This text can be used to replace Section 4.3 i Bretschers book.

1. Representing linear maps with matrices

1.1. **Linear maps.** Throughout the text we let $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^n$ be two subvector spaces of Euclidan spaces. Recall that a map $T\colon V \longrightarrow W$ is *linear* if

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

for all vectors $\vec{x}$ and $\vec{y}$ in V, and all scalars a and b. What we want explain is how a linear map $T\colon V \longrightarrow W$ between vector spaces is to be represented by matrices. In the course we have earlier discussed the situations with $V = W = \mathbb{R}^n$.

1.2. **Bases.** In order to represent the linear map $T\colon V \longrightarrow W$ with a matrix, we need to fix bases for the vector spaces V and W. We know that bases exists, and therefore we imagine that we have fixed bases. Let $\mathfrak{B}$ be a basis of V, and let $\mathfrak{C}$ be a basis for W. Thus, if V is of dimension p, we have that $\mathfrak{B}$ consists of p vectors $(\vec{e}_1, \dots \vec{e}_p)$ that are linearly independent and that span V. Similiarly if W is of dimension q, then $\mathfrak{C} = (\vec{f}_1, \dots, \vec{f}_q)$.

1.3. **Coordinate matrix.** Recall that for any vector $\vec{w}$ in W we will with $\big[w\big]_{\mathfrak{C}}$ mean the coordinate matrix of the vector $\vec{w}$ with respect to the basis $\mathfrak{C}$. Thus, since $\mathfrak{C} = (\vec{f}_1, \dots, \vec{f}_q)$ is a basis of W we know that there are unique scalars $a_1, \dots, a_q$ such that

$$\vec{w} = a_1\vec{f}_1 + a_2\vec{f}_2 + \cdots + a_q\vec{f}_q.$$

The coordinate matrix of $\vec{w}$ with respect to the basis $\mathfrak{C}$ is simply

$$\big[\vec{w}\big]_{\mathfrak{C}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \end{bmatrix}.$$

1.4. **Matrix representing the map.** Returning to our linear map $T\colon V \longrightarrow W$, we construct the matrix

$$A_T = \Big[\big[T(\vec{e}_1)\big]_{\mathfrak{C}}\,\big[T(\vec{e}_2)\big]_{\mathfrak{C}} \cdots \big[T(\vec{e}_p)\big]_{\mathfrak{C}}\Big] \tag{1.4.1}$$

Note that the matrix A_T has p columns, where p is the dimension of V. The first coloumn in the matrix is the coordinate matrix of the vector $T(\vec{e}_1)$, in W, with respect to the basis $\mathfrak{C}$. The second coloumn is the coordinate matrix of $T(\vec{e}_2)$, and so on. Note also that matrix is of size $(q \times p)$, where $q = \dim(W)$ and $p = \dim(V)$.

Proposition 1.5. *Let $T\colon V \longrightarrow W$ be a linear map between vector spaces. Let $\mathfrak{B} = (\vec{e}_1, \dots, \vec{e}_p)$ be a basis of V, and let $\mathfrak{C}$ be a basis of W, and let A_T be the matrix 1.4.1. Then we have, for any vector $\vec{x} \in V$ that*

$$\big[T(x)\big]_{\mathfrak{C}} = A_T \left[\vec{x}\right]_{\mathfrak{B}}.$$

Proof. The proof is just a matter of writing down what the formula in the Proposition states, and we leave that for the reader. □

Example 1.6. Let $V = \mathbb{R}^m$, and $W = \mathbb{R}^n$, and that the bases we have fixed for V and W are both the standard bases. Then, for any linear map $T\colon V \longrightarrow W$ the matrix representation given in the Proposition, is the standard matrix discussed in the very beginning of the course.

Example 1.7. Let $V = \mathbb{R}^m$, and $W = \mathbb{R}^n$, and let $\mathfrak{B}$ be a basis for V and $\mathfrak{C}$ be a basis for W. For any linear map $T\colon V \longrightarrow W$ the matrix in the proposition is the matrix of a linear transformation discussed in Section 3.4 in the book.

Example 1.8. Let $V = \mathbb{R}^m = W$, and let $T\colon V \longrightarrow V$ be the identity map. Then, by fixing one basis $\mathfrak{B}$ for the domain V, and another basis $\mathfrak{C}$ for the target V, the proposition gives a matrix A_T that is called the change of basis matrix.

Example 1.9. For further examples, see Exercises 60, 61, 62, and 63 in Bretscher. See also Uppgift 7, Tentamen 2010 10 22, and Uppgift 10, Tentamen 2010 06 05.

1.10. **Further ahead.** In the end of the course, we will be looking at the following situation. We have a linear map $T\colon V \longrightarrow V$, and in fact we will have $V = \mathbb{R}^n$. Then we choose one basis $\mathfrak{B}$ both for the target and the domain. And by the proposition we get a matrix A representing the linear map $T\colon V \longrightarrow V$ with respect to the fixed basis $\mathfrak{B}$, for target and domain. Then we fix another basis $\mathfrak{C}$ for target and domain, and we get a matrix D representing the same linear map $T\colon V \longrightarrow V$, but with respect to the basis $\mathfrak{C}$ in target and domain. We are interested in the relation between these two matrices A and D.

1.11. **A commutative diagram.** Before, giving that relation, we should agree on that the following diagram obviously commute

(1.11.1)
$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ V & \xrightarrow{T} & V \end{array}$$

where $\mathrm{id}\colon V \longrightarrow V$ is the identity map. And commute means simply that if start in the left upper corner V, and the following the arrows down to the lower right corner, it does not matter which arrows we follow. Indeed, let $\vec{x}$ be a vector, then the commutativity of the diagram reads

$$\mathrm{id}T(\vec{x}) = T\mathrm{id}(\vec{x}).$$

1.12. **Commutative diagram and matrices.** However, in the commutative diagram we have that the four arrows represent a linear map. When we fix a basis for the target and domain, the linear map is represented by a matrix, the proposition above tells us. In the upper horizontal line we fix the basis $\mathfrak{C}$ for both target and domain, and then the linear map $T\colon V \longrightarrow V$ is represented by the matrix D. In the lower horizontal line we fix the basis $\mathfrak{B}$, and then the linear map $T\colon V \longrightarrow V$ is represented by the matrix A. The vertical arrow $\mathrm{id}\colon V \longrightarrow V$ is represented by the matrix P, which is the change of basis matrix from the basis $\mathfrak{C}$ to the basis $\mathfrak{B}$. Now, the commutativity above is rephrased as the identity of matrices

$$PD = AP.$$

Remember that composition of functions is done from the right. So, PD means that first apply the function representing D, and thereafter P. Thus PD corresponds to the upper right arrow composed with the rightmost vertical arrow. And AP represents the left most vertical arrow, composed with the lower horizontal arrow.

The change of basis matrix P is invertible, and the invers is simply the change of basis from the basis $\mathfrak{B}$ to $\mathfrak{C}$. Thus, the identity $PD = AP$ we can also write as

$$D = P^{-1}AP.$$

1.13. **Here is a claim.** Remembering the identity $D = P^{-1}AP$ will not be of much help, however remembering the trivial fact that the diagram 1.11.1 commutes will be very useful when dealing with eigenvectors and the diagonalization problem.