Last lecture (3)

- Solar activity
- Solar wind

Today's lecture (4)

- Ionosphere

  - layers

  - radio wave reflection

  - electrical conductivity in magnetized plasma

Today

Activity	Date	Time	Room	Subject	Litterature
L1	28/8	15-17	Q21	Course description, Introduction, The Sun 1	**CGF** Ch 1.1,1.2, 1.4, 5, (p 110-113), 6.3
L2	29/8	13-15	Q2	The Sun 2, Plasma physics 1	**CGF** Ch 1.3, 5 (p 114-121)
L3	4/9	10-12	E2	Solar wind, The ionosphere and atmosphere 1, Plasma physics 2	**CGF** Ch 6.1, 2, 3.1-3.2, 3.5, **LL** Ch III, Extra material
T1	6/9	8-10	Q21	Mini-group work 1	
L4	6/9	15-17	Q2	The ionosphere 2, Plasma physics 3	**CGF** Ch 3.4, 3.7, 3.8
T2	10/9	15-17	Q21	Mini-group work 2	
L5	11/9	10-12	E3	The Earth's magnetosphere 1, Plasma physics 4	**CGF** 4-1-4.3, **LL** Ch I, II, IV.A
T3	17/9	8-10	Q21	Mini-group work 3	
L6	18/9	13-15	Q33	The Earth's magnetosphere 2, Other magnetospheres	**CGF** Ch 4.6-4.9, **LL** Ch V.
L7	19/9	13-15	Q2	Aurora, Measurement methods in space plasmas and data analysis 1	**CGF** Ch 4.5, 10**, LL** Ch VI, Extra material
T4	24/9	8-10	Q2	Mini-group work 4	
L8	24/9	15-17	V3	Space weather and geomagnetic storms	**CGF** Ch 4.4, **LL** Ch IV.B-C, VII.A-C
T5	2/10	8-10	Q31	Mini-group work 5	
L9	2/10	13-15	Q2	Alfvén waves, Interstellar and intergalactic plasma, Cosmic radiation	**CGF** Ch 7-9, Extra material
T6	8/10	15-17	Q21		
L10	9/10	10-12	Q2	Guest Lecture by Swedish astronaut Christer Fuglesang	
Written examination	16/10	14-19	L21, L22, L31		

Mini groupwork 1

a)

$$h = \frac{42\ \text{mm}}{7\ \text{mm}} \cdot 6378\ \text{km} \cdot 2 = 77000\ \text{km}$$

The thermal energy is divided into motion in the three dimensions, two of which only give rise to a gyro motion around the magnetic field lines, with the motion along the magnetic field corresponding to an energy

$$E = \frac{k_B T}{2} = \frac{1.38 \cdot 10^{-23} \cdot 1.5 \cdot 10^{6}}{2} = 1 \cdot 10^{-17}\ \text{J}$$

$$v = \sqrt{\frac{2E}{m_e}} = \sqrt{\frac{2 \cdot 10^{-17}}{0.91 \cdot 10^{-30}}} = 4.7 \cdot 10^{6}\ \text{ms}^{-1}$$

Approximating the loop with a quarter-circle, the electron has to travel a length

$s = \pi h/2 = 120\ 000$ km

Then we get $t = 25$ s.

Energy - temperature

Average energy of molecule/atom:

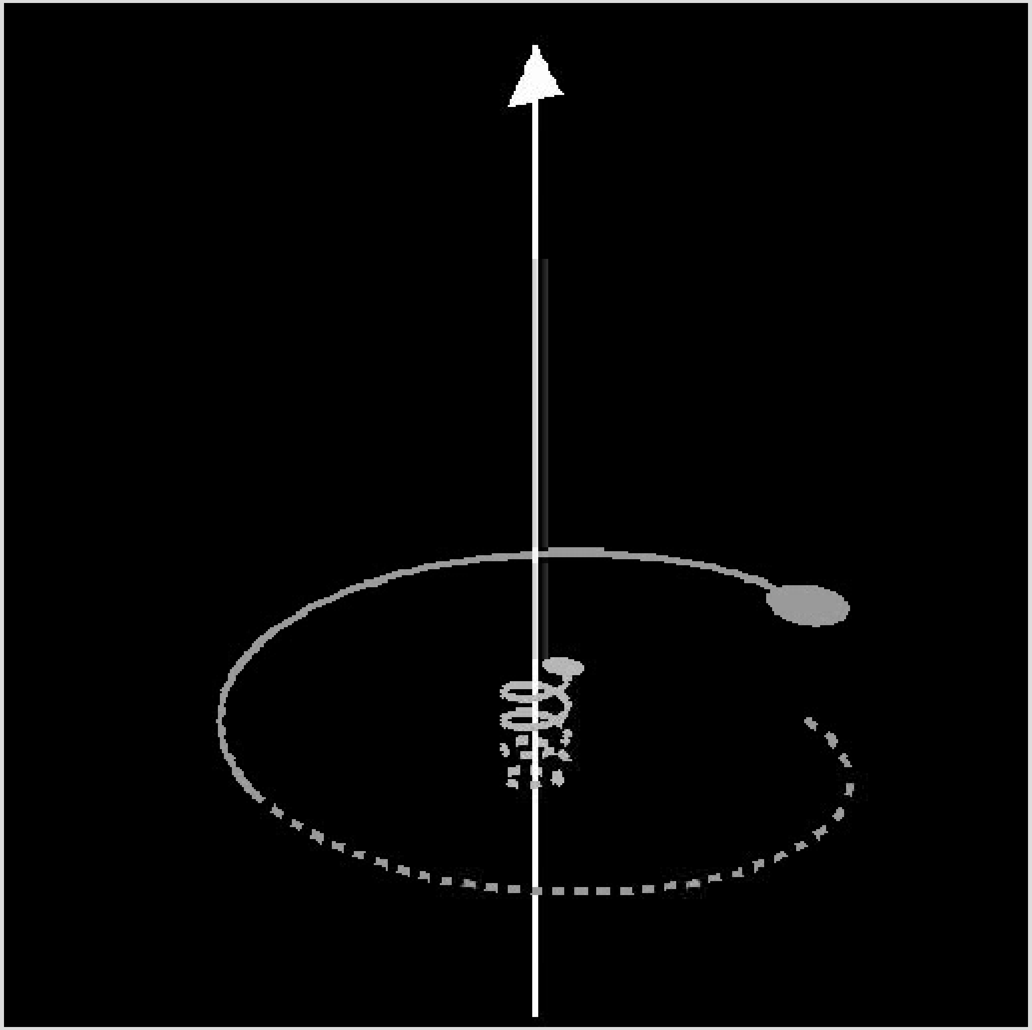
$$E = \frac{3}{2} k_B T \quad \Rightarrow$$

$$T = \frac{2E}{3k_B}$$

$$1 \text{ eV} = 1.6 \cdot 10^{-19} \text{ J} \Rightarrow$$

$$T = \frac{2E}{3k_B} = \frac{2 \cdot 1.6 \cdot 10^{-19} \text{ J}}{3 \cdot 1.38 \cdot 10^{-23} \dfrac{\text{J}}{\text{K}}} = 7729 \text{ K}$$

Gyro motion



Equipartion principle

Statistically the kinetic energy is equally distributed along the three dimensions:

$$E_{\parallel} = \frac{1}{2} k_B T$$

$$E_{\perp} = \frac{2}{2} k_B T$$

Mini groupwork 1

b)

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\frac{qB}{m} \Rightarrow$$

$$B = \frac{2\pi f_c m}{q} = \frac{2\pi \cdot 1 \cdot 10^{10} \cdot 0.91 \cdot 10^{-30}}{1.6 \cdot 10^{-19}} = 0.36\,\mathrm{T}$$

The perpendicular energy is given by

$$E = 2 \cdot \frac{k_B T}{2} = 2 \cdot \frac{1.38 \cdot 10^{-23} \cdot 1.5 \cdot 10^{6}}{2} = 2 \cdot 10^{-17}\,\mathrm{J}$$

$$\Rightarrow$$

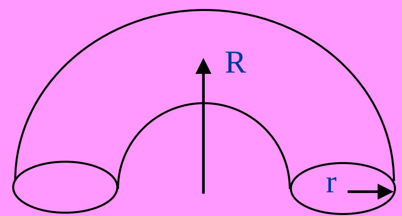
$$v = \sqrt{\frac{2E}{m_e}} = \sqrt{\frac{4 \cdot 10^{-17}}{0.91 \cdot 10^{-30}}} = 6.6 \cdot 10^{6}\ \mathrm{ms^{-1}}$$

$$\rho = \frac{m_e v_\perp}{qB} = \frac{0.91 \cdot 10^{-30} \cdot 6.6 \cdot 10^{6}}{1.6 \cdot 10^{-19} \cdot 0.36} = 1.0 \cdot 10^{-4}\ \mathrm{m}$$

Mini groupwork 1

c)

Model the flare by a half torus with minor axis r, and major axis. From the figure, estimate R = 2·6 R_E, and r = 2 R_E.

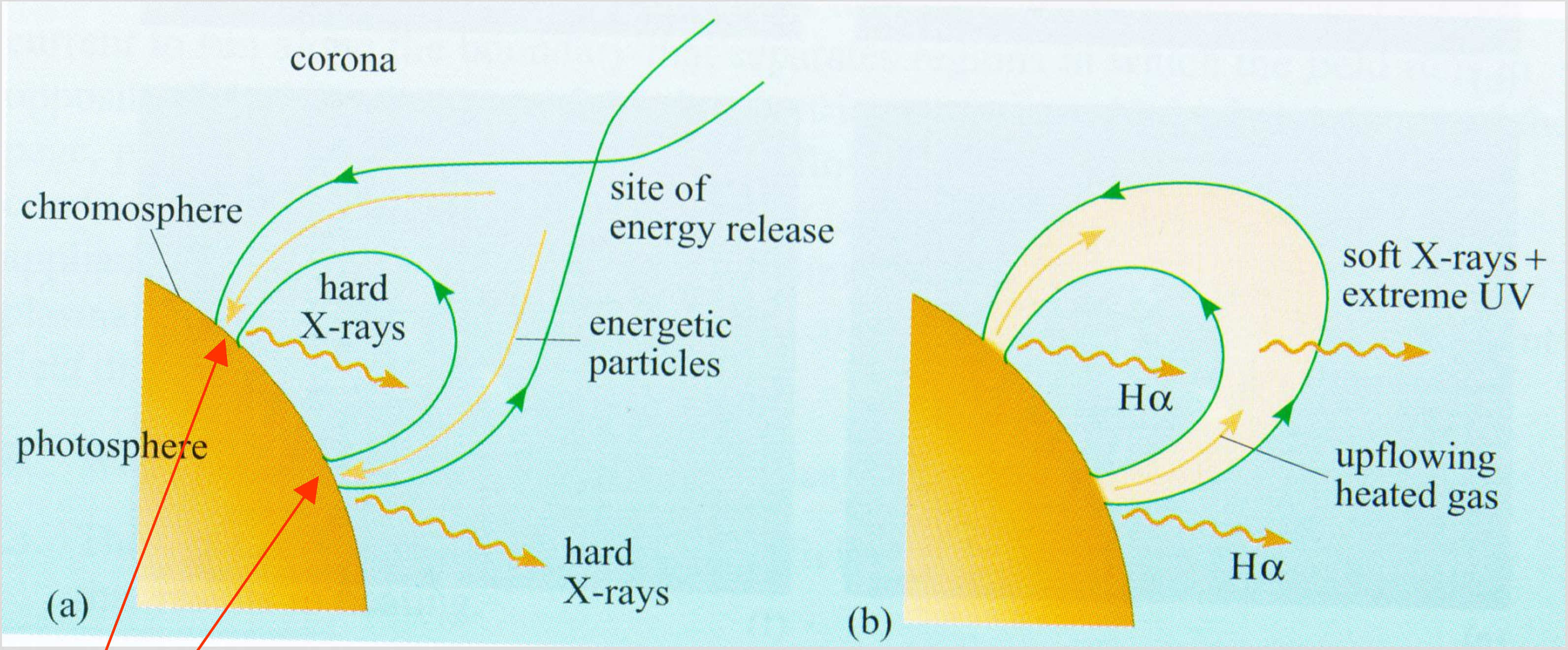


Let this half-torus be filled with a magnetic field of strength B ~ 0.36 T (using the value in b)). If the volume of the half-torus is V and the magnetic energy density is p_B, the total energy is

$$W = Vp_B = \pi R \pi r^2 \frac{B^2}{2\mu_0} = \pi^2 \cdot 2 \cdot 12^2 {R_E}^3 \frac{B^2}{2\mu_0}$$

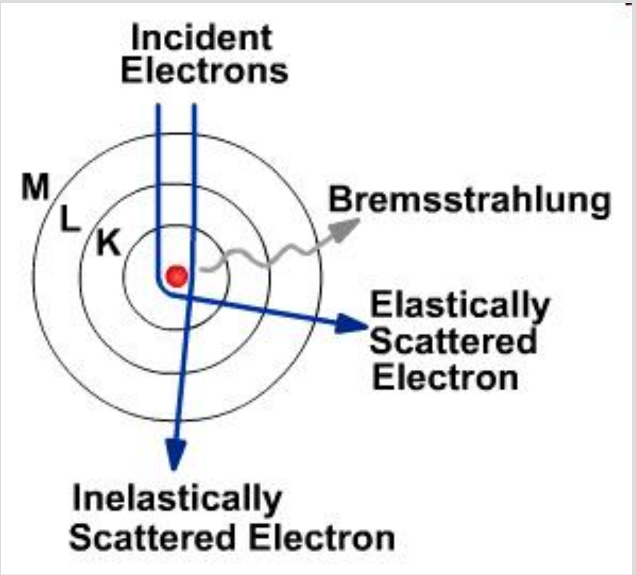
$$= \pi^2 \cdot 2 \cdot 12^2 \, (6378 \cdot 10^3)^3 \frac{(0.36)^2}{2\mu_0} = 3.8 \cdot 10^{28} \text{ J}$$

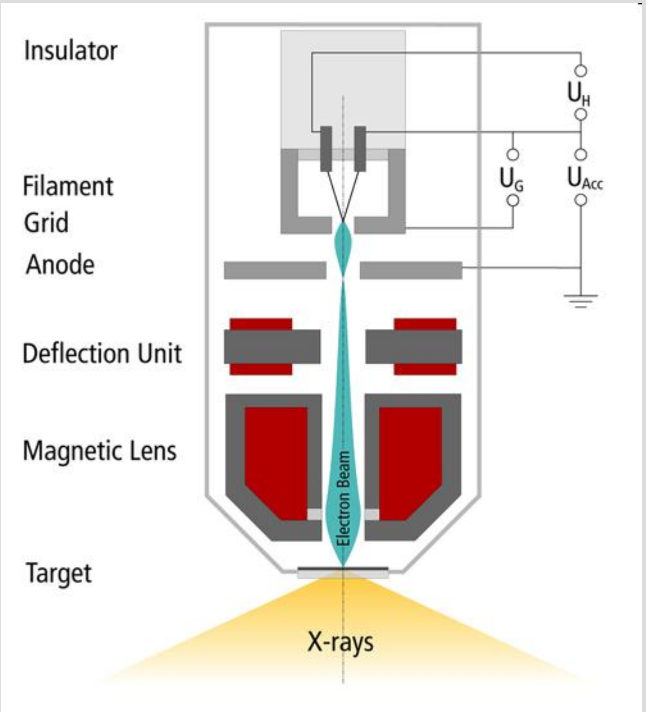
Solar flare mechanism

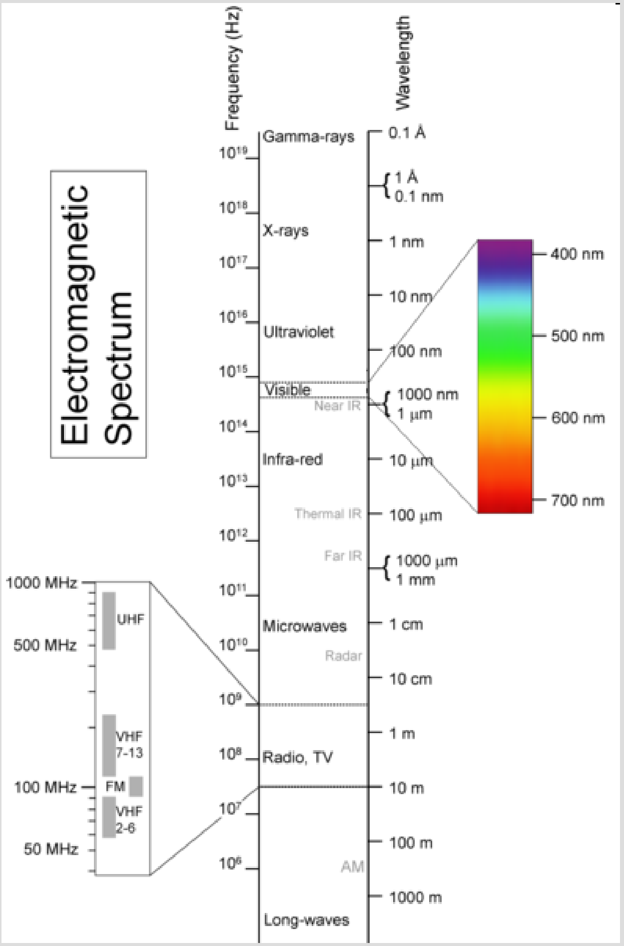


Electrons are accelerated, collide with solar surface (photosphere) and emit bremsstrahlung (X-rays).

X-rays, Bremsstrahlung



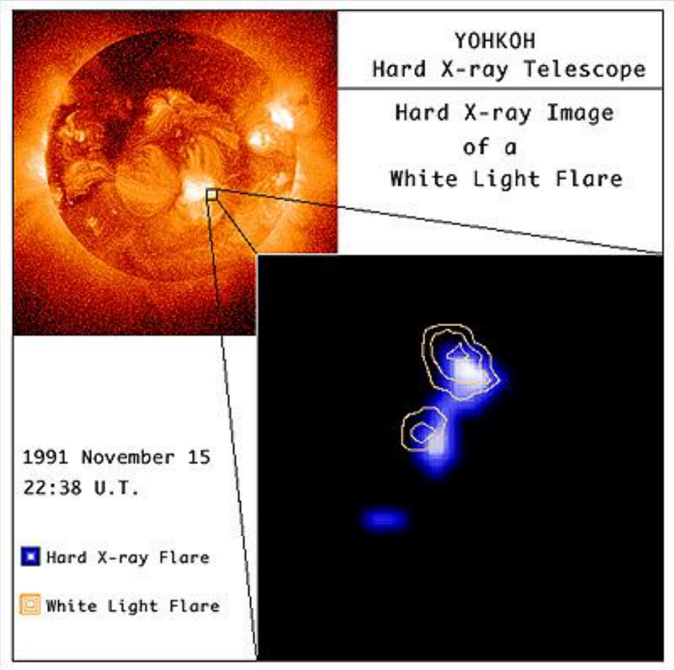




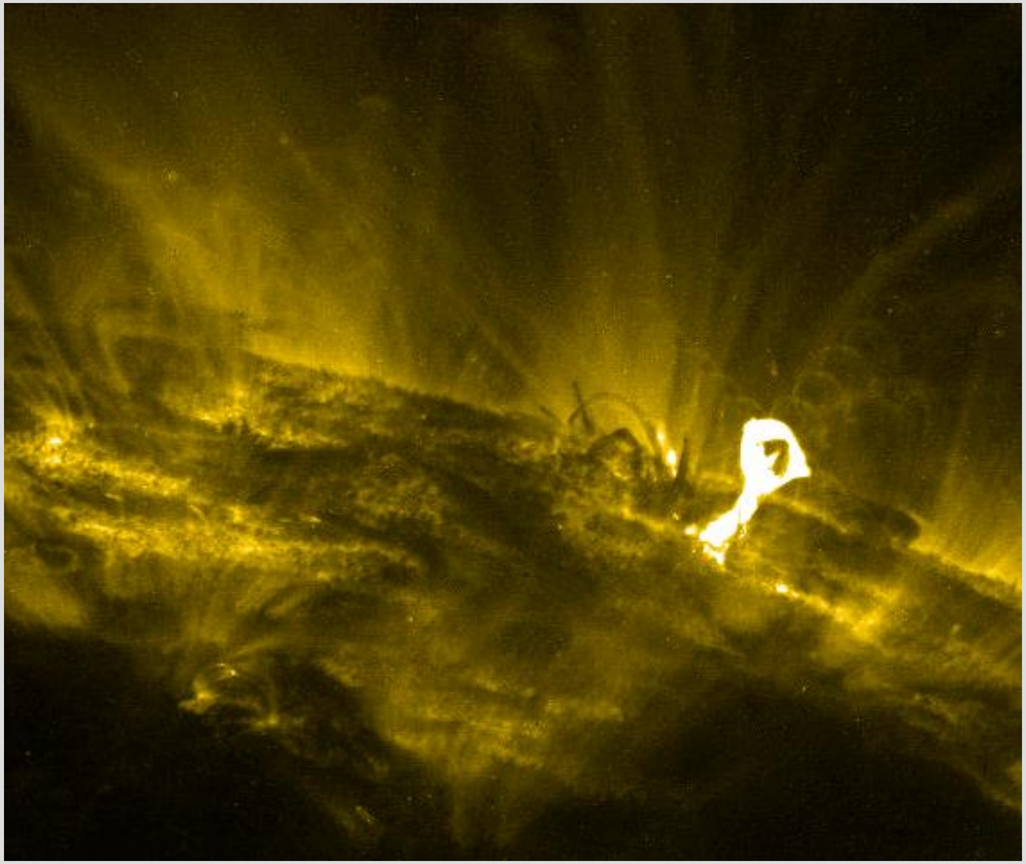
$$U_{acc} = hf = \frac{hc}{\lambda} \quad \Rightarrow$$

$$\lambda = \frac{hc}{U_{acc}} = \frac{6.62\cdot 10^{-34}\cdot 3\cdot 10^{8}}{100\cdot 10^{3}\cdot 1.6\cdot 10^{-19}} = 1.2\cdot 10^{-11}\,\mathrm{m} = 0.012\,\mathrm{nm}$$

Solar flare observations



(a) double signature of x-ray emissions at foot of flare



(b) coronal loop filled with hot gas

Frozen in magnetic flux *PROOF II*

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{v} \times \mathbf{B})}_{A} + \underbrace{\frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}}_{B}$$

Order of magnitude estimate:

$$\frac{A}{B} = \frac{\nabla \times (\mathbf{v} \times \mathbf{B})}{\frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}} \approx \frac{\frac{v \Delta B}{L}}{\frac{\Delta B}{\mu_0 \sigma L^2}} = v L \mu_0 \sigma \equiv R_m$$

Magnetic Reynolds number R_m:

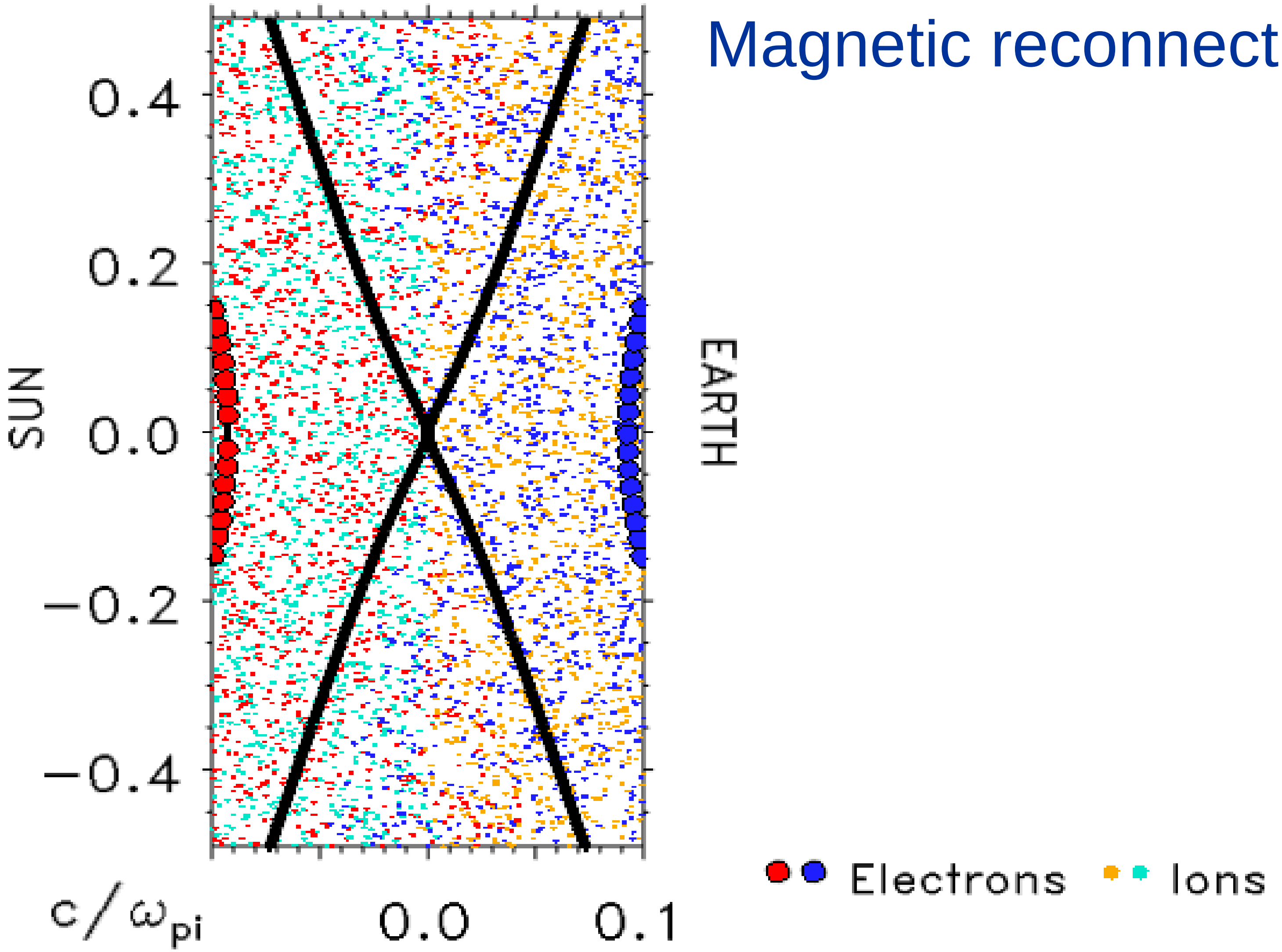
$$\mathrm{R_m} >> 1 \Rightarrow \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Frozen-in fields!

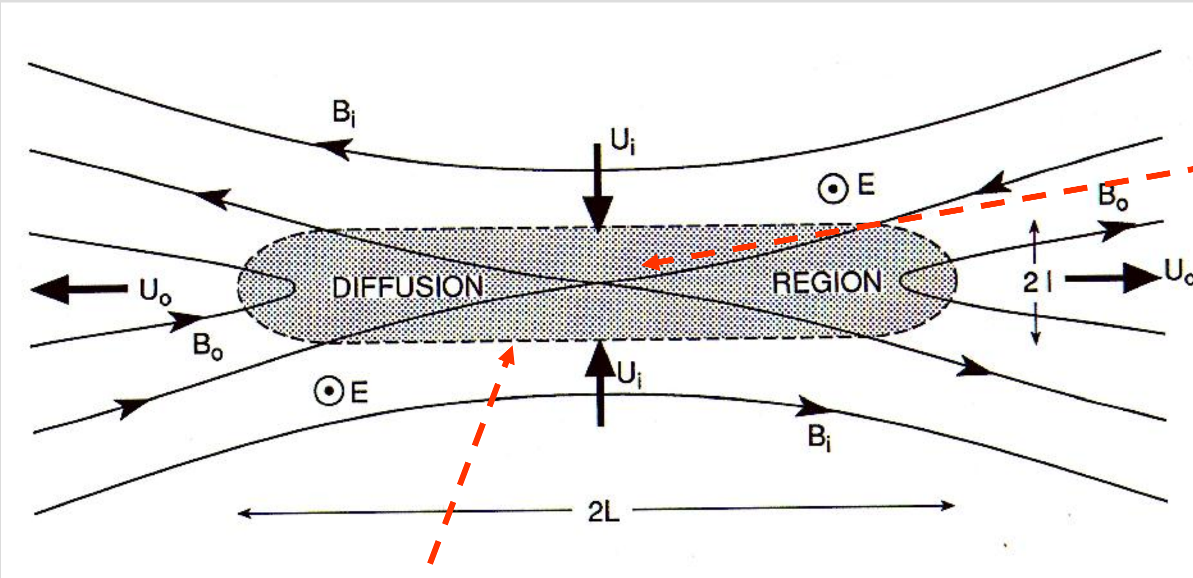
$$\mathrm{R_m} << 1 \Rightarrow \quad \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

Diffusion equation!

Magnetic reconnection



Reconnection



- Field lines are "cut" and can be re-connected to other field lines
- **Magnetic energy is transformed into kinetic energy ($U_o >> U_i$)**

In 'diffusion region':

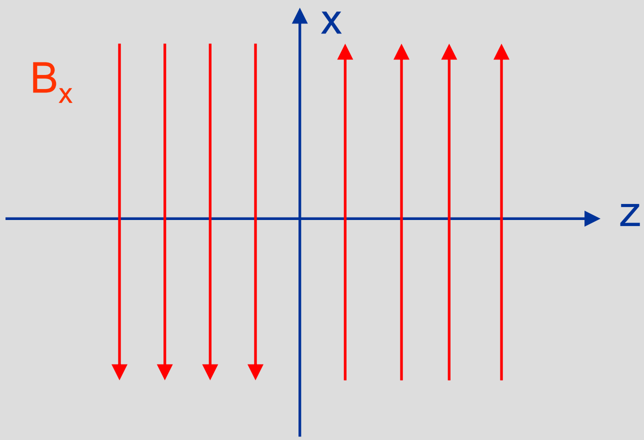
$R_m = \mu_0 \sigma l v \sim 1$

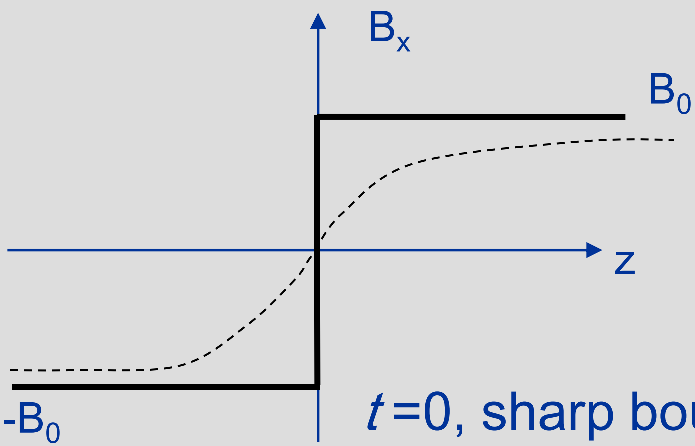
Thus: condition for frozen-in magnetic field breaks down.

A second condition is that there are two regions of magnetic field pointing in *opposite* direction:

- **Plasma from different field lines can mix**

Reconnection in 1D





t =0, sharp boundary

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \longrightarrow \frac{\partial B_x}{\partial t} = \frac{1}{\mu_0 \sigma} \frac{\partial^2 B_x}{\partial z^2}$$

Diffusion equation! Has solution

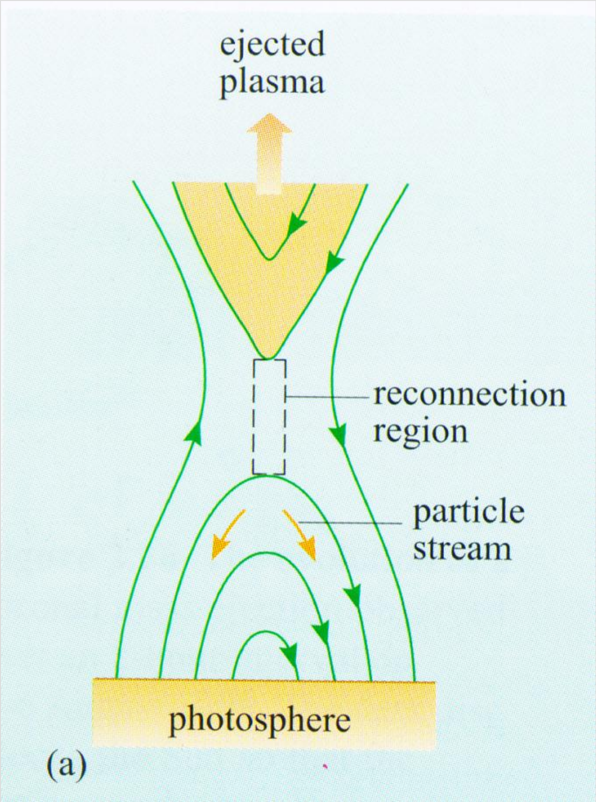
$$B_x(z,t) = B_0 erf\left(\left[\frac{\mu_0 \sigma}{4t} \right]^{1/2} z \right)$$

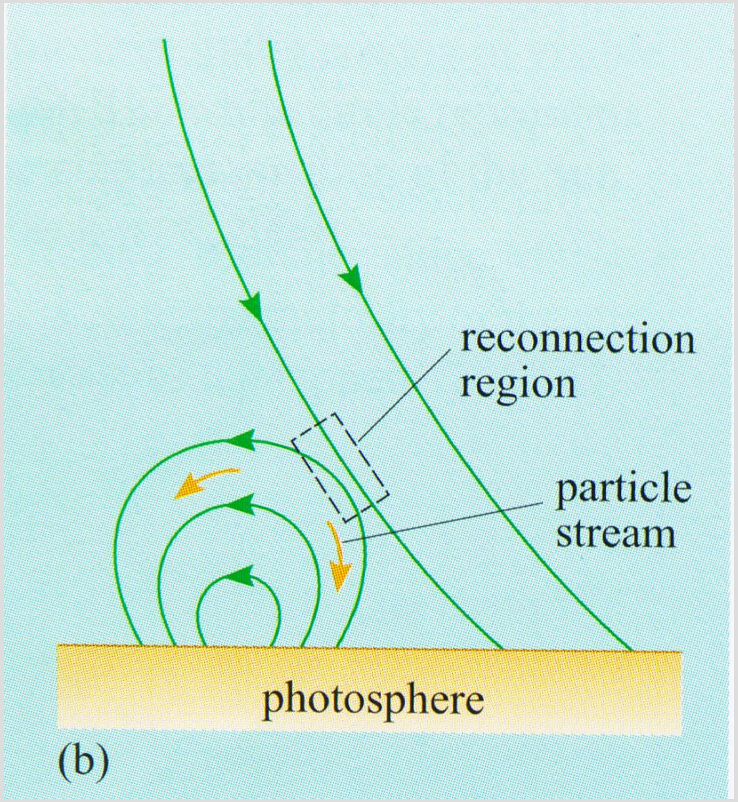
The total magnetic energy then decreases with time:

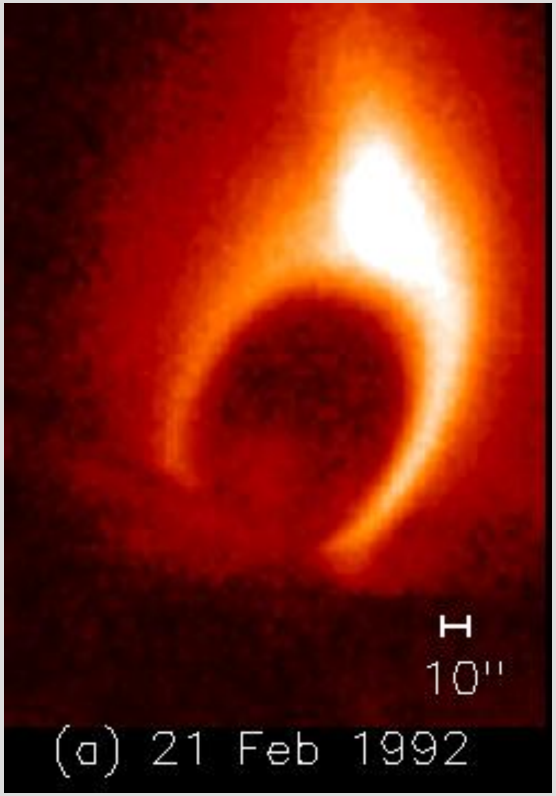
$$W_B = \int_{-\infty}^{\infty} \frac{B^2}{2\mu_0} dz$$

The magnetic energy is converted into heat and kinetic energy in 2D

Solar flare
energization mechanism



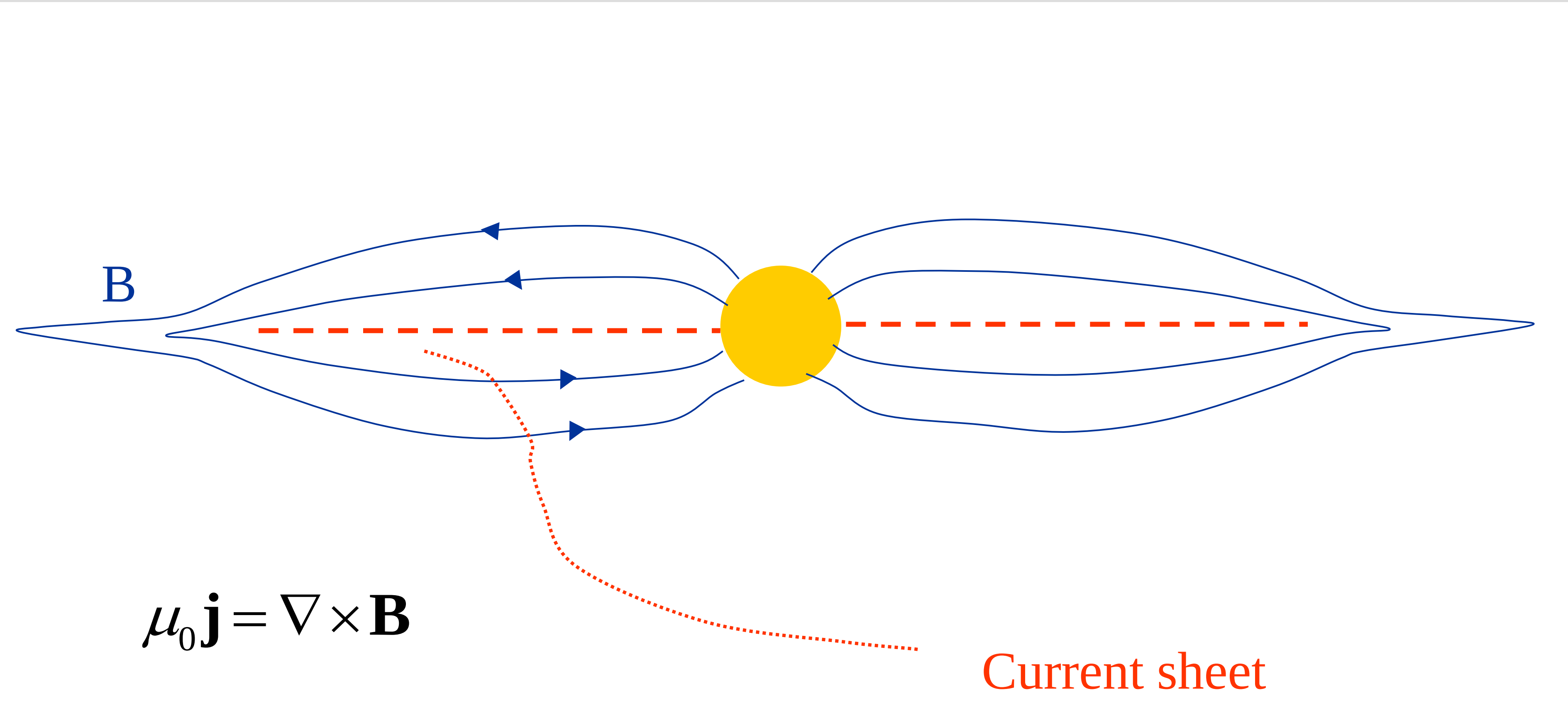




Two possible reconnection geometries

Solar wind

Interplanetary current sheet



Solar wind

Some basic facts

Average values

$n_p = 8\ cm^{-3}$

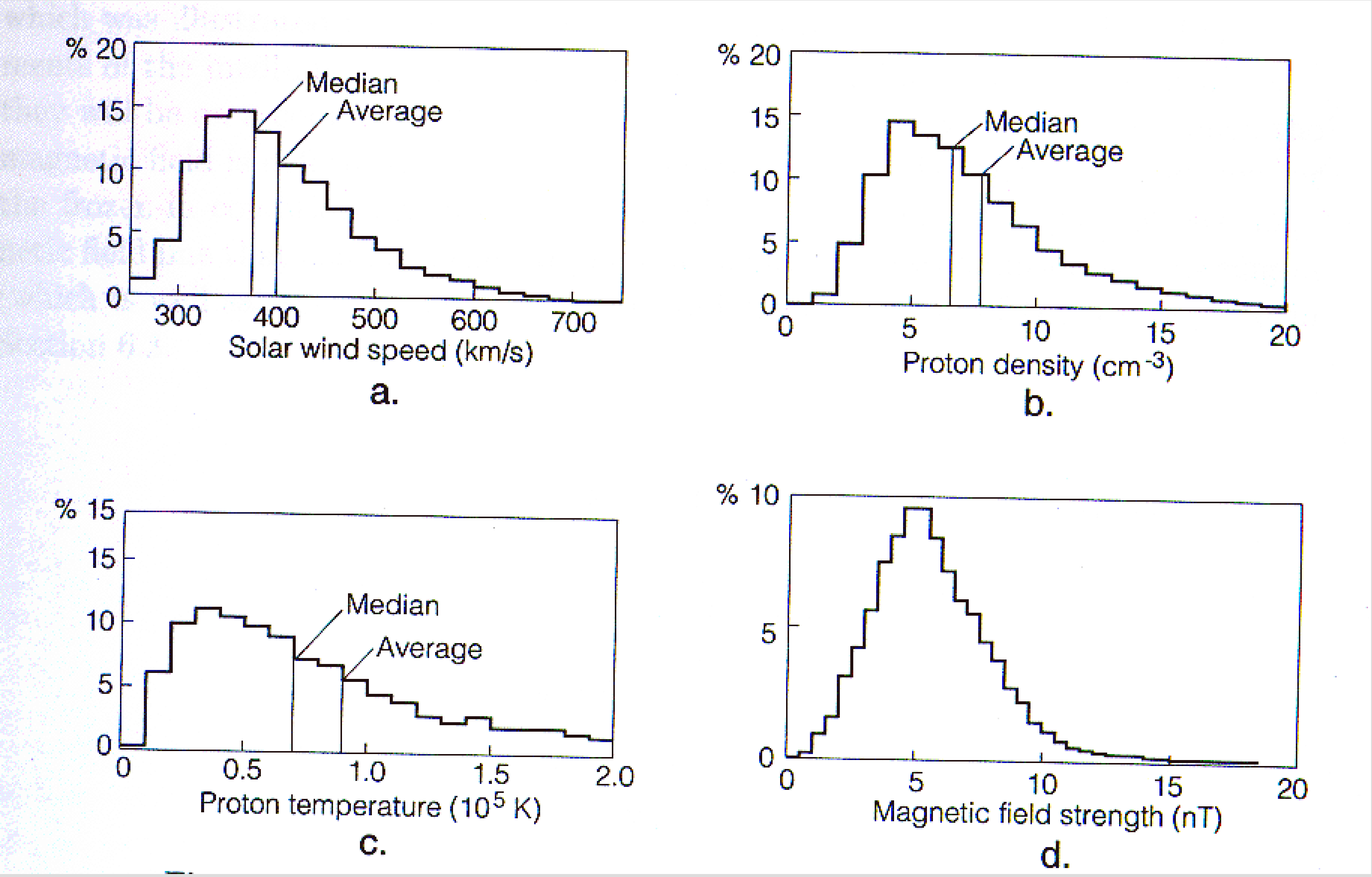
$v = 320\ km/s$

$T_p = 4{\cdot}10^4\ K$

$T_e = 10^5\ K$

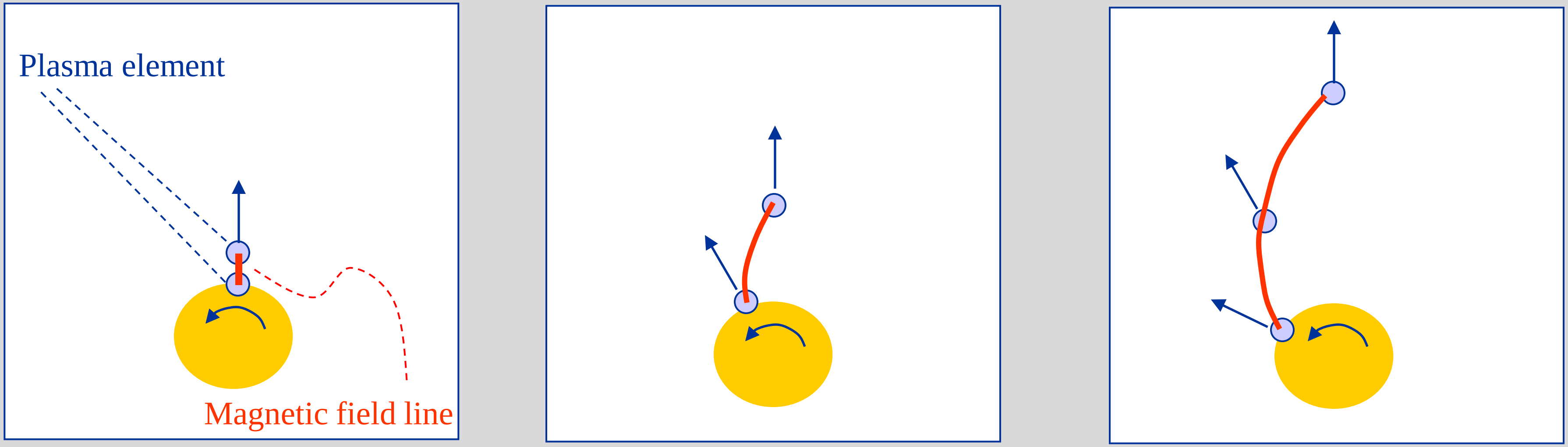
$B = 5\ nT$

$\Phi_K = \rho v^3/2 = 0.22\ mW/m_2$



Solar wind

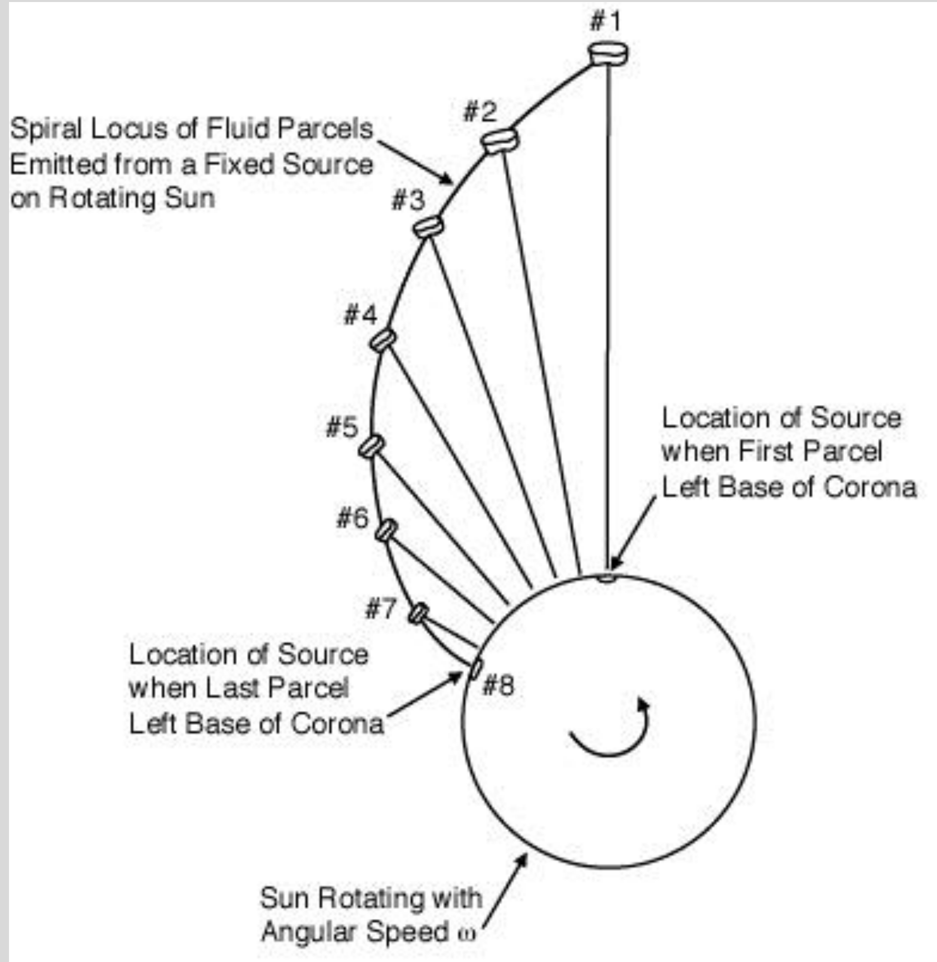
Magnetic field frozen into solar wind



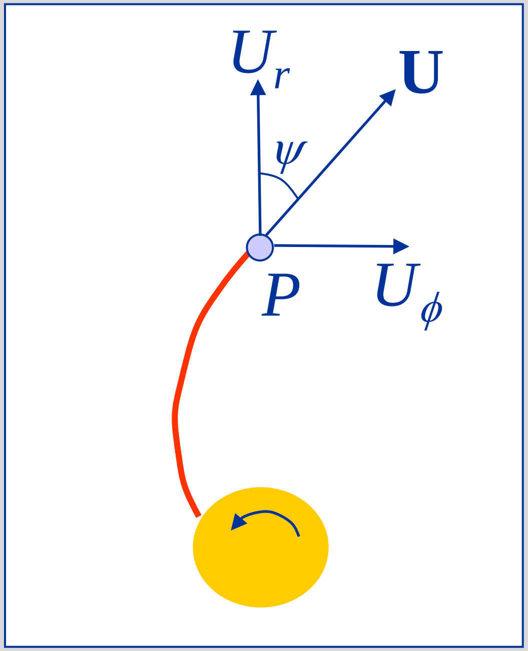
This is now seen from ”above”! (Looking down on the ecliptic plane from the pole.)

Solar wind

Parker spiral



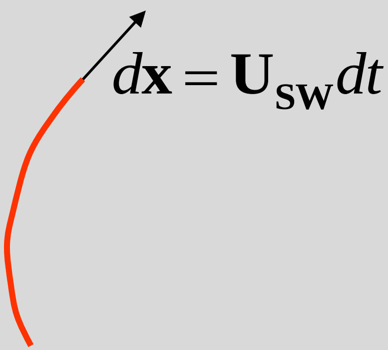
Parker spiral



Derivation of Ψ (Parker angle)

Consider a coordinate system rotating with the sun. The plasma element P in this coordinate system has two velocity components: U_r and U_ϕ.

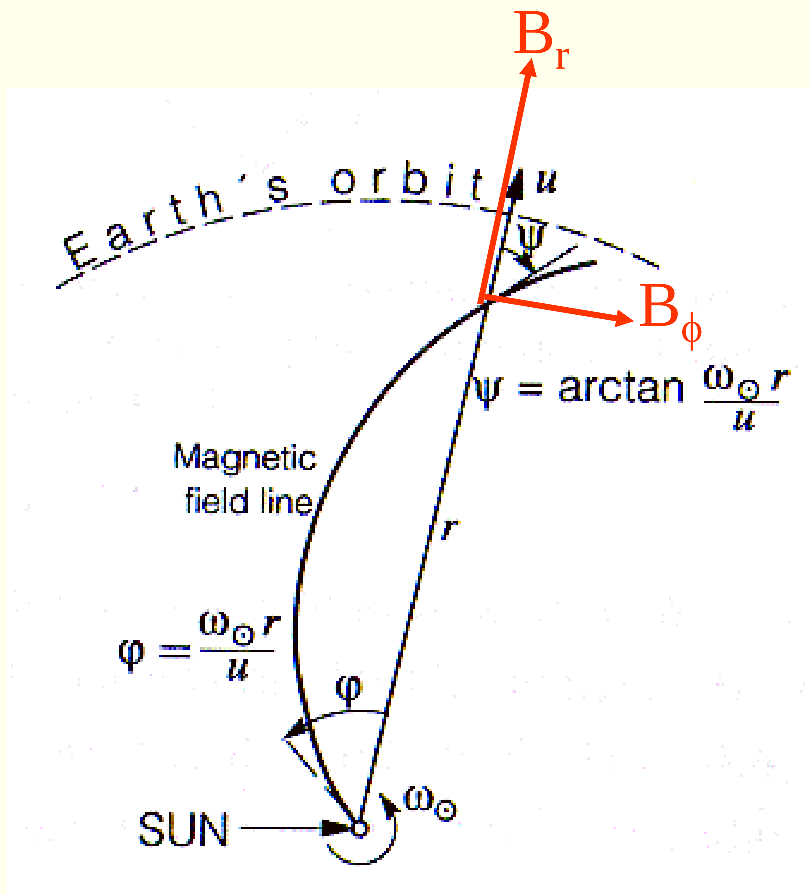
Since the magnetic field is frozen into the solar wind, and follows the orbit of the plasma element P, at any time B has to be parallel to U. Then we have:



$$\tan \psi = \frac{B_\phi}{B_r} = \frac{U_\phi}{U_r} = \left(\frac{\omega r}{u_{SW}} \right)$$

Solar wind

Parker spiral



Archimedean spiral:

$$\frac{B_\phi}{B_r} = \tan\psi = \left(\frac{\omega r}{u_{SW}}\right)$$

Archimedean spiral

An Archimedean spiral (also arithmetic spiral), is a spiral named after the 3rd-century-BC Greek mathematician Archimedes; it is the locus of points corresponding to the locations over time of a point moving away from a fixed point with a constant speed along a line which rotates with constant angular velocity. Equivalently, in polar coordinates (r,ϕ) it can be described by the equation

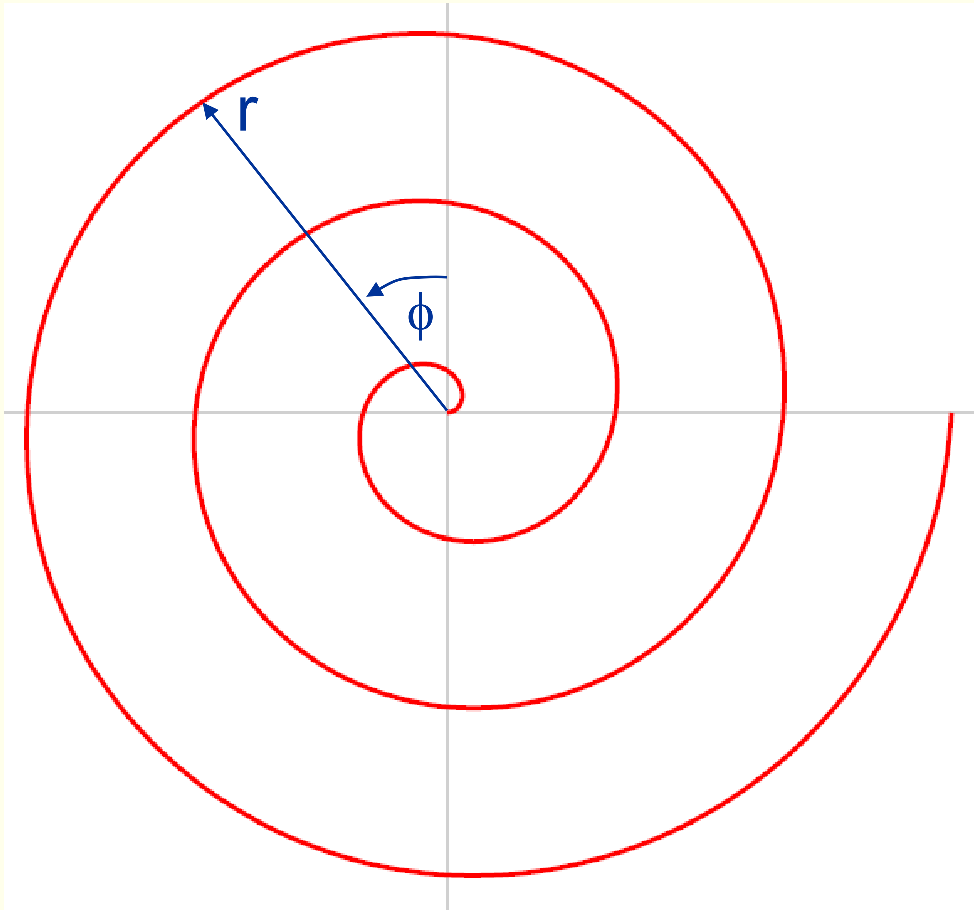
$$r = a + b\phi$$

$$r = a + b\omega t$$

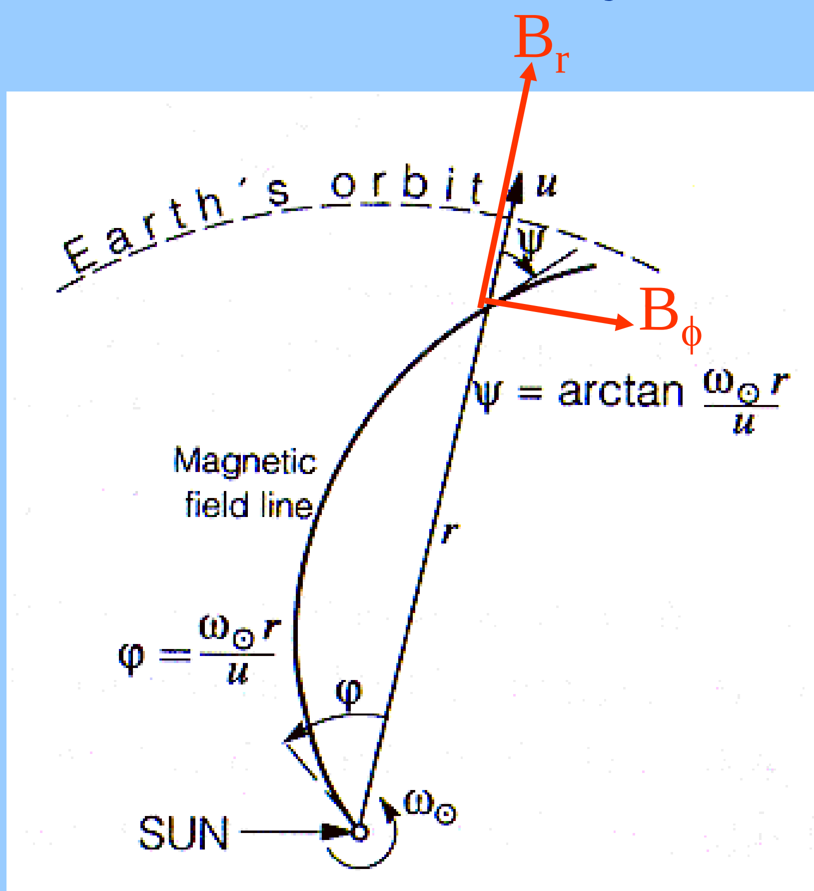
$$\frac{dr}{dt} = b\omega = u_{SW}$$

$$b = \frac{u_{SW}}{\omega}$$

$$r = R_{sun} + \frac{u_{SW}}{\omega}\phi$$



Use rotation period
T of sun: $T = 27$ days



$r = 1$ A.U.

What is the angle Ψ at Earth's orbit for a typical solar wind speed?

Yellow	$\approx 50°$
Red	$\approx 80°$
Blue	$\approx 1°$
Green	$\approx 10°$

$$\Psi = \arctan(\frac{\omega r}{u})$$

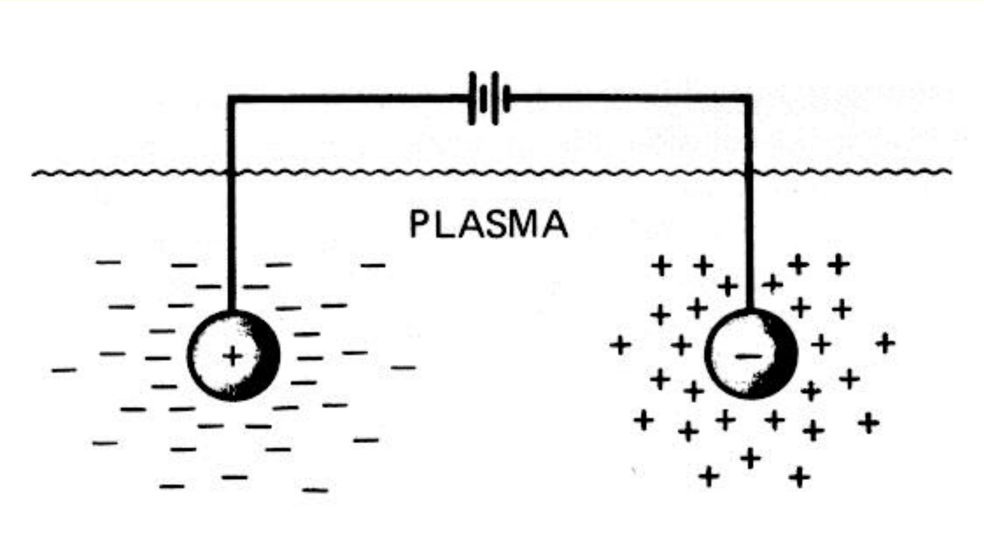
What is ω?

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{27 \cdot 24 \cdot 60 \cdot 60} = 2.7 \cdot 10^{-6} s^{-1}$$

$$\Psi = \arctan(\frac{\omega r}{u}) = \arctan(\frac{2.7 \cdot 10^{-6} \cdot 1.5 \cdot 10^{11}}{320 \cdot 10^{3}}) = \arctan(1.27) = 52^{\circ}$$

Yellow

Quasineutrality



$$\frac{\Delta n}{n} = \frac{(n_e - n_i)}{n_e} < \left(\frac{\lambda_D}{l_c}\right)^2$$

$$\Phi = \Phi_0 e^{-x/\lambda_D}$$

$$l_C >> \lambda_D \quad \Rightarrow$$

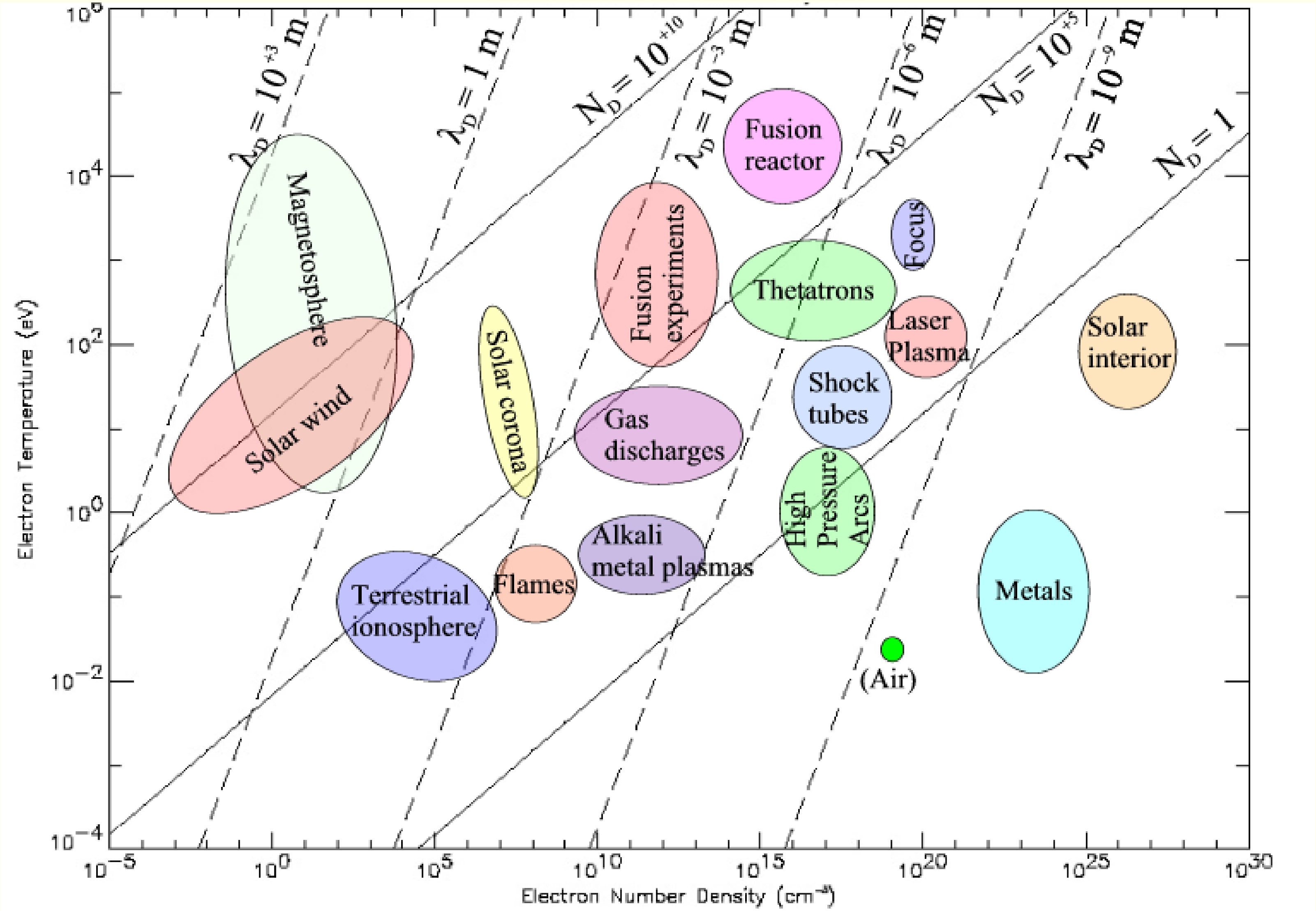
Debye length

Plasma close to neutral:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_e e^2}}$$

$$n_e \approx n_i$$

Debye lengths



Plasma models/descriptions

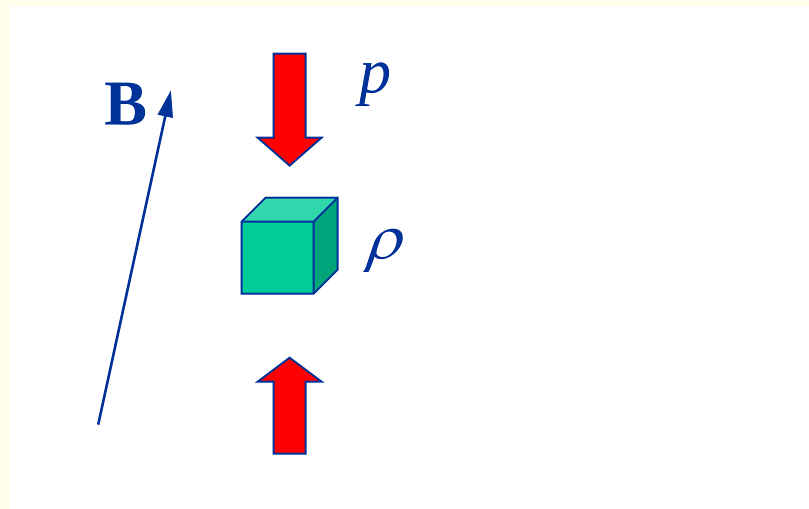
- Single particle motion
- Computer simulations of many-particle dynamics
- Generalization of statistical mechanics (kinetic theory)
- Generalization of fluid mechanics: *Magneto-hydrodynamics (MHD)*

Plasma physics
Magnetohydrodynamics (MHD)

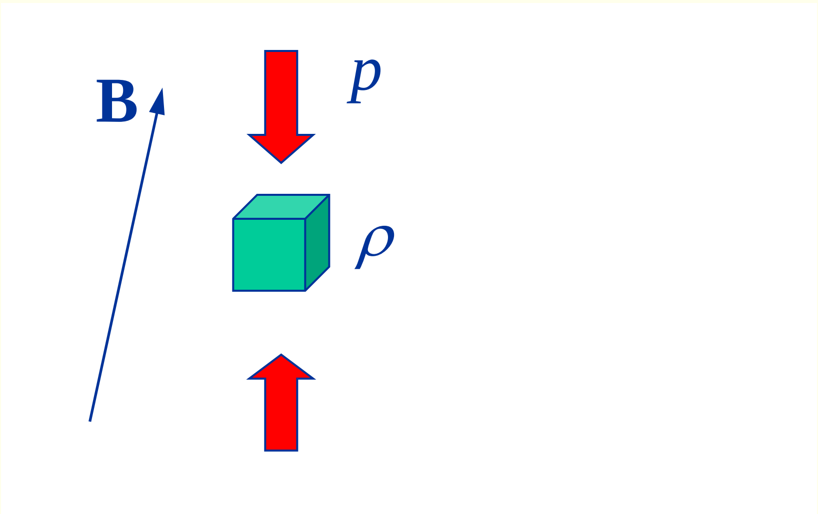
MHD is a combination of

- *fluid-/hydrodynamics* (which is based on Newton's laws of motion)
- *Maxwell's equations* (electrodynamics)

applied on a plasma volume element.



Magnetohydrodynamics (MHD)



For a volume element of plasma:

$$\mathbf{F} = m\mathbf{a} \Rightarrow$$

quasineutrality

$$-\nabla p + n_e q \mathbf{v}_e \times \mathbf{B} + \rho_q \mathbf{E} = \rho \frac{d\mathbf{v}}{dt} \Rightarrow$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} \qquad (1)$$

Magnetohydrodynamics (MHD)

(1) $$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}$$

This together with two of Maxwell's equations and Ohm's law make up the most common MHD equations:

(2) $$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(3) $$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Only consider slow variations

(4) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Magnetohydrodynamics (MHD)

(1) $$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B}$$

In equilibrium:

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} \quad \Longleftrightarrow$$

$$-\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$

$$-\nabla p - \nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} = 0$$

Represents tension in magnetic field

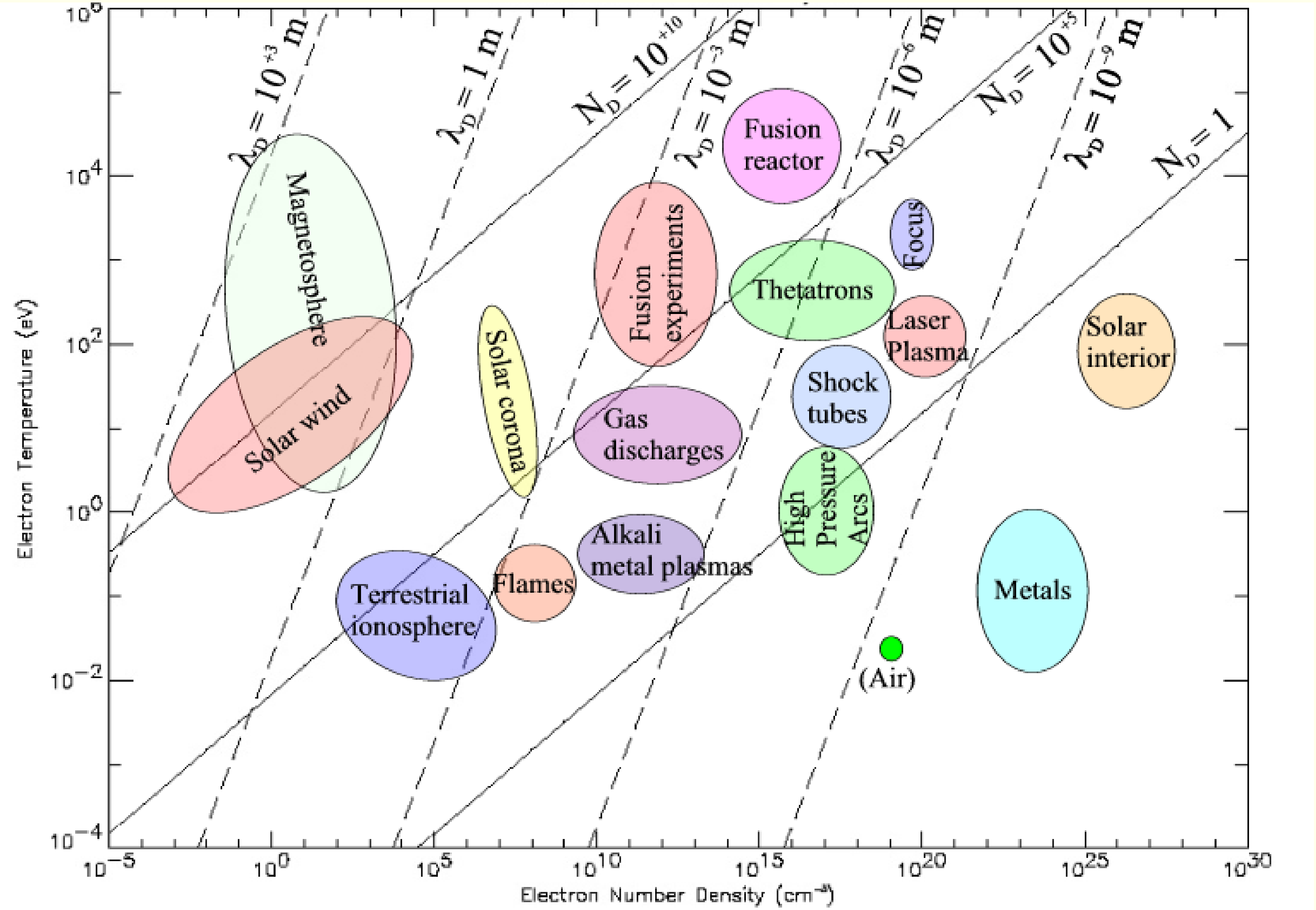
If magnetic tension = 0

$$p + \frac{B^2}{2\mu_0} = konst$$

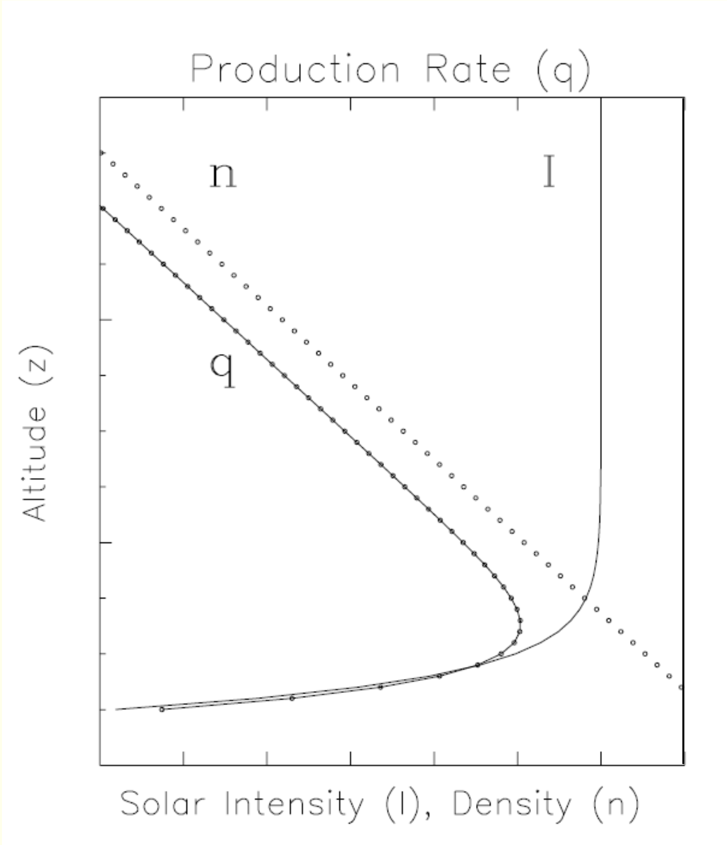
Magnetic pressure

The ionosphere

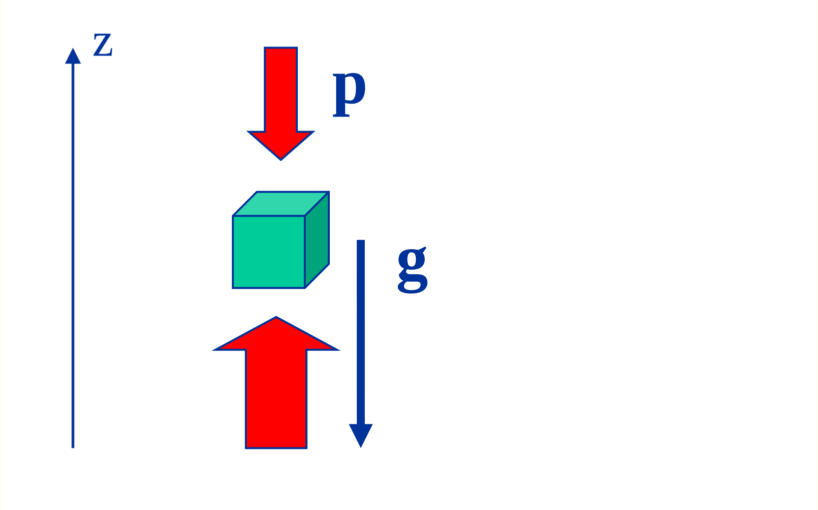
Debye lengths



Basic principle for creation of ionospheric layer



Atmospheric scale height



$$-\frac{dp}{dz} = g\rho_m$$ hydrostatic equilibrium for a volume element

$$p = nk_B T = \frac{\rho k_B T}{m}$$ ideal gas law

$$-\frac{k_B T}{m}\frac{d\rho_m}{dz} = g\rho_m$$ if T is constant

$$\rho_m = const \cdot e^{-z/(k_B T/gm)} = const \cdot e^{-z/H}$$

Scale height

$H = k_B T/gm$

Scale height

$$H = k_B T/gm$$

What is the approximate scale height in the atmosphere right here, right now?

(0° C = 273 K)

Blue	1 km	Yellow	30 km
Green	9 km	Red	100 km

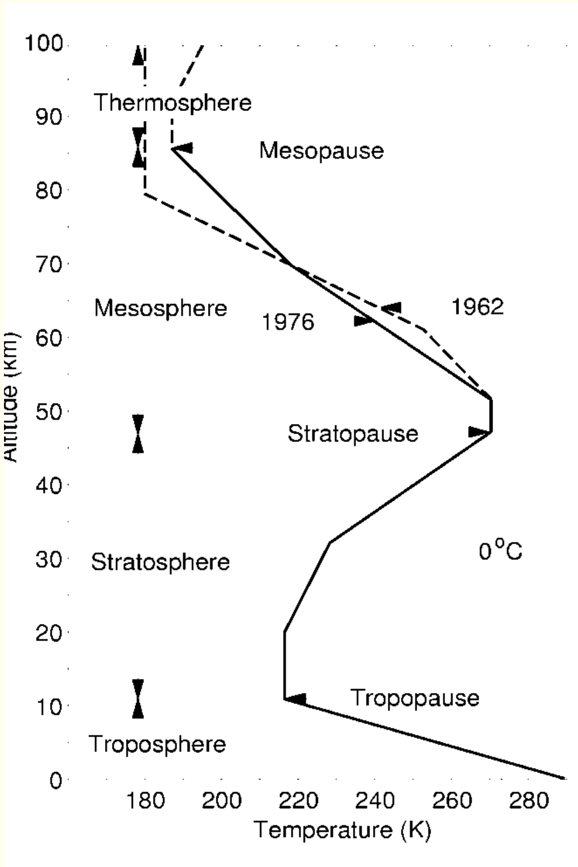
$H = k_B T/gm = (1.38 \cdot 10^{-23} \cdot 290)/(9.81 \cdot 14 \cdot 2 \cdot 1.67 \cdot 10^{-27}) =$

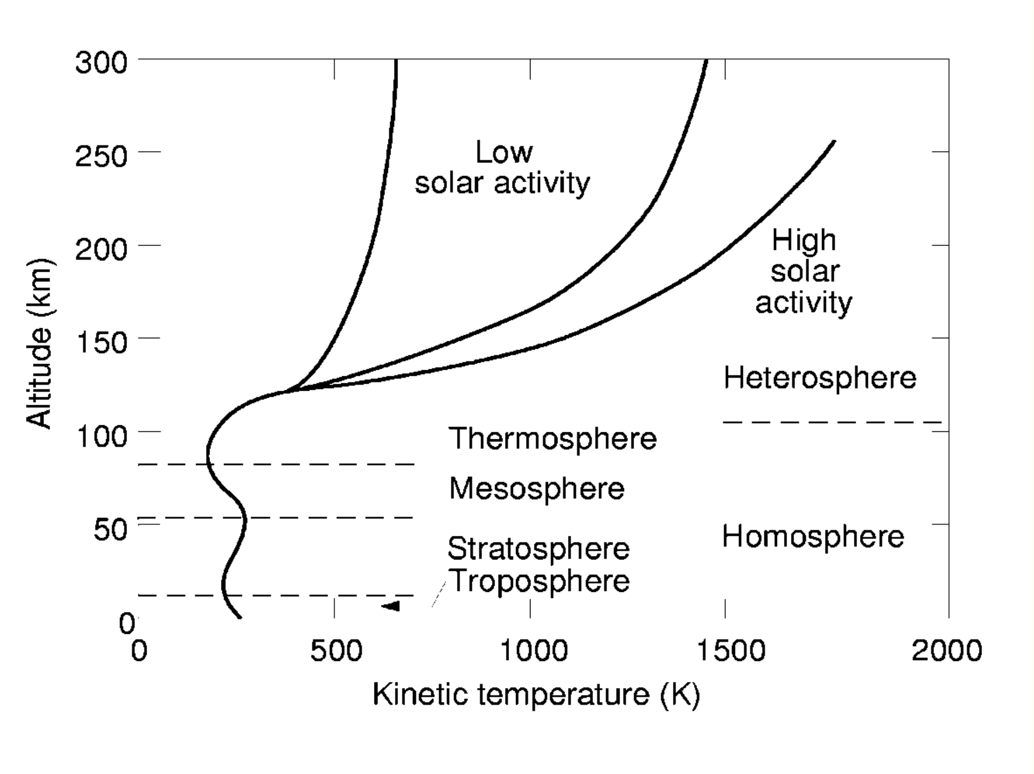
$= 8724 \text{ m} \approx 9 \text{ km}$

Green

What did we neglect when we derived the scale height?

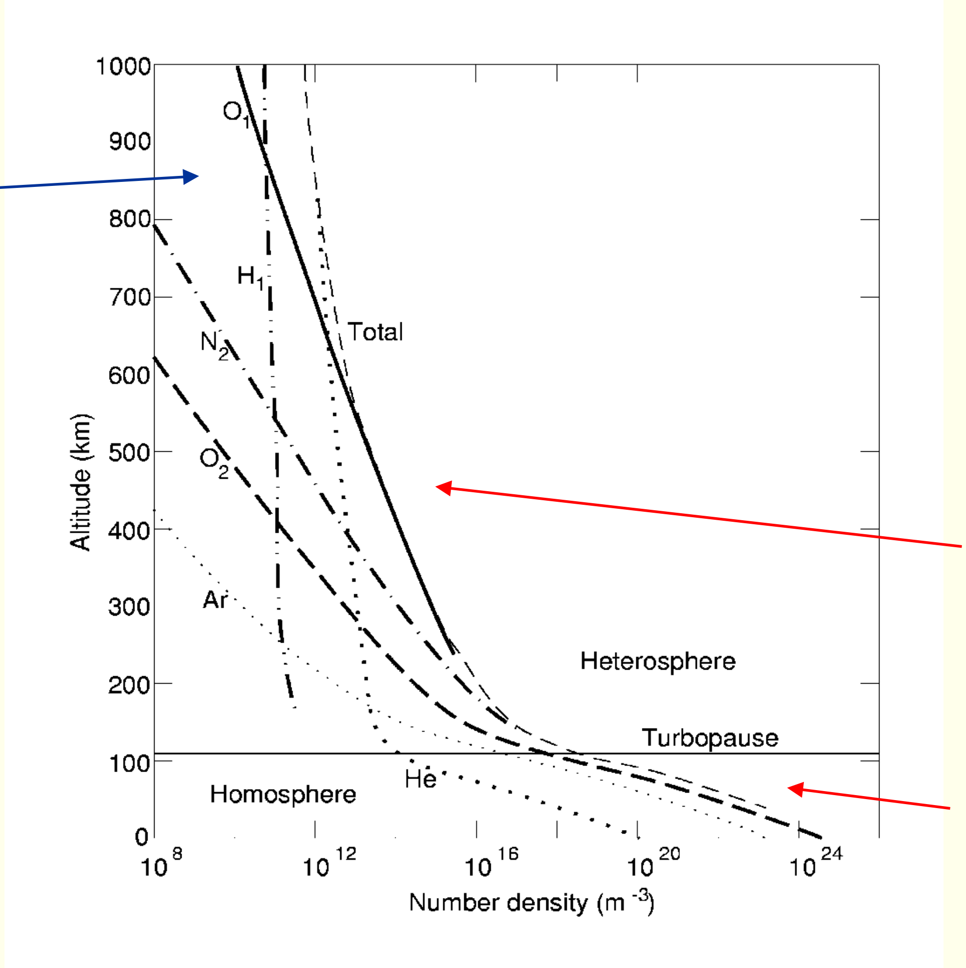
Temperature profile





Atmospheric composition

Longer scale height due to higher temperature



Separate scale heights for different components

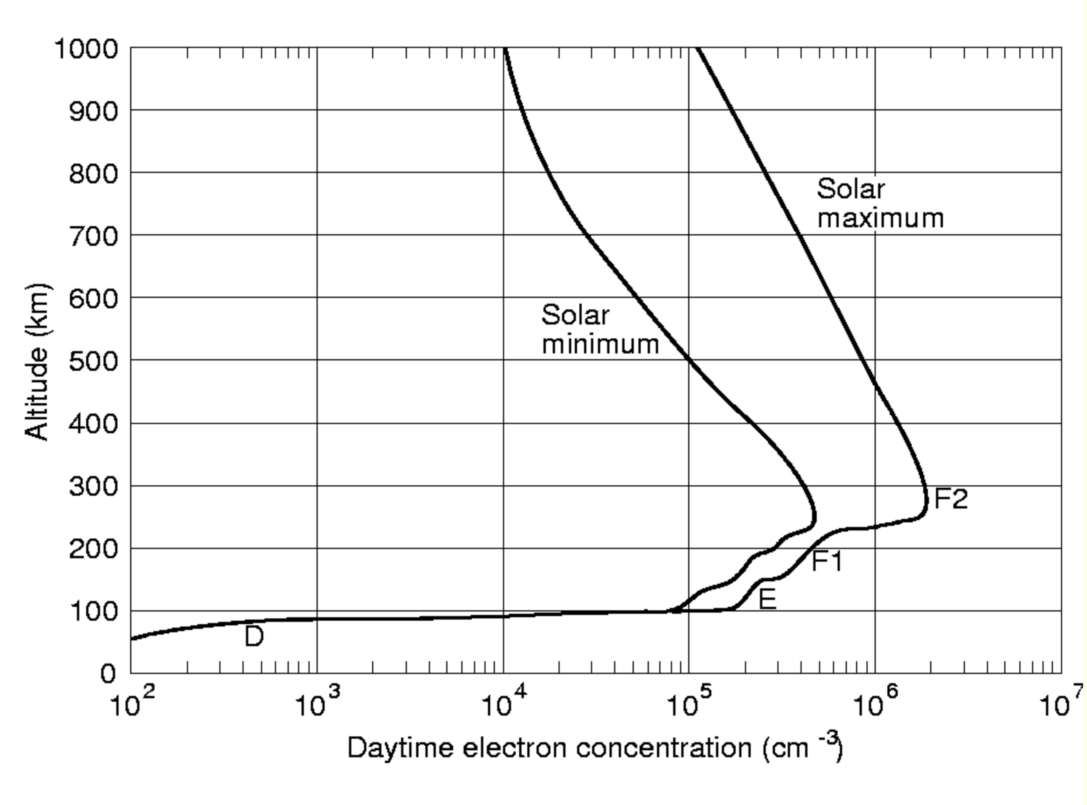
Turbulent mixing – one scale height

Ionosphere History

- The ionized, electrically conducting part of the upper atmosphere
- The ionosphere is a **plasma**

- Stewart, 1882: Explained variations in the geomagnetic field
- Kenelly & Heavyside, 1902: explained Marconis transatlantic radio communication experiments
- Appleton & Barnett: experimental proof

Altitude distribution of electron density (n_e)



Continuity equation = conservation of ?

$$\frac{\partial n_e}{\partial t} = q - r - \nabla \cdot (n_e \mathbf{v}_e)$$

Ionization ($m^{-3}s^{-1}$)

Recombination ($m^{-3}s^{-1}$)

Flow ($m^{-3}s^{-1}$)

Continuity equation

$$\frac{dn_e}{dt} = q - r$$

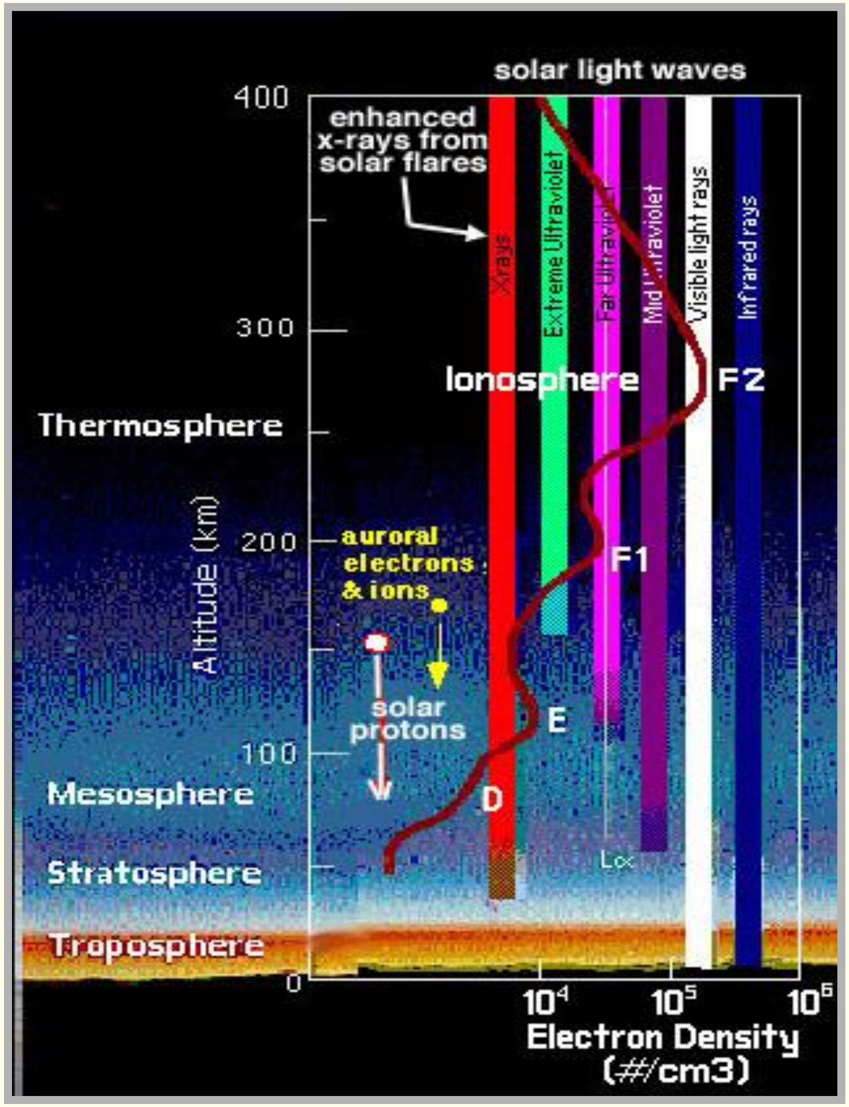
$$q = a_i I n_n$$

Ionization ($m^{-3}s^{-1}$)

Recombination ($m^{-3}s^{-1}$)

$$r = a_r n_e n_i = a_r n_e^2$$

Example: $e + O_2^+ \rightarrow O + O$ (dissociative recombination)



UV and X–ray radiation

$$\frac{dI}{dz} = I n_n a_a$$

Derive Chapman layer

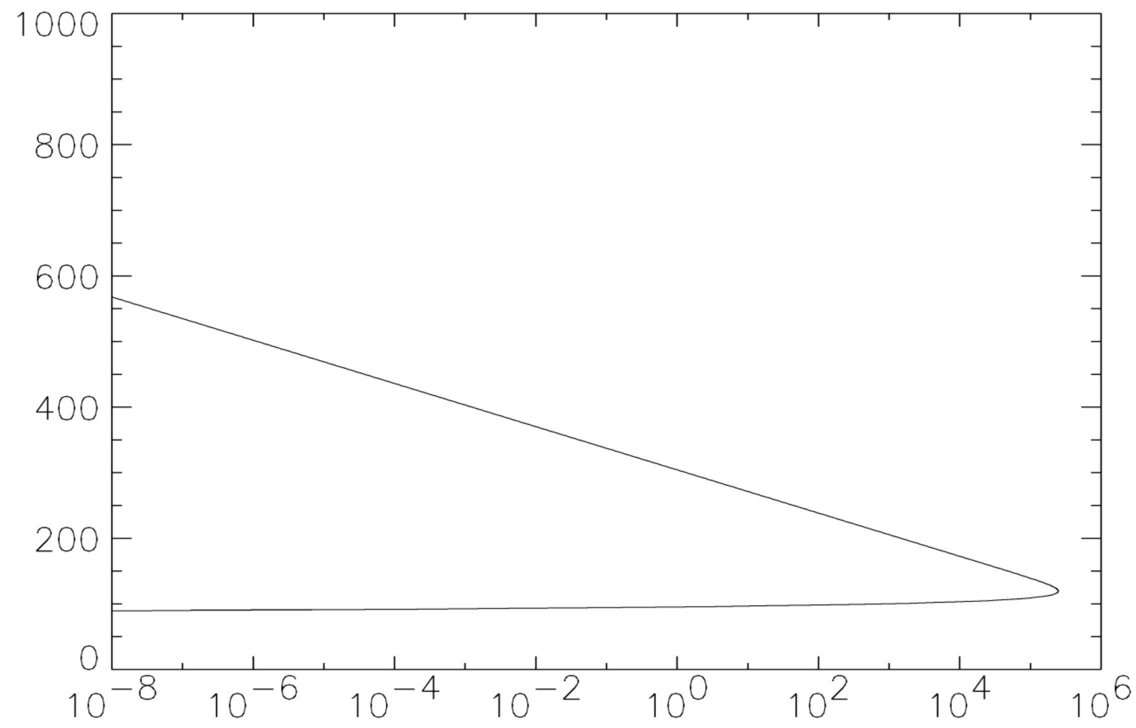
What does it look like in reality?

- Temperature not constant
- Many different wavelengths in solar radiation
- Several different molecules and atoms in neutral atmosphere. Composition also depends on altitude.

"E-region" - simple model calculation

O_2 dominating species, 10 nm X-ray radiation

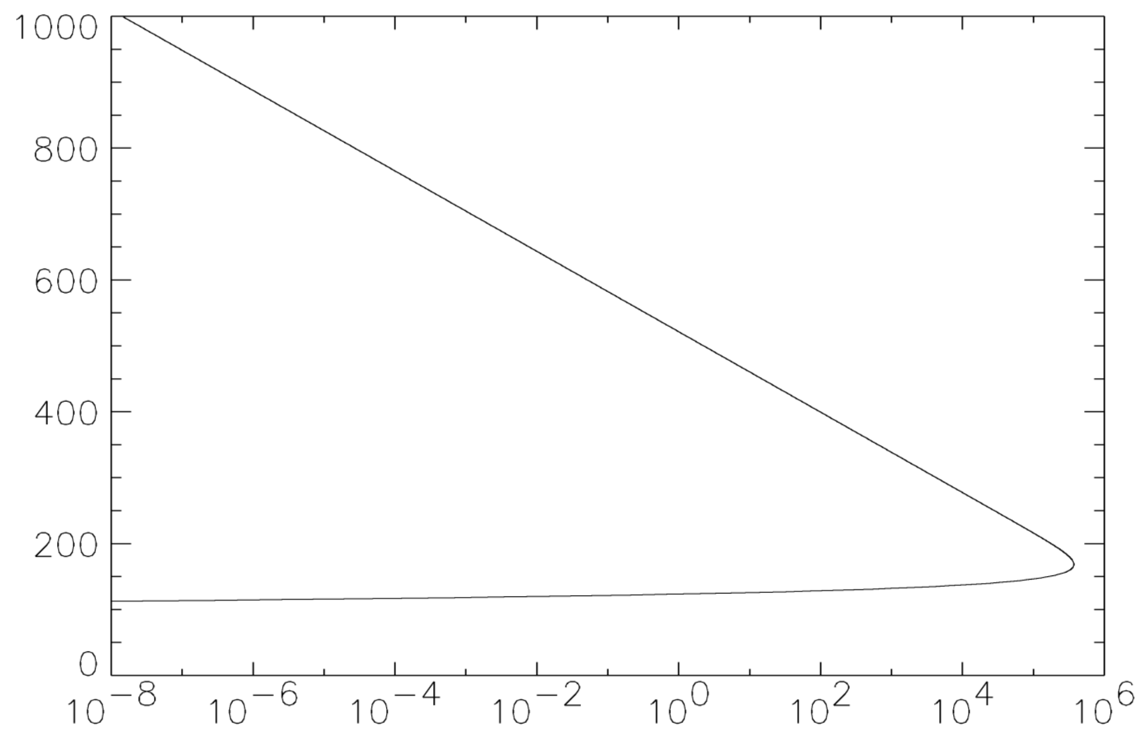
$a_a = 9.3 \times 10^{-23}$
$a_j = 9.3 \times 10^{-23}$
$a_r = 3.0 \times 10^{-14}$
$T = 270$
$m = 16*2*amu$
$n_0 = 2.7 \times 10^{25}\ m^{-3}$
$I_0 = 3.6 \times 10^{13}$ photons/m^2/s



”F1-region” - simple model calculation

O_2 dominating species, 30 nm UV radiation

$a_a = 9.3\times10^{-23}$
$a_j = 9.3\times10^{-23}$
$a_r = 3.0\times10^{-14}$
$T = 500$
$m = 16*2*amu$
$n_0 = 2.7\times10^{25}\ m^{-3}$
$I_0 = 1.5\times10^{14}$ photons/m²/s



”F2-region” - simple model calculation

O dominating species, 30 nm UV radiation

$a_a = 9.3\times10^{-23}$

$a_j = 9.3\times10^{-23}$

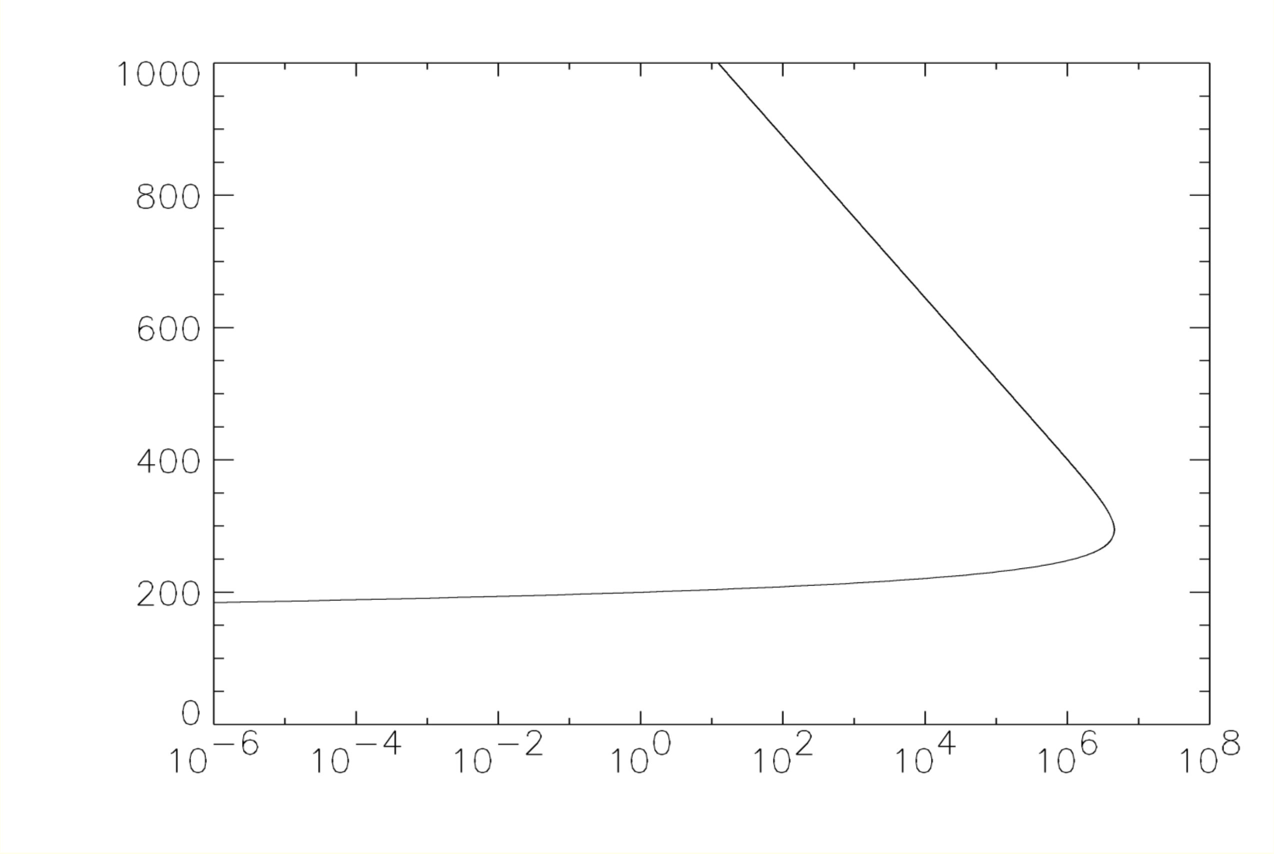
$a_r = 1.0\times10^{-16}$

T=500

m = 16*amu

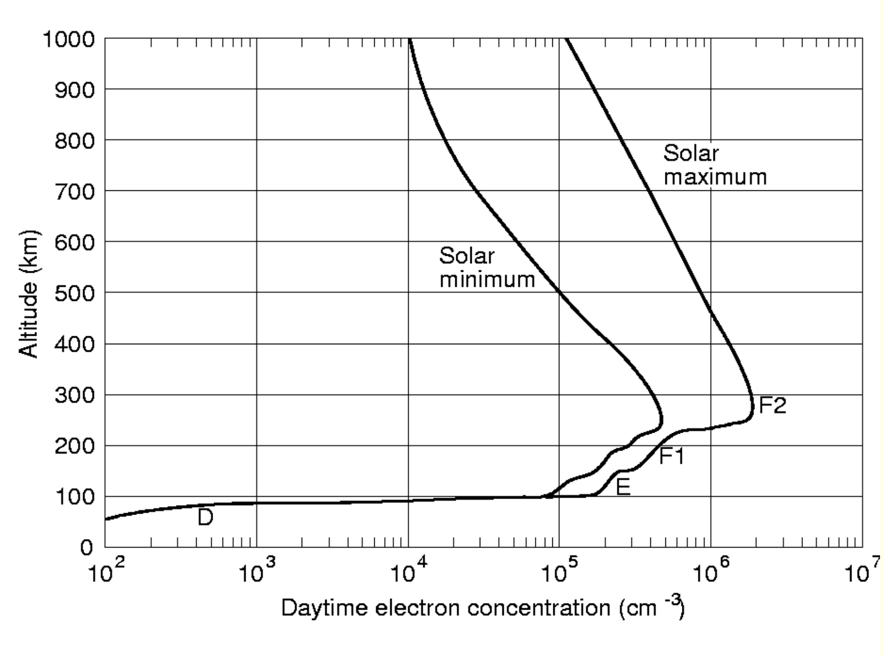
$n_0 = 2.7\times10^{25}$ m^{-3}

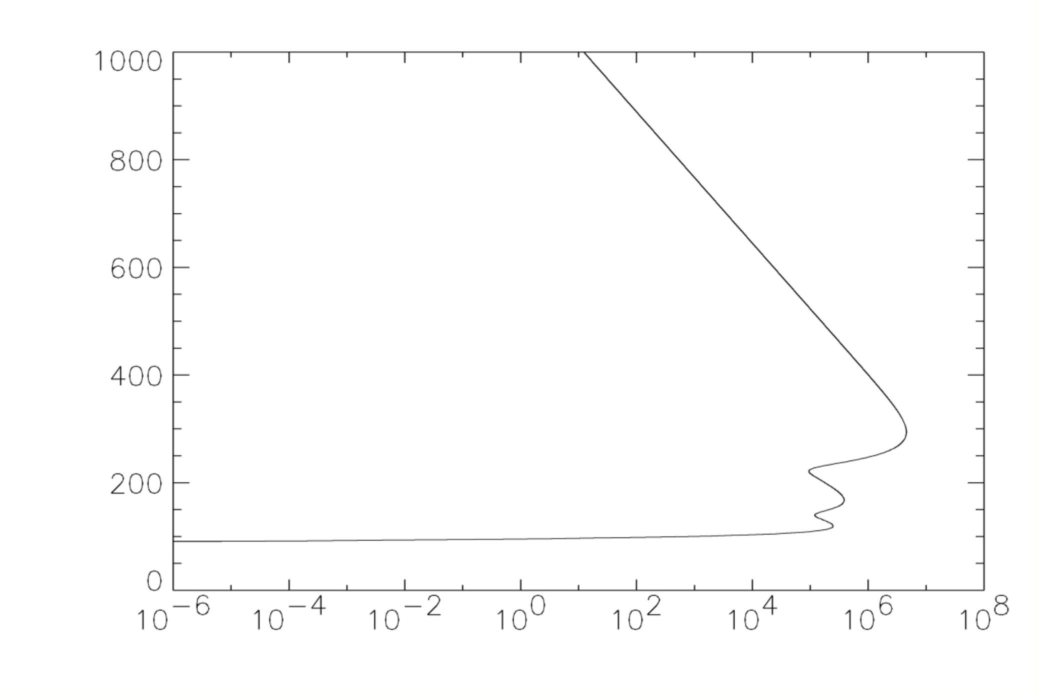
$I_0 = 1.5\times10^{14}$ photons/m²/s



Measurements

”E” + ”F1” + ”F2”





Ionospheric layers

Layer	D	E	F_1	F_2
Altitude (km)	60-85	85-140	140-200	200 - ca 1500
Nighttime electron density (cm^{-3})	$<10^2$	$2 \cdot 10^3$	—	$2 - 5 \cdot 10^5$
Daytime electron density (cm^{-3})	10^3	$1 - 2 \cdot 10^5$	$2 - 5 \cdot 10^5$	$0.5 - 2 \cdot 10^6$
Ion species	NO^+ O_2^+	NO^+ O_2^+	NO^+ O_2^+ O^+	O^+ He^+ H^+
Cause of ionization	Lyman α (1215 Å) + cosmic radiation	Lyman β (1025 Å) X-rays	UV	UV

Propation of radio waves in the ionosphere

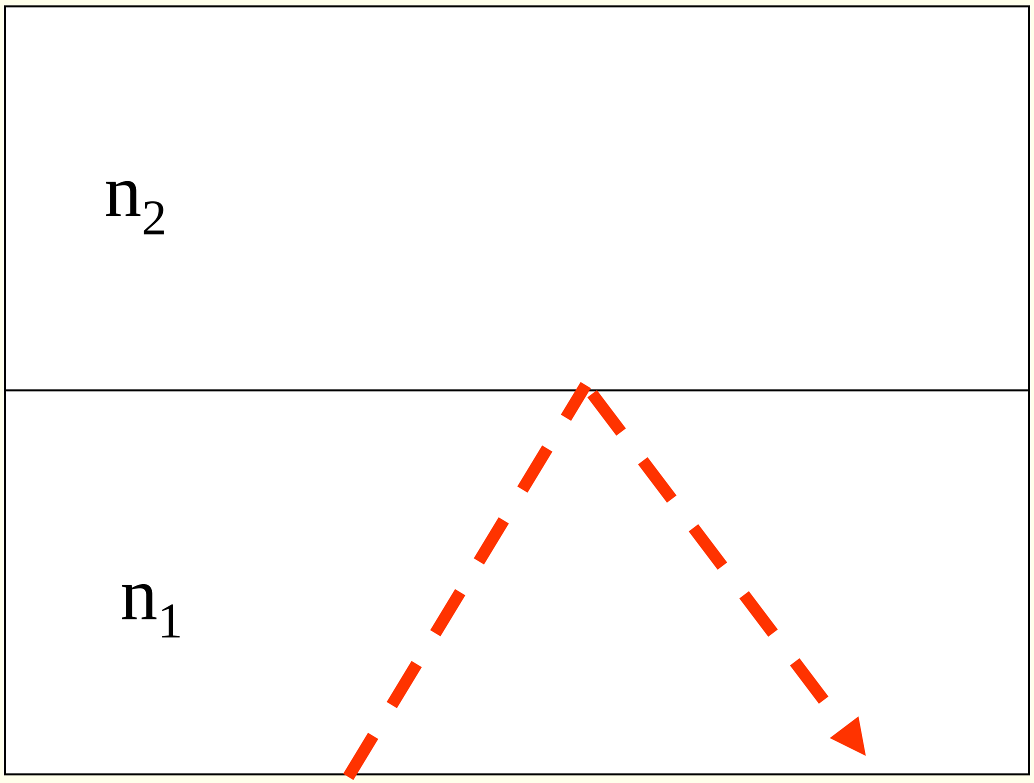
1. Absorption/damping

Takes place in the D-region due to high collision frequency. (Collisions with neutral atoms.)

2. Reflection

takes place in the F-region due to large gradients in the refraction index.

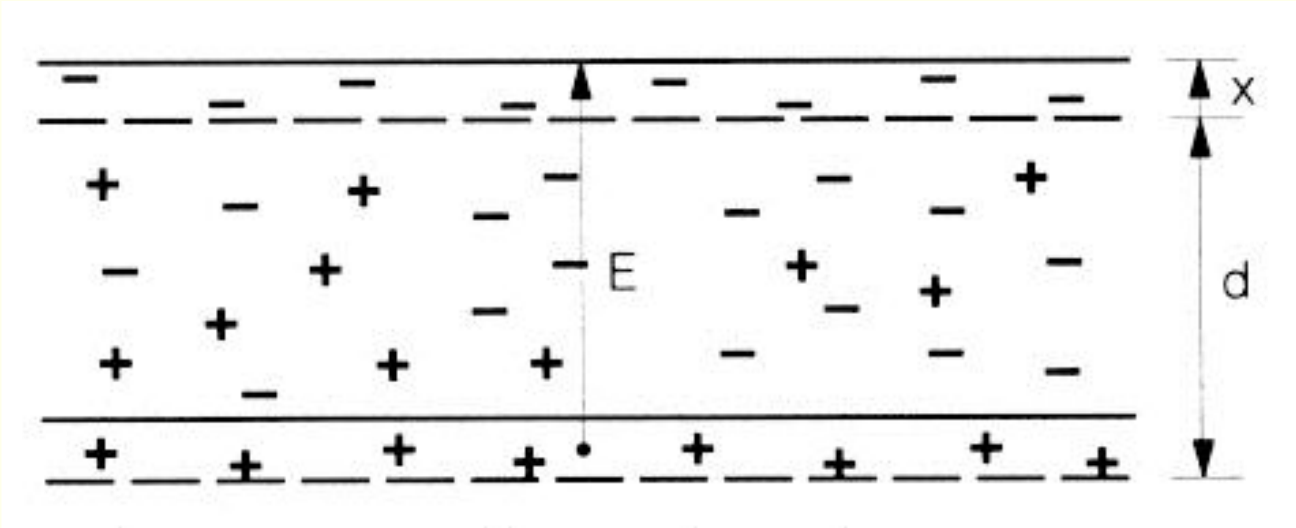
Reflection of radio waves



Total reflection at a sharp boundary (or large gradient) if

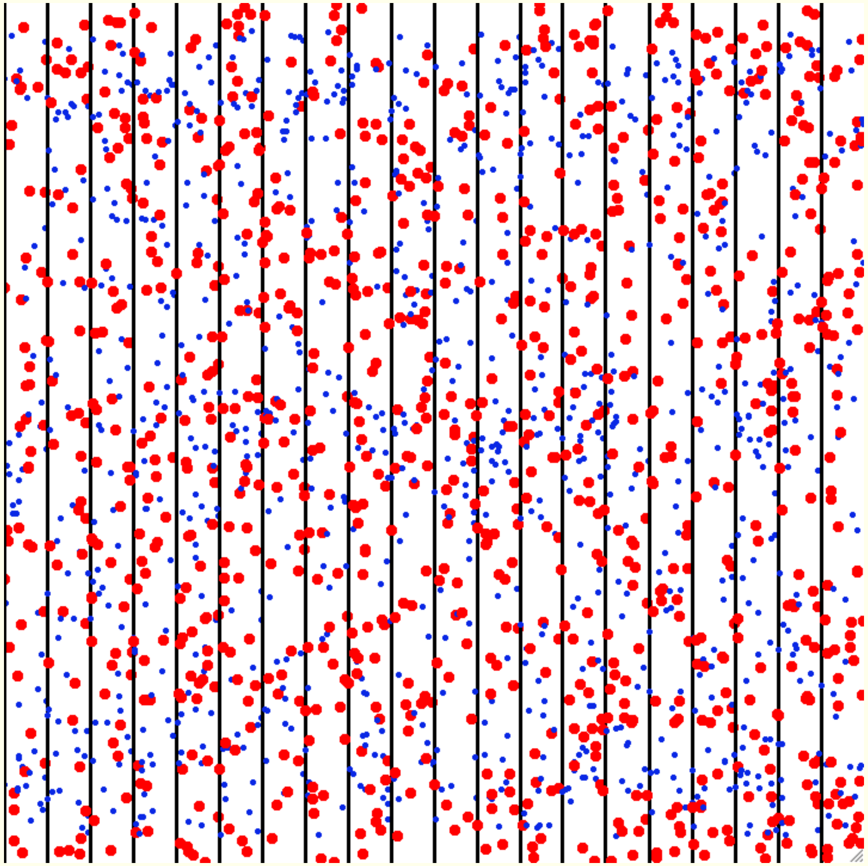
$$n_2 < n_1$$

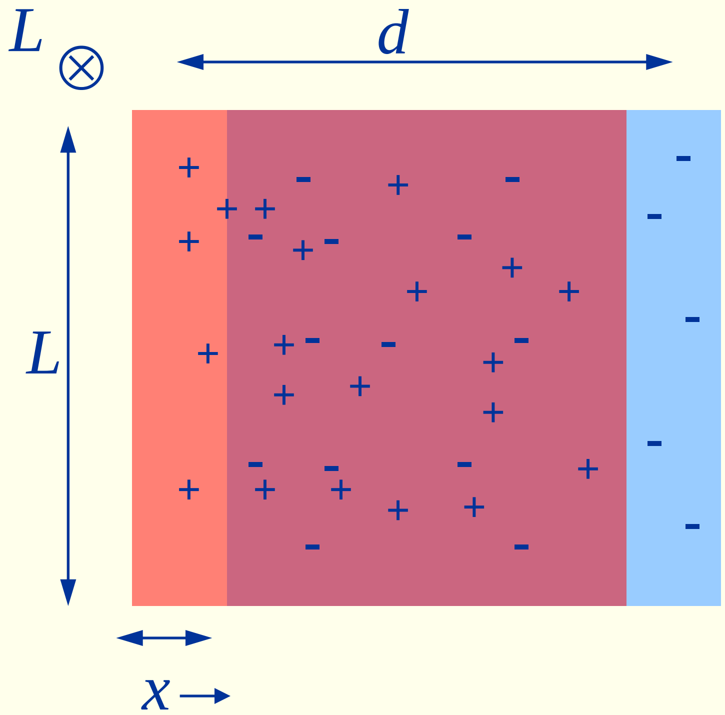
Plasma frequency



Charge imbalance creates an electric field which tends to even out the imbalance.

Plasma oscillations parallel to B





Newtons law on an individual electron inside the slab:

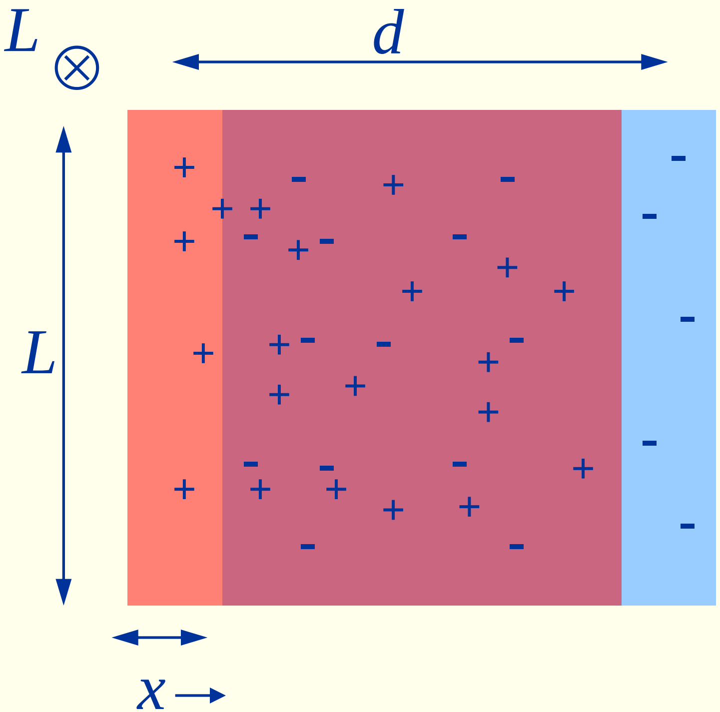
$$F = m_e a$$

$$F = -eE$$

Surface charge density

$$E = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = -e n_e x$$



$$-\frac{n_e e^2 x}{\varepsilon_0 m_e} = \frac{d^2 x}{dt^2}$$

$$x = \sin(\omega_{pe} t)$$

$$F = m_e a \qquad E = \frac{\sigma}{\varepsilon_0}$$

$$F = -eE \qquad \sigma = e n_e x$$

$$\omega_{pe} \equiv \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

What is the plasma frequency f_{pe} at the daytime E-region, close to solar minimum? (see Fälthammar p 28)

$$f_{pe} = \frac{\omega_{pe}}{2\pi} \equiv \frac{1}{2\pi}\sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

Blue	7 kHz	Yellow	400 MHz
Green	3 MHz	Red	2 GHz

$$f = \frac{\omega_{pe}}{2\pi} \equiv \frac{1}{2\pi}\sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}} = \frac{1}{2\pi}\sqrt{\frac{(1.6\cdot 10^{-19})^2}{8.854\cdot 10^{-12} 0.91\cdot 10^{-30}}}\sqrt{n_e} =$$

$$8.97\sqrt{n_e} = 8.97\sqrt{10^5\cdot 10^6} = 2.8\cdot 10^6\, Hz = 2.8\, MHz$$

Green

Index of refraction for electromagnetic waves in a plasma

(1) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

(2) $$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

(3) $$\mathbf{j} = -en_e \mathbf{v}_e$$

(4) $$m_e \frac{\partial \mathbf{v}_e}{\partial t} = -e\mathbf{E}$$

Assume all quantities vary sinusoidally, with frequency ω, *e.g.*:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

(1) $\Rightarrow$ $$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t}$$

(2) $\Rightarrow$ $$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\therefore \nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial}{\partial t}(en_e \mathbf{v}_e) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$\Rightarrow$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 e n_e \frac{\partial \mathbf{v}_e}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$\Rightarrow$

Index of refraction for electromagnetic waves in a plasma

$$i\mathbf{k}\left(i\mathbf{k}\cdot\mathbf{E}\right)-k^2\mathbf{E}=\mu_0\left(-i\omega\right)en_e\mathbf{v}_e+\frac{1}{c^2}\left(-i\omega\right)^2\mathbf{E}$$

Does not represent E.M. wave

(4) ⇨

$$-k^2\mathbf{E}=\mu_0\left(-i\omega\right)en_e\frac{ie\mathbf{E}}{\omega m_e}+\frac{1}{c^2}\left(-i\omega\right)^2\mathbf{E}$$

⇨

$$c^2k^2=-c^2\frac{\mu_0 n_e e^2}{m_e}+\omega^2=\frac{-1}{\mu_0\varepsilon_0}\frac{\mu_0 n_e e^2}{m_e}+\omega^2$$

$$\therefore\quad \omega^2=c^2k^2+\omega_p^2$$

$$n^2=\frac{c^2}{v_{ph}^2}=\frac{c^2k^2}{\omega^2}=\frac{\omega^2-\omega_p^2}{\omega^2}=1-\frac{\omega_p^2}{\omega^2}$$

$\therefore$

$$n=\sqrt{1-\frac{\omega_p^2}{\omega^2}}=\sqrt{1-\frac{f_p^2}{f^2}}$$