Optimal Filtering, 2012–09–24
Due, 2012–10–01
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Homework # 5

1. Consider the following signal model with a direct feedthrough

$$\begin{aligned} x_{k+1} &= F_k x_k + G_k w_k \\ y_k &= H_k x_k + v_k + J_k w_k \end{aligned}$$

where v_k and w_k are independent, zero-mean, white Gaussian processes with covariances $R_k \delta_{kl}$ and $Q_k \delta_{kl}$, respectively. Show that this arrangement is equivalent to one in which $y_k = H_k x_k + \bar{v}_k$. Characterize $\bar{v}_k$.

2. **a)** Consider the signal model in the problem above. Let $R_k = 0$ for all k, and assume that x_0 is known to be zero. Show that if J_k is nonsingular, $P_{k|k-1} = 0$ for all k.

   b) Explain why this should be so.

3. We wish to estimate the position in a plane of an aircraft from noisy observations from a radar. This is often referred to as a tracking problem. To start with we will construct a process and measurement model. The aircraft moves

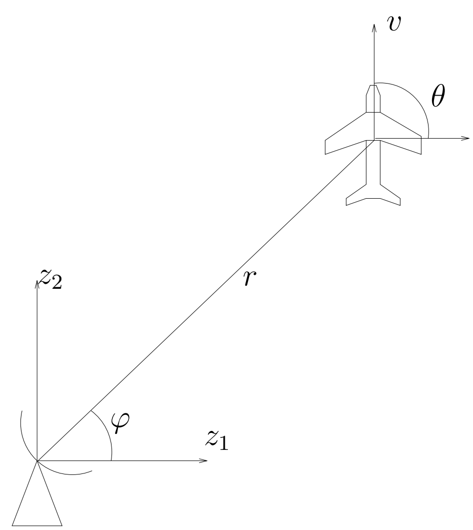

Figure 1: Geometry of the tracking problem.

with velocity v at an angle θ. We measure r and φ.

Let

$$x(n) = \begin{pmatrix} z_1(n) \\ \dot{z}_1(n) \\ z_2(n) \\ \dot{z}_2(n) \end{pmatrix}$$

be the state vector. Assume a constant velocity $\dot{z}_i(n+1) = \dot{z}_i(n)$ and assume that the time between samples is T.

a) Derive the state space model of the scenario in Figure 1, *i.e.*, find the matrices F and H. In the measurement equation, do a variable transformation so that the measurements are z_1 and z_2. The process and measurement noise are assumed independent and white with constant covariance matrices R_1 and R_2.

b) We can improve the model by using time-variable covariance matrices. Assume that the disturbances of v and θ are modeled by

$$v(n+1) = v(n) + e_v(n)$$
$$\theta(n+1) = \theta(n) + e_\theta(n)$$

where the noises $e_v(n)$ and $e_\theta(n)$ are independent with variances σ_v^2 and σ_θ^2 respectively.

The measurements are given by

$$\hat{r}(n) = r(n) + e_r(n)$$
$$\hat{\varphi}(n) = \varphi(n) + e_\varphi(n)$$

where $r(n)$ and $\varphi(n)$ are assumed independent, $e_r(n)$ and $e_\varphi(n)$ have variances σ_r^2 and σ_φ^2.

Show that the process noise covariance, R_1, can be modeled by:

$$R_1(2,2) = \mathrm{E}\{w_2^2(n)\} = \sigma_v^2 \cos^2\theta(n) + \sigma_\theta^2 v^2(n) \sin^2\theta(n)$$
$$R_1(4,4) = \mathrm{E}\{w_4^2(n)\} = \sigma_v^2 \sin^2\theta(n) + \sigma_\theta^2 v^2(n) \cos^2\theta(n)$$
$$R_1(2,4) = \mathrm{E}\{w_2(n)w_4(n)\} = (\sigma_v^2 - v^2(n)\sigma_\theta^2)\cos\theta(n)\sin\theta(n) \ .$$

Show that the measurement noise covariance, R_2, can be modeled by:

$$R_2(1,1) = \mathrm{E}\{e_1^2(n)\} = \sigma_r^2 \cos^2\varphi(n) + r^2(n)\sigma_\varphi^2 \sin^2\varphi(n)$$
$$R_2(2,2) = \mathrm{E}\{e_2^2(n)\} = \sigma_r^2 \sin^2\varphi(n) + r^2(n)\sigma_\varphi^2 \cos^2\varphi(n)$$
$$R_2(1,2) = \mathrm{E}\{e_1(n)e_2(n)\} = (\sigma_r^2 - r^2(n)\sigma_\varphi^2)\sin\varphi(n)\cos\varphi(n) \ .$$

HINT: There are no disturbances in position, and regard the noises as small.

c) Write MATLAB scripts that plot the states and the outputs, the initial state is $x(0) = [10, 1, 20, 2]^T$, (N=100).

Use the following values, for the two cases:

a) $T = 1 \;\; R_1 = 0.1\boldsymbol{I} \;\; R_2 = 5\boldsymbol{I}$

b) $T = 1 \;\; \sigma_v^2 = 0.01 \;\; \sigma_\theta^2 = 0.01 \;\; \sigma_r^2 = 1 \;\; \sigma_\varphi^2 = 0.1$

Plot the trajectory and the measured trajectory in the same figure.

4. Design and implement (in MATLAB) the Kalman filter associated with the radar problem given above. For case **a)**, compare the optimal filter, $K(n)$, with the suboptimal implementation, the stationary Kalman filter, $\bar{K}$ (you may want to try with higher noise variance). Hint: the command `dare` in Matlab may be useful.

   For case **b)** compare the exact time-varying Kalman filter with a Kalman filter using fixed covariance matrices.

   Try the filters with different initial values, $P(0)$ and x_0 . How will the choice of $P(0)$ and x_0 effect the behavior?