OPTIMAL FILTERING

LECTURE 3

1. Wiener filtering, causal and discrete time

2. Kalman filter, discrete time and state space model

Reading instructions: Kailath, Sect. 7.3-7.8, 1.1-1.2

WIENER FILTERING, DISCRETE TIME

Given observations $\{y_i, \quad -\infty < i \leq k\}$, find the linear least squares estimate (l.l.s.e.) of $x_{k+\lambda}$.

The processes $\{x_k\}$ and $\{y_k\}$ are assumed jointly stationary with exponentially bounded covariance (and cross covariance) sequences

$$|r(k)| < K\alpha^{|k|} \quad \text{for some} \quad K > 0 \quad \text{and} \quad 0 < \alpha < 1.$$

Thus find

$$\hat{x}_{k+\lambda} = \sum_{i=0}^{\infty} h_{k,i} y_{k-i}$$

such that $\mathrm{E}\{|x_{k+\lambda} - \hat{x}_{k+\lambda}|^2\}, \quad -\infty < k < \infty$ is minimized.

Use the orthogonality property.

$$\begin{aligned} & x_{k+\lambda} - \hat{x}_{k+\lambda} \;\perp\; y_j \quad j \le k \\ \Longrightarrow \quad & \mathrm{E}\{(x_{k+\lambda} - \hat{x}_{k+\lambda}) y_j^*\} = 0 \quad j \le k \\ \Longrightarrow \quad & r_{xy}(k+\lambda-j) = \sum_{i=0}^{\infty} h_{k,i} r_y(k-i-j) \quad j \le k \end{aligned}$$

Change of variables $k - j \to k$

$$r_{xy}(k+\lambda) = \sum_{i=0}^{\infty} h_{k+j,i} r_y(k-i) \quad k \ge 0$$

We see that $h_{k+j,i}$ does not depend on j (since $r_{xy}(k+\lambda)$ and $r_y(k-i)$ do not depend on j) and thus the resulting filter will be time invariant.

Now let

$$g_k = r_{xy}(k+\lambda) - \sum_{i=0}^{\infty} h_i r_y(k-i) \quad \text{for all} \quad k$$

We know that

$$g_k = 0, \quad k \ge 0$$

Z-transform

$$\Longrightarrow \quad G(z) = S_{xy}(z) z^{\lambda} - H(z) S_y(z)$$

where $S_y(z) = S_y^*(z^{-*})$ is the *Z-spectrum* of $\{y_i\}$ with ROC $\alpha < |z| < \alpha^{-1}$ (rational polynomial in z). $S_y(e^{i\omega})$ is real and nonnegative.

$$G(z) = \sum_{-\infty}^{\infty} g_k z^{-k}$$

SPECTRAL FACTORIZATION

Spectral Factorization of $S_y(z)$:

Rational spectra in z

$$S_y(z) = r_e \frac{\Pi_{i=1}^m (z-\alpha_i)(z^{-1}-\alpha_i^*)}{\Pi_{i=1}^n (z-\beta_i)(z^{-1}-\beta_i^*)} \quad \text{with} \quad |\alpha_i| < 1, \quad |\beta_i| < 1, \quad r_e > 0$$

with no zeros on the unit circle.

$$S_y(z) = \underbrace{\sqrt{r_e} L(z)}_{S_y^+(z)} \underbrace{\sqrt{r_e} L^*(z^{-*})}_{S_y^-(z)}$$

where the spectral factor is

$$S_y^+(z) = \sqrt{r_e} L(z) = \sqrt{r_e} z^{n-m} \frac{\Pi_{i=1}^m (z-\alpha_i)}{\Pi_{i=1}^n (z-\beta_i)} \quad \text{with} \quad |\alpha_i| < 1, \quad |\beta_i| < 1$$

The factor z^{n-m} ensures a canonical factorization with $L(\infty) = 1$.

SPECTRAL FACTORIZATION

$$S_y^+(z) = \sqrt{r_e} L(z) = \sqrt{r_e} z^{n-m} \frac{\Pi_{i=1}^m (z-\alpha_i)}{\Pi_{i=1}^n (z-\beta_i)} \quad \text{with} \quad |\alpha_i| < 1, \quad |\beta_i| < 1$$

$S_y^+(z)$ and $L(z)$ has the stable poles and zeros (inside the unit circle) of $S_y(z)$. $S_y^+(z)$ and $L(z)$ are causal and causally invertible.

For real processes $L^*(z^{-*}) = L(z^{-1})$.

ADDITIVE DECOMPOSITION

Let $\{f_k\}$ have a Z-transform that exists in an annulus containing the unit circle (exponentially bounded which gives a rational Z-transform).

Define

$$\{F(z)\}_+ = \sum_{k=0}^{\infty} f_k z^{-k}$$

$$\{F(z)\}_- = \sum_{k=-\infty}^{-1} f_k z^{-k}$$

and of course

$$F(z) = \underbrace{\{F(z)\}_+}_{\text{causal}} + \underbrace{\{F(z)\}_-}_{\text{strictly anticausal}}$$

DISCRETE TIME WIENER–HOPF

Back to

$$G(z) = S_{xy}(z)z^{\lambda} - H(z)S_y(z)$$

$$\Longrightarrow \quad \frac{G(z)}{S_y^-(z)} = \frac{S_{xy}(z)z^{\lambda}}{S_y^-(z)} - H(z)S_y^+(z)$$

Note that $G(z)$ is strictly anticausal since

$$g_k = 0 \quad k \geq 0$$

and $S_y^-(z)$ is anticausal and anticausally invertible. Hence,

$$\left\{G(z)\frac{1}{S_y^-(z)}\right\}_+ = \left\{\left(\sum_{k=-\infty}^{-1} g_k z^{-k}\right)\left(\sum_{k=-\infty}^{0} f_k z^{-k}\right)\right\}_+$$

$$= \left\{\sum_{k=-\infty}^{-1} c_k z^{-k}\right\}_+ = 0$$

Thus

$$\frac{G(z)}{S_y^-(z)}$$

is strictly anticausal.

We also have

$$\{H(z)S_y^+(z)\}_+ = H(z)S_y^+(z)$$

and thus

$$\boxed{H(z) = \frac{1}{S_y^+(z)} \left\{ \frac{S_{xy}(z)z^{\lambda}}{S_y^-(z)} \right\}_+}$$

THE KALMAN FILTER

We shall introduce a finite dimensional state space model for a process. This allows recursive and efficient computation of the linear least squares estimate.

Discrete Time State Space Model

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k w_k \qquad k \geq 0 \\ y_k &= H_k x_k + v_k \end{aligned}$$

x_k – $(n \times 1)$ state vector	x_0 – initial state vector
w_k – $(m \times 1)$ process noise	F_k – $(n \times n)$ system matrix
v_k – $(p \times 1)$ measurement noise	G_k – $(n \times m)$
y_k – $(p \times 1)$ observation vector	H_k – $(p \times n)$

THE KALMAN FILTER

x_k, w_k, and v_k are stochastic quantities which we will assume are Gaussian processes with

$$E\{x_0\} = E\{w_k\} = E\{v_k\} = 0$$

$$E\{x_0 x_0^*\} = P_0 \quad E\{x_0 v_k^*\} = 0 \quad E\{x_0 w_k^*\} = 0$$

$$E \begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_l^* & v_l^* \end{bmatrix} = \begin{bmatrix} Q_k & S_k \\ S_k^* & R_k \end{bmatrix} \delta_{kl}$$

where

$$\delta_{kl} = \begin{cases} 1 & k = l \\ 0 & \text{otherwise} \end{cases}$$

THE KALMAN FILTER

Notation:

$\hat{x}_{k|m} =$ l.l.s.e. estimate of x_k given the observations $\{y_0, y_1, \ldots, y_m\}$.

$P_{k|m} =$ Covariance of $\hat{x}_{k|m}$.

The basic Kalman filtering problem:

Determine the estimate

$$\hat{x}_{k|k-1} = E\{x_k | y_0, \ldots, y_{k-1}\}$$

based on the observations y_l, $0 \le l \le k-1$ and knowledge of the model $\{F_k, G_k, H_k, Q_k, S_k, R_k, P_0\}$.

THE KALMAN FILTER

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After tedious derivations (assuming $S_k = 0$)

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1} (y_k - H_k \hat{x}_{k|k-1}) \\
\hat{x}_{k+1|k} &= F_k \hat{x}_{k|k} \\
P_{k|k} &= P_{k|k-1} - P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1} H_k P_{k|k-1} \\
P_{k+1|k} &= F_k P_{k|k} F_k^* + G_k Q_k G_k^*
\end{aligned}
$$