Optimal Filtering, 2012–09–17
Due, 2012–09–24
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Homework # 4

1. We have the following *measurement equation* $y_k = H_k x_k + v_k$, where v_k is zero-mean Gaussian with covariance R_k. Show that if R_k is singular, some linear functional of x_k is determined by y_k with zero error, i.e., there exists a vector a_k such that y_k determines $a_k^T x_k$.

2. Consider the standard state-space model with $S_k = 0$ and assume $\Pi_0 > 0$.

   a) Show that the covariance of a two-step predictor is given by

   $$P_{k+1|k-1} = F_k P_k F_k^* + G_k Q_k G_k^*$$

   b) Find a relationship between $\tilde{x}_{k+1|k-1}$ and $\tilde{x}_{k+1}$ and use it to show that

   $$P_{k+1|k-1} = P_{k+1} + F_k P_k H_k^* (R_k + H_k P_k H_k^*)^{-1} H_k P_k F_k^*$$

3. Your task is to derive a Kalman based solution to estimate the initial state value x_0 given measurements $\{y_0, y_1, \ldots, y_k\}$. Assume the standard state-space model with $S_k = 0$.

   Hint: Introduce an extended state vector $\begin{bmatrix} x_k \\ x_k^0 \end{bmatrix}$, where $x_{k+1}^0 = x_k^0$ for all $k \geq 0$, in addition to the standard state-space model. Show that the resulting estimate is given by the standard Kalman filter for x_k, extended with the following equations,

   $$\begin{aligned} K_k^0 &= P_k^0 H_k^* (H_k P_{k|k-1} H_k^* + R_k)^{-1} \\ \hat{x}_{0|k} &= \hat{x}_k^0 = \hat{x}_{k-1}^0 + K_k^0 (y_k - H_k \hat{x}_{k|k-1}) \\ P_{k+1}^0 &= P_k^0 F_k^* - K_k^0 H_k P_{k|k-1} F_k^* \\ P_{k+1}^{00} &= P_k^{00} - K_k^0 H_k P_k^0 \end{aligned}$$

   which are initialized by

   $$\begin{aligned} \hat{x}_{-1}^0 &= \hat{x}_{0|-1} \\ P_0^{00} &= P_0^0 = P_{0|-1} \end{aligned}$$

   Explain also what P_k^0 and P_k^{00} denote.

4. Filtered residuals μ_k are defined as

$$y_k = H_k x_k + v_k$$
$$\mu_k = y_k - H_k \hat{x}_{k|k}$$

Show that

a) $\mu_k = R_k R_{e_k}^{-1} e_k$

b) $\{\mu_k\}$ is a white sequence with covariance matrix

$$R_k^\mu = R_k - H_k P_{k|k} H_k^*$$