Optimal Filtering, 2012–09–10
Due, 2012–09–17
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Homework # 3

1. Assume the continuous time signals $x(t)$ and $y(t)$ are jointly wss.

   a) If $y(t)$ passes through a filter with transfer function $1/S_y^+(s)$, determine the spectrum of the output signal $e(t)$.

   b) Show that the non-causal Wiener filter for estimating $x(t+\lambda)$ from $\{e(\tau),\ -\infty < \tau < \infty\}$, has the transfer function

   $$H_{nc}(s) = \frac{S_{xy}(s)e^{s\lambda}}{S_y^-(s)}$$

   c) As the name suggests, $H_{nc}(s)$ will in general be non-causal. A seemingly ad-hoc approach to turn this into a causal filter is to truncate it and only keep the causal part $\{H_{nc}(s)\}_+$. Show that the combination of the two filters $1/S_y^+(s)$ and $\{H_{nc}(s)\}_+$ is equivalent to the causal Wiener filter for estimating $x(t+\lambda)$ directly from $\{y(\tau),\ -\infty < \tau \le t\}$.

   d) Try to give an intuitive explanation why the above procedure gives the causal Wiener filter and why it is worse to take the non-causal Wiener filter for estimating $x(t+\lambda)$ from $y(t)$ and truncate that, i.e. to use the filter $\{S_{xy}(s)e^{s\lambda}/S_y(s)\}_+$.

2. Consider the discrete time Wiener problem of estimating $x_{k+\lambda}$ from $\{y_i,\ -\infty < i \le k\}$.

   a) Show that

   $$\begin{aligned}
   \min \text{mse} &= \mathrm{E}\{x_{k+\lambda}^2\} - \mathrm{E}\{x_{k+\lambda}\hat{x}_{k+\lambda}\} \\
   &= r_x(0) - \mathrm{E}\{\hat{x}_{k+\lambda}^2\} \\
   &= r_x(0) - \int_{-\pi}^{\pi} \left| \left\{ \frac{S_{xy}(e^{j\omega})e^{j\omega\lambda}}{S_y^-(e^{j\omega})} \right\}_+ \right|^2 \frac{d\omega}{2\pi} \\
   &= r_x(0) - \int_{-\pi}^{\pi} \frac{|S_{xy}(j\omega)|^2}{S_y(e^{j\omega})} \frac{d\omega}{2\pi} + \int_{-\pi}^{\pi} \left| \left\{ \frac{S_{xy}(j\omega)e^{j\omega\lambda}}{S_y^-(e^{j\omega})} \right\}_- \right|^2 \frac{d\omega}{2\pi}.
   \end{aligned}$$

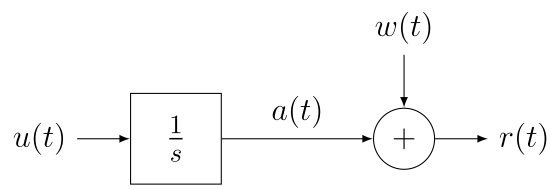
   b) Interpret the results in a).

   Note: If you are browsing for hints in the book, there is a typo on page 257, where the hint in 7.19 (f) should say

   $$\int_{-\pi}^{\pi} \left|a(e^{j\omega})\right|^2 d\omega = \int_{-\pi}^{\pi} \left|\{a(e^{j\omega})\}_+\right|^2 d\omega + \int_{-\pi}^{\pi} \left|\{a(e^{j\omega})\}_-\right|^2 d\omega$$

   If you wish to use this result, first prove that it's true.

3. Consider the following system where



$$
\begin{aligned}
\mathrm{E}\{u(t)\} = \mathrm{E}\{w(t)\} &= \mathrm{E}\{u(t)w(\tau)\} = 0 \\
\mathrm{E}\{u(t)u(\tau)\} &= \sigma^2\delta(t-\tau) \\
\mathrm{E}\{w(t)w(\tau)\} &= \frac{N_0}{2}\delta(t-\tau)
\end{aligned}
$$

a) Use the Wiener solution to find the transfer function of the optimal linear filter for the estimation of $a(t)$ from $\{r(\tau),\ -\infty < \tau < t\}$.

   Show that mse $= \sqrt{\frac{N_0\sigma^2}{2}}$.

b) Same as a) but estimate $\dot{a}(t) = \frac{da(t)}{dt}$.

Hint: Start by considering the case without a pure integration, i.e., a pole off the imaginary axis. Then take the limit as the pole approaches the imaginary axis.

4. If

$$S_y(z) = \frac{-\beta z^{-1} + 1 + |\beta|^2 - \beta^* z}{-\alpha z^{-1} + 1 + |\alpha|^2 - \alpha^* z}, \qquad |\alpha| < 1, \quad |\beta| > 1$$

show that

$$\hat{y}(k+\lambda) = (\alpha - 1/\beta^*)\alpha^{\lambda-1}\sum_{i=0}^{\infty}(1/\beta^*)^i y(k-i)$$

with MSE

$$|\beta|^2 + |\alpha\beta - 1|^2 \frac{1-|\alpha|^{2\lambda-2}}{1-|\alpha|^2}$$