10^{-6} = 61 \times 10^{-6}$

$\bar{S^2} = p\bar{S_b^2} + (1-p)\bar{S_c^2} = (0.6 \times 244 + 0.4 \times 61) \times 10^{-6} = 170.8 \times 10^{-6}$

$\bar{w} = \frac{\lambda\bar{S^2}}{2(1-\rho)} = \frac{100 \times 170.8 \times 10^{-6}}{2(1-0.96)} = 0.2135$ sec

$\bar{N_q} = \lambda\bar{w} = 21.35$

$\bar{T} = \bar{w} + \bar{S} = 0.2135 + 0.0096 = 0.2231$ sec

c) $V \sim Exp(u_v)$ $\quad \bar{V} = \frac{1}{u_v} = 2$ ms

-With probability ρ, the server is busy. $\bar{R_{v|busy}} = 0$

-With probability $1-\rho$, the server is in maintenance:

$\bar{R_{v|maintenance}} \xlongequal{V\sim Exp(u_v),\ memoryless} \bar{V} = \frac{1}{u_v} = 2$ ms

$\bar{R_v} = \rho \cdot \bar{R_{v|busy}} + (1-\rho) \cdot \bar{R_{v|maintenance}} = (1-\rho) \cdot \frac{1}{u_v} = 0.08$ ms $= 80$ us

$\because R_{v|empty} = R_{v|maintenance} \sim Exp(u_v)$

$\therefore \bar{R_{v|empty}} = \frac{1}{u_v} = 2$ ms

d) $\overline{T} = \overline{W_v} + \overline{S}$

$$\overline{W_v} = \frac{\lambda \overline{S^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}} \qquad V \sim \mathrm{Exp}(u_v),\ \overline{V} = \frac{1}{u_v}$$

$$\overline{V^2} = V[V] + \overline{V}^2 = \frac{2}{u_v^2}$$

$$= 0.2135 + \frac{\frac{2}{u_v^2}}{2 \cdot \frac{1}{u_v}}$$

$$= 0.2135 + \frac{1}{u_v}$$

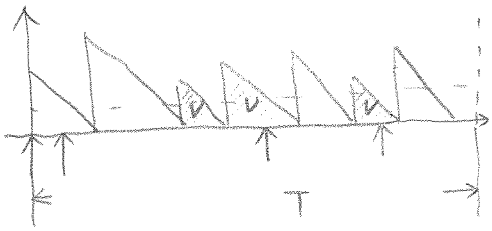
$$= 0.2155 \text{ sec}$$

$$\overline{T} = \overline{W_v} + \overline{S} = 0.2155 + 0.0096 = 0.2251 \text{ sec}$$

$$\overline{T_b} = \overline{W_v} + \overline{S_b} = 0.2155 + 0.012 = 0.2275 \text{ sec}$$

$$\overline{T_c} = \overline{W_v} + \overline{S_c} = 0.2155 + 0.006 = 0.2215 \text{ sec}$$

Another way to get $\overline{R_v}$ in c):



$$\overline{R_v} = \frac{\sum_{i=1}^{n} \frac{1}{2} V_i^2}{T}$$

$$T \to \infty: \ T(1-\rho) = \sum_{i=1}^{n} V_i$$

$$\Rightarrow \frac{1}{T} = \frac{1-\rho}{\sum_{i=1}^{n} V_i}$$

$$\therefore \overline{R_v} = \frac{1-\rho}{\sum_{i=1}^{n} V_i} \cdot \sum_{i=1}^{n} \frac{1}{2} V_i^2$$

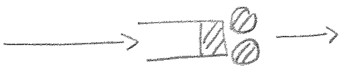
$$= \frac{1-\rho}{2} \cdot \frac{\frac{1}{n}\sum_{i=1}^{n} V_i^2}{\frac{1}{n}\sum_{i=1}^{n} V_i}$$

$$= \frac{1-\rho}{2} \cdot \frac{\overline{V^2}}{\overline{V}}$$

$$= \frac{1-\rho}{2} \cdot \frac{\frac{2}{u_v^2}}{\frac{1}{u_v}} = \frac{1-\rho}{u_v} = 80\ \mu s$$

Problem 5 (each for 2.5 points)

a)



Service: Exponential, $x = 2$ min, $\mu = \frac{1}{2}$

- Time between completed service, if both "servers" are busy:

  $\min[Exp(\mu), Exp(\mu)] = Exp(2\mu)$

- I have to wait for 2 finished services to get a free server

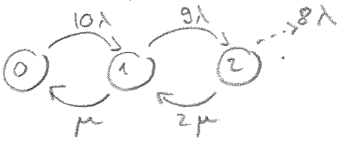
$\Rightarrow \bar{W} = \{2 \text{ finished service}\} = 2 \cdot \frac{1}{2\mu} = 2$ min

P(wait more than 3 min) = P(∅ or 1 service in 3 min, according to a Poisson (2μ) service process) $= e^{-2\mu t} + 2\mu t e^{-2\mu t} = \ldots = 4 \cdot e^{-3}$

b) M/M/2/2/10

$\bar{x} = 2$ min $\Rightarrow \mu = 30$ req/hour

$\lambda = 1$ req./hour



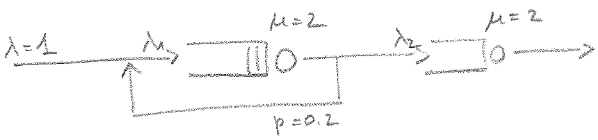
$$\left.\begin{array}{l} p_0 \cdot 10\lambda = p_1 \cdot \mu \\ p_1 \cdot 9\lambda = p_2 \cdot 2\mu \\ p_0 + p_1 + p_2 = 1 \end{array}\right\} (p_0, p_1, p_2) = \left(\frac{60}{83}, \frac{20}{83}, \frac{3}{83}\right)$$

P(arriving customer is blocked) = P(call blocking) =

$$\frac{8\lambda p_2}{10\lambda p_0 + 9\lambda p_1 + 8\lambda p_2} = \ldots = \frac{24}{804} \approx 0.03$$

[Alt. solution: Engset form]

c)



$\lambda_1 = \lambda + 0.2\lambda_1 \Rightarrow \lambda_1 = \frac{\lambda}{0.8} = \frac{5}{4} \Rightarrow \rho_1 = \frac{\lambda_1}{\mu} = \frac{5}{8}$

$\lambda_2 = 0.8\lambda_1 = \lambda = 1 \Rightarrow \rho_2 = \frac{\lambda_2}{\mu} = \frac{1}{2}$

P(the queuing network is empty) = P(queue 1 empty)P(queue 2 empty)

$= (1-\rho_1)(1-\rho_2) = \ldots = \frac{3}{16}$

d)

$$\left.\begin{array}{l} \lambda = 10 \text{ arrival/min} \\ \bar{x} = 60 \text{ min} \\ + \text{no blocking} \end{array}\right\} \bar{N} = \lambda \bar{x} = 600$$

This is an M/M/∞ system. We have shown, that

$$p_k = \frac{a^k}{k!} e^{-a} \Rightarrow p_0 = e^{-600} \text{ (very small)}$$